Handwritten Homework Assignments - Exploration for MATH 1554

For each assignment, complete the questions by hand on paper/tablet. Write neatly and use complete sentences where necessary. **You must submit original work**, but y'all can share ideas. Handwritten homework is due on Sunday in Gradescope; no late submissions accepted.

Week 1: *Practice sketching in 2D and 3D.* In $\mathbb{R}^2$, using the coordinates x_1, x_2, sketch the following making a new sketch for each part (a)-(b):

(a) sketch two *random* lines in $\mathbb{R}^2$ and label each line with the equation describing it, and draw and label the point of intersection,

(b) sketch one *random* line and a parallel line and label each line with the equation describing it.

In $\mathbb{R}^3$, using the coordinates x_1, x_2, x_3, sketch the following making a new sketch for each part (a)-(e):

(a) use the exact same equations from part (a) above in $\mathbb{R}^2$ above, but with the variables x_1, x_2, x_3 and understanding that the coefficient of x_3 should be understood to be zero; sketch the two **planes** and the **line** of intersection,

(b) the plane consisting of points satisfying the condition that $x_1 = 0$ (the *back wall*),

(c) the plane $x_3 = 0$ and the plane $x_1 = 0$ on the same axes, and sketch and label the line of intersection of these two planes,

(d) a line passing through the origin which is not contained in any of the three coordinate planes, include and label at least three points on the line,

(e) the plane defined by $2x + 3y + z = 6$, include and label at least four points in this plane no three of which are colinear, and at least one of the points should not lie in any of the coordinate axes or coordinate planes.

In each case, you are practicing drawing an accurate, representative graph of the plane of points which satisfy the given equation in the variables x_1, x_2, and x_3.

Week 2: *Practice with span, linear combination, and inconsistent systems.* ***(a)*** Choose two vectors v, w in $\mathbb{R}^2$ and a third vector b also in $\mathbb{R}^2$, and express b as a linear combination of v, w by finding scalars c_1, c_2 such that $c_1 v + c_2 w = b$. Sketch the situation in $\mathbb{R}^2$ with an illustration that uses the parallelogram rule. ***(b)*** Repeat part *(a)* with **new** vectors in $\mathbb{R}^3$ such that the augmented matrix $[v\ w \mid b]$ gives a consistent system, again illustrating by graphing but this time in $\mathbb{R}^3$. ***(c)*** Why is it harder to find a consistent system for part *(b)* compared to part *(a)*? Explain your idea clearly using complete sentences. *See next page!*

Warning!

PLEASE NOTE: Your submissions for this and future exploration assignments need to contain vectors which are *general looking*. Choosing vectors which are scalar multiples of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, or $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, or have too many zeros or ones, or are otherwise too simple and miss the point of the exploration will receive a deduction of points.

Please, do not ask on Piazza if your vectors are general enough to get full credit. The explorations are assignments which require you to make a *judgement call*, to **explore** a particular concept of the course and NOT to come up with the simplest example which satisfies the minimum requirements of the assignment.

Week 3: *Learn some basics of MATLAB.* See Sal's personal website for links to the MATLAB #1 Exploration. (requires MATLAB installation)

Week 4: *Propositional logic.* This week we will explore the basics of *propositional logic* in order to help develop some framework to practice True/False questions when studying for the exam. The assignment this week is to complete the three questions **Q1, Q2, Q3** and upload to Gradescope. The additional (non-question) text is just to help you understand the assignment.

Let's start with the main definitions:

Definition: A **mathematical statement** (aka, a **proposition**) is a sentence which is either true or false.

Each of the following are statements; they can be true or false, depending on the choice of vectors and/or matrices in each statement.

1. $A\vec{x} = \vec{b}$ is consistent.

2. $[A \mid \vec{b}]$ is row equivalent to $[C \mid \vec{d}]$.

3. $A\vec{x} = \vec{0}$ has a non-trivial solution.

