LINEAR ALGEBRA

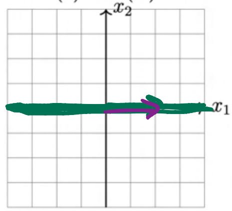
week 8

Midterm 2 Lecture Review Activity, Math 1554

a nonzero v in the

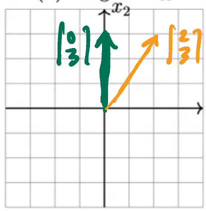
1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, ~~the~~ range of the transform, and the column space of A. How are the range and column space related to each other?

(a) Null(A)

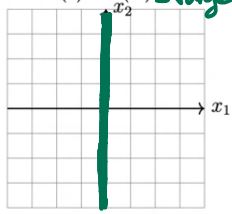


a nonzero v in the

(b) range of T_A



(c) Col(A) = range of T



Course Schedule

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Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
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7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
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9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
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$$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}0\\3\end{bmatrix}$$

↑ in the range.

Exam @ 6:30 pm tonight.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2)\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$\text{Col}A = \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

Nul A: $A \sim \begin{array}{c} \overset{s}{\downarrow} \\ \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\end{array}$ $\quad \vec{x} = s\begin{bmatrix}1\\0\end{bmatrix}$

$$A\vec{x} \stackrel{*}{=} x_1\begin{bmatrix}0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = \vec{b}$$

2. Indicate **true** if the statement is true, otherwise, indicate **false**.

a=1.

	true	false
a) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = a, x_2 = 4a, x_3 = x_1x_2\}$ is a subspace for any ~~$a \in$~~ $\mathbb{R}$.	○	●
b) If A is square and non-zero, and $\boxed{A\vec{x} = A\vec{y} \text{ for some } \vec{x} \neq \vec{y},}$ then det(A) $\neq 0$.	○ ✓	●

$A(x-y)=0$ $\qquad$ $\det A = 0$

$$S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = 1, x_2 = 4, x_3 = 1\cdot 4 = 4\} = \left\{\begin{bmatrix}1\\4\\4\end{bmatrix}\right\}$$

Not a subspace.

$$3\begin{bmatrix}1\\4\\4\end{bmatrix} = \begin{bmatrix}3\\12\\12\end{bmatrix} \notin S$$

e.g. S is not closed under scalar mult.

(a') $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 = a, \ x_3 = a\}$ $\qquad$ a=1.

$$= \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 = 1, \ x_3 = 1\} =$$

(a') $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 = a,\ x_3 = a\}$ $\quad \underline{a=1}$.

$= \{\vec{x} \in \mathbb{R}^3 \mid \underline{x_1 + x_2 = 1},\ \underline{x_3 = 1}\}$

(0) $\vec{0} \in S$
(1) $\vec{v} + \vec{w} \in S$ if $\vec{v}, \vec{w} \in S$
(2) $c\vec{v} \in S$ if $c \in \mathbb{R}$, $\vec{v} \in S$

$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ldots$ are in S.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$

X closed under scalr mult?

$2\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} \notin S$?

IMT $\quad$ A not invertible $\Rightarrow \det A = 0$

2. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = a, x_2 = 4a, x_3 = x_1 x_2\}$ is a subspace for any $a \in \mathbb{R}$.	○	○
b) If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then $\det(A) \neq 0$.	○	●

Ex. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = A\vec{x}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = A\vec{y}$

$A\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ has two solns.

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

$\text{Nul}A = \text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

(a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix}2\\3\end{pmatrix}$ and ~~$\dim(\text{Null}(A)) = 1$~~. $A = \begin{pmatrix}2 & -1\\ 3 & -3/2\end{pmatrix}$

(b) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix}2\\3\end{pmatrix}$ and $\dim(\text{Null}(A)) = 0$. $A = \begin{pmatrix}NP\end{pmatrix}$

(c) A is in RREF and $T_A:\ \mathbb{R}^3 \to \mathbb{R}^3$. The vectors u and v are a basis for the range of T.