4. The columns of A are linearly dependent.

5. $T(\vec{x}) = A\vec{x}$ is onto.

For example, the first statement (1.) is true if $A = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\2\\0\end{bmatrix}$, but this statement is false if $A = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

Q1: *You try it!* Pick **one** of the other statements (2.)-(5.) above and come up with choices for the vectors/matrices that make the statement true, and choices that make the same statement false. Check with calculations that your example works for each.

Note: please write the problem statement of the problem you are solving, to help the grader.

Statements can be combined in a variety of ways to create new statements, using for example AND, OR, or IMPLIES.

For example, each of the following are statements. (connecting word highlighted for emphasis)

6. $[A \mid \vec{b}]$ is row equivalent to $[C \mid \vec{d}]$ **and** $A\vec{x} = \vec{b}$ is consistent.

7. $A\vec{x} = \vec{0}$ has a non-trivial solution **or** $T(\vec{x}) = A\vec{x}$ is onto.

8a. The columns of A are linearly dependent **implies** $A\vec{x} = \vec{0}$ has a non-trivial solution.

8b. **If** the columns of A are linearly dependent, **then** $A\vec{x} = \vec{0}$ has a non-trivial solution.

The most common way that an IMPLIES statement is written is to use **if** and **then**. The statements (8a.) and (8b.) say exactly the same thing using different wording.

An implication statement is true whenever knowing that the "if part" is true forces the "then part" to also be true. So (8a.) is a true implication because whenever the columns of A are linearly dependent there is a free variable in the system $A\vec{x} = \vec{0}$, and assigning a non-zero value to the free variable gives a non-zero $\vec{x}$ which satisfies $A\vec{x} = \vec{0}$.

On the other hand, an implication statement is false if for some choice making the "if part" true, the "then part" is false. For example consider the following false implication.

9. If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, then $A(\vec{x} + \vec{y}) = \vec{b}$.

This implication is false because there is a counterexample for which the first part is true, but the second part is false. For example, choosing $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\vec{b} = \left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right]$ we see that $\vec{x} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\vec{y} = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ both satisfy $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, but $\vec{x} + \vec{y} = \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]$ and so $A(\vec{x} + \vec{y}) = \left[\begin{smallmatrix} 6 \\ 6 \end{smallmatrix}\right]$ (and $\left[\begin{smallmatrix} 6 \\ 6 \end{smallmatrix}\right]$ is not $\vec{b}$).

Q2: *You try it!* Select any one TRUE and any one FALSE true/false question from any of the practice exams that use IMPLIES (aka an if-then statement). For each of the two problems, identify the propositions in the problem (the "if part" and the "then part"). If the implication is false, provide a counter-example with explanation/calculations to show why it works. If the implication is true, give a short *general** explanation using precise and correct terminology from class. (* a short general proof - not an example)

Note: please write the problem statement of the problem you are solving, to help the grader.

Finally, some statements can have one or more **mathematical quantifiers**. There are two kinds of mathematical quantifiers, which are the universal quantifier **for all** (aka **for every**), and the existential quantifier **for some** (aka **there exists**).

For example, each of the following statements has a quantifier. (quantifier highlighted)

10. If $A\vec{x} = \vec{b}$ is consistent **for some** $\vec{b} \in R^m$, then A has a pivot in every row.

11. If A does not have a pivot in every column and $T(\vec{x}) = A\vec{x}$, then **for every** $\vec{x}_1$ and $\vec{x}_2$ with $\vec{x}_1 \neq \vec{x}_2$ we have that $T(\vec{x}_1) = T(\vec{x}_2)$.

Q3: *You try it!* In (10.) and (11.) identify the propositions in each problem. The implications (10.) and (11.) are both false; provide a counterexample to **one** of the implications. Give a short description (one or two short sentences) as to why your counterexample works, and verify any assertions you make with calculations.

Note: please write the problem statement of the problem you are solving, to help the grader.

Hint: For (10.) a counterexample will be a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent, but A does not have a pivot in every row.

Hint: For (11.) a counterexample will be a matrix A that does not have a pivot in every column, and such that there exist vectors $\vec{x}_1$ and $\vec{x}_1$ such that $\vec{x}_1 \neq \vec{x}_2$ and $T(\vec{x}_1) \neq T(\vec{x}_2)$.

Week 5: *Practice with transformations.* For each matrix A below, ***(a)*** state the domain and codomain of T_A, ***(b)*** find $T_A(e_1), T_A(e_2)$, ***(c)*** find $T_A(v), T_A(w)$, ***(d)*** describe in a few words what the transformation is doing, and ***(e)*** state whether the associated transformation is one-to-one/onto, and ***(f)*** give the matrix an appropriate "name" (fine to be silly name like, e.g., "the x-zero-er" for projection to y-axis"). For the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ \text{-3} \end{bmatrix}$$

(1a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(1b) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(1c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(1d) $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

(1e) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

(1f) $A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

(1g) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(1h) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

Next, for the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

(2a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2c) $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

(2d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Sal says: For the "name your matrix" this is a bit silly, and that's ok. Just come up with a creative name that makes sense to you, and don't worry about it too much!

Hint: For part **(2c)** above, an appropriate name could be 'the total mystery' or the 'crazy transformation', or maybe 'some kind of weird shear embedding'. It's not always easy to see what the matrix is doing! Don't overthink this one, please.

Week 6: *Walkthrough Nul(A) is a subspace.*

Step 1: Pick a matrix and find Nul(A). Pick a matrix A of size 3×5 (to get a good feel for the problem). Choose entries not all positive, and not too many zeros, and your matrix shouldn't be rref (but it's ok to start with a matrix in rref and do some row operations to jumble up the numbers to get your A matrix). Find the null space Nul(A) by finding the parametric vector form of the general solution x to $A\mathrm{x} = 0$, and write $\mathrm{Nul}(A) = \mathrm{span}\{v_1, v_2, \ldots, v_k\}$ where $v_1, v_2, \ldots, v_k$ are the vectors appearing in parametric vector form.

Step 2: An example that Nul(A) is closed under vector addition. Choose two vectors $\mathrm{w}_1, \mathrm{w}_2$ in the span $\mathrm{Nul}(A) = \mathrm{span}\{v_1, \ldots, v_k\}$ from step 1. Do this by taking two or more vectors in the basis from step 1 and adding them to each other using some scalars, *i.e.* chose a random linear combination of the vectors $v_1, \ldots, v_k$ from step 1. Do this twice with different weights each time, once to get w_1 and once to get w_2. Add these vectors together to get $\mathrm{z} = \mathrm{w}_1 + \mathrm{w}_2$. **Verify** that z is in the null space of A **using the definition of null space**, by multiplying A times z and verifying that $A\mathrm{z} = 0$.

Step 2b: Answer the following question. What are the weights of z if you write z as a linear combination of $v_1, \ldots, v_k$, and how are these weights related to the weights of $\mathrm{w}_1, \mathrm{w}_2$?

Step 3: An example that Nul(A) is closed under scalar multiplication. Let w be either w_1 or w_2 from step 2, one of the random vectors in the null space of A. Choose a random scalar c. Check that $c\mathrm{w}$ is in the null space of A by verifying $A(c\mathrm{w}) = 0$.

Step 3b: Answer the following question. What are the weights of $c\mathrm{w}$ if you write $c\mathrm{w}$ as a linear combination of $v_1, \ldots, v_k$, and how are these weights related to the weights of w?