$u = \begin{pmatrix}1\\0\\0\end{pmatrix}, v = \begin{pmatrix}1\\1\\0\end{pmatrix}, A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ $\text{Col}A = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ $\dim \text{Col}A + \dim \text{Nul}A = \#\text{cols}.$

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix}2 & a\\ 3 & b\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$2+2a=0 \quad a=-1$

$3+2b=0 \quad b=-3/2$

$$\boxed{A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}\ ?$$

(b') Col$A$ is spanned by $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$, $\boxed{A = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}}$

$$\boxed{\text{Col}A = \mathbb{R}^2}$$

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

$Av = 2v \quad Av = 3v$

$2v = 3v$

$\Rightarrow 3v - 2v = 0$

$\Rightarrow (3-2)v = 0$

		possible	impossible
4.i)	Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u}+\vec{v}$ is also an eigenvector of A.	●	○
4.ii)	$T_A = A\vec{x}$ is one-to-one, $\dim(\text{Col}(A)) = 4$, and $T_A:\ \mathbb{R}^3 \to \mathbb{R}^4$.	○	●

$\vec{u}, \vec{v}$ need same λ!

$\dim \text{Col}A = 3$ if T_A is 1-1.

$$A v = 2v$$

$$\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} \quad \lambda = 2$$

$$\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix} = 2\begin{bmatrix}1\\3\end{bmatrix} \quad \lambda = 2$$

$$A\vec{u} = 2\vec{u}$$

$$\begin{bmatrix}* & * & *\\ * & * & *\\ * & * & *\\ * & * & *\end{bmatrix}$$

$$A(v+u) = Av + Au$$

$$= 2v + 2u = 2(v+u)$$

4x6 rank A = rank A^T.

5. (2 points) Fill in the blanks.

   (a) If A is a 6×4 matrix in RREF and $\text{rank}(A) = 4$, what is the rank of A^T? $\boxed{4}$

   (b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? $\boxed{0}$

$\text{Vol}(T(S)) = 0$ for any shape S.

$= |\det A| \, \text{Vol}(S)$

6. (3 points) A virus is spreading in a lake. Every week,

   - 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
   - 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

   Initially there are exactly 1000 fish in the lake.

   a) What is the stochastic matrix, P, for this situation? Is P regular?

   b) Write down any steady-state vector for the corresponding Markov-chain.

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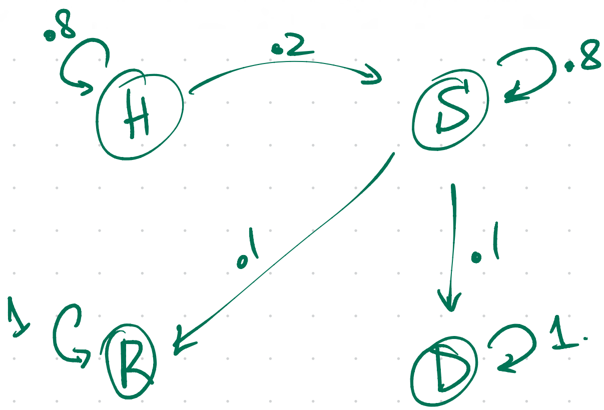

$$\begin{array}{c|cccc|c} & H & S & R & D & \\ P = & .8 & 0 & 0 & 0 & H \\ & .2 & .8 & 0 & 0 & S \\ & 0 & .1 & 1 & 0 & R \\ & 0 & .1 & 0 & 1 & D \end{array}$$

$$q_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = q_2$$

$$q_3 = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

are all steady state prob vectors!

$$P - I = \begin{bmatrix} -.2 & 0 & 0 & 0 \\ .2 & -.2 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & .1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} & & & s & t \\ \sim & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{array}$$

$$\vec{X} = s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$s = 2$
$t = 3$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ 2/5 \\ 3/5 \end{pmatrix} = \vec{q}_4$$

Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Motivation: it can be useful to take large powers of matrices, for example A^k, for large k.