Step 4: The general case. Try to convince yourself that no matter how A is chosen, Nul(A) is always closed under scalar multiplication and vector addition. *Hint: one way is to use the facts about matrix-vector muliplication that $A(x + y) = Ax + Ay$ and $A(cx) = c(Ax)$, and another option is to think about span and how if x, y are in span$\{v_1, \ldots, v_k\}$ the weights of $x + y$ are related to the weights of x, y and the weights of cx are related to the weights of x.*

Week 7: *Transformations and the determinant.*

Step 1: Sketch a parallelogram somewhere in $\mathbb{R}^2$ such that none of the vertices of the parallelogram lie on the origin. Label the vertices of the parallelogram a, b, c, d and also label the coordinates (x_1, x_2) for each of the four points. Label the parallelogram S for *shape*.

Step 2b: Showing your work, compute the area area(S) using the content of Section 3.3, by finding a pair of vectors $\vec{v}_1, \vec{v}_2$ which determine the parallelogram from *(Step 1)*. Go back and label the vectors $\vec{v}_1, \vec{v}_2$ in your sketch from *(Step 1)*.

Step 2: Choose a (somewhat random) linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which can be anything but must satisfy:

(i) the transformation should not have a diagonal standard matrix,

(ii) the transformation should be invertible (one-to-one and onto),

(iii) the transformation should not be a single rotation or a single reflection*.

(*) it is ok for the transformation you pick to be a rotation followed by a reflection, for example. However, if your transformation is 'two rotations' or 'two reflections' but can be represented by a single rotation or single reflection, then you will lose points.

Step 3: Transform your shape S from *(Step 1)* using your transformation T from *(Step 2)*. That is, sketch the **image of** S in $\mathbb{R}^2$. Make a new sketch for this part and label the images $T(a), T(b), T(c), T(d)$ and give the new coordinates for all four points.

Step 3b: Showing all work, compute the area area$(T(S))$ using the same method as *(Step 2b)* by finding the images of $\vec{v}_1, \vec{v}_2$ under the transformation T. Label the images $T(\vec{v}_1), T(\vec{v}_2)$ in your sketch from *(Step 3)*.

Step 4: Compute $\det A$ where A is the standard matrix of $T(\vec{x}) = A\vec{x}$, and compare the areas area(S) and area$(T(S))$ with the value of $\det A$. Write a general formula which relates these three quantities.

Week 8: MATLAB #2 Basis of Eigenvectors and Markov chains. *Note: see Sal's webpage.*

Week 9: Three separate parts.

1. Find a 2×2 matrix A *with real entries* with **no real eigenvalues** and show that it is correct by finding the characteristic polynomial and explaining why it has no real roots.
2. Find all eigenvalues and a corresponding eigenvector for each matrix below **without calculations** by thinking it out using the linear transformation's **geometric interpretation**.

For each matrix, graph each eigenvector and it's image after the transformation as well as a random NON-eigenvector. Check your graph is accurate with matrix multiplication. Confirm your geometric intuition by finding the eigenvalues, algebraic and geometric multiplicity of each eigenvalue, and a basis of eigenvectors for each eigenspace (you may need to do this step first for any transformations that you can not visualize).

(a) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, T_A =zero.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, T_A =projection onto x-axis.

(c) $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, T_A =rotation by θ.
(and state the values of θ for which A has (real) eigenvectors)

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, T_A =reflect about the line "$y = x$".

(e) $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, T_A =stretch in x-direction.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, T_A =shear.

(g) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, T_A =project to the floor.

(h) $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, T_A =rotate by 90° about the z-axis.

3. Pick any two transformations of $\mathbb{R}^2$ above other than (a), do one row operation to the matrix to obtain a matrix B and compute the eigenvalues of the new matrix. Make sure to pick A and do a row operation so that B has different eigenvalues than A (at least one eigenvalue should be different - this is **not** guaranteed). Compute the eigenvectors for each eigenvalue, are they the same or different? Finish the sentence below.

When you do a single row operation to a matrix A then the eigenvalues ... and the eigenvectors ....

Week 10: NOTE: This week's assignment doesn't have to be handwritten. (For 1pt) Write a paragraph which explains how eigenvectors/eigenvalues or some other topic from the course are used in a field which interests you. Be specific and put some thought into this. (For 2pt) If your paragraph is not essentially the thing on wikipedia about how bridges have something to do with eigenvalues, but actually give some details or original content then you get 2pts instead of 1pt. (For 3pt) If you also support your research with real math or alternately something creative. This has to include *some mathematical content* but can take *any form whatsoever*. Last year's submissions included a poem, several posters, a few slide-show presentations like using PowerPoint, etc., and quite a few of them actually were pretty decent research project results that I was quite impressed by, but I remember the poem the best; it was funny and it used the right math ideas about linear algebra to *be* funny, which essentially forces that the person *understood* the concepts. It was brilliant.

WARNING: Your submission must **not** be something already covered this semester, or about something we will cover later. So you can not write about the Google matrix, for instance. If you want to write about a topic that was cut but appeared previously then that's ok (e.g., Leontief, homogeneous coordinates, computer graphics.)

Please understand the point of this exercise, should you choose to do it: Pick any scientific discipline. I mean ANY. If it is scientific it's ok: how to build a bridge, how do design a new chemical, how to solve some hard algorithmic problem using computers (like how many stars the Netflix algorithm should predict for your enjoyment of the 1977 original version of Disney's Pete's Dragon, for example). Take 5 steps into your chosen scientific field and you will bump into linear algebra. That's the exercise. 1pt is essentially "write down in your own words what wiki has to say about it", 2pts is essentially "do something a little better but without any real math content", and then 3pts is "a pretty good job explaining how a specific concept from linear algebra is used in a scientific field you are interested in", where it is up to the grader to determine if what you present with the math explaining qualifies for +3pt.

Week 11: MATLAB #3 Least-squares exploration - see Sal's website for instructions and supplementary documents.

Week 12: MATLAB #4 Google PageRank - see Sal's website for instructions.

Week 13: *Spectral decomposition and constrained optimization.*

Part 1: Pick two random vectors in $\mathbb{R}^3$ (no zeros - few repeated numbers) and call them $\vec{u}$ and $\vec{v}$. Compute the matrix $\vec{u}\vec{v}^T$ which is size 3×3. Write out the entire sentence below, with the boxes, and fill them in with the appropriate word/phrase to complete the sentence.

The rows of the matrix $\vec{u}\vec{v}^T$ are scalar multiples of [ ] with scalars that are [ ].

Part 2: Let

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix}$$

Compute (a) an orthogonal decomposition $A = PDP^T$ where $D = \left[\begin{smallmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right]$, and (b) the spectral decomposition of A, namely express A as the sum of rank 1 matrices

$$A = \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T + \lambda_3 \vec{v}_3 \vec{v}_3^T$$

where $P = \left[\, \vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \,\right]$ so that $\vec{v}_i$ are the normalized eigenvectors of A. Last, (c) write down $Q_A(\vec{x}) = \vec{x}^T A \vec{x}$ the quadratic form defined by A. You will analyze this quadratic form next.

Part 3: First (a) Write $Q_A(\vec{x})$ as a quadratic form with no cross terms $Q_A(\vec{x}) = Q_D(\vec{y})$ using the orthogonal diagonalization in Part 2. Then (b) find a unit vector $\vec{y}$ which attains the maximum $M = \max\{Q_D(\vec{y}) : \|\vec{y}\| = 1\}$, and (c) find a corresponding vector $\vec{x}$ which maximizes $Q_A(\vec{x})$. Finally, (d) find a vector $\vec{x}$ which attains the same value of as $Q_D(1,1,0) = 15$ when plugged into Q_A, check your answer by plugging in the vector $\vec{x}$ into $Q_A(\vec{x})$ and make sure you get 15 as the output.

Week 14: MATLAB #5 - see Sal's website for instructions and supplementary documents.