But: multiplying two $n \times n$ matrices requires roughly n^3 computations. Is there a more efficient way to compute A^k?

Topics and Objectives

Topics

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

Learning Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

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https://strawpoll.com/eJnvvpDYanv

Diagonal Matrices

A matrix is **diagonal** if the only non-zero elements, if any, are on the main diagonal.

The following are all diagonal matrices.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad [2], \quad I_n, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll only be working with diagonal square matrices in this course.

Powers of Diagonal Matrices

If A is diagonal, then A^k is easy to compute. For example,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & .5 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & .25 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3^k & 0 \\ 0 & (0.5)^k \end{bmatrix}$$

But what if A is not diagonal?

A^k got a way to do this?

Diagonalization

why!

Suppose $A \in \mathbb{R}^{n\times n}$. We say that A is **diagonalizable** if it is similar to a diagonal matrix, D. That is, we can write

$$A = PDP^{-1}$$

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k \text{ times}}$$

$$= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$$

$$= PD\cancel{P^{-1}P}DP\cancel{^{-1}P}\cdots\cancel{P^{-1}P}DP^{-1}$$

$$= PD\cdots DP^{-1} = PD^kP^{-1}$$

Diagonalization

what/how.

Theorem

If A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

Note: the symbol $\Leftrightarrow$ means " if and only if ".

Also note that $A = PDP^{-1}$ if and only if

$$A = \underset{P}{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]} \overset{D}{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}} [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]^{-1}$$

where $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent eigenvectors, and $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues (**in order**).

$$A = PDP^{-1}$$

eigenvalues

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}^{-1}$$

cols are eigenvectors

Distinct Eigenvalues

Theorem

If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.

Why does this theorem hold?

eigenvectors w/ different eigenvalues are lin ind.

Is it necessary for an $n \times n$ matrix to have n distinct eigenvalues for it to be diagonalizable?

For repeated eigenvalues

Non-Distinct Eigenvalues

(alg mult > 1)

Theorem. Suppose

- A is $n \times n$
- A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$, $k \le n$ $\quad p(\lambda) = (\lambda-1)(\lambda-2)^2$ $\quad$ alg mult of $\lambda=2$ is 2.
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity") = # free vars of $A - \lambda I$.

Then

1. $d_i \le a_i$ for all i $\quad$ alg $\geq$ geo.
2. A is diagonalizable $\Leftrightarrow$ $\Sigma d_i = n \Leftrightarrow d_i = a_i$ for all i
3. A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

A has these eigenvectors

has these eigenvalues.

Example 1

Diagonalize if possible.

$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$ $\quad A = PDP^{-1}$

$= [\vec{v}_1\ \vec{v}_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}[\vec{v}_1\ \vec{v}_2]^{-1}$

Find $\vec{v}$'s and λ's.

Step 1: Find λ's.

$\det(A - \lambda I) =$

$p(\lambda) = \det\begin{bmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 0$

$= -2 - 2\lambda + \lambda + \lambda^2$

$= \lambda^2 - \lambda - 2 = 0$

$\Rightarrow (\lambda-2)(\lambda+1) = 0$

$\boxed{\lambda = 2, -1}$

Example 2

Diagonalize if possible.

$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}}$$

Next:

Step 2

$\lambda_1 = 2$ $\quad \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} - 2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$ $\quad \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} - (-1)I = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{x} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Ex $\quad A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)^2 - 1$

$= \lambda^2 - 6\lambda + 8$

$= (\lambda-4)(\lambda-2) = 0$

$\boxed{\lambda = 4, 2}$

$\lambda_1 = 4$

$A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}$$

??

$\lambda_2 = 2$ $\quad A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{x} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Example 3 $p(\lambda) = -(\lambda-1)(\lambda-3)(\lambda-c)$

$c = 1$ or 3

The eigenvalues of A are $\lambda = 3, 1$. If possible, construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the n^{th} number in this sequence.

$\lambda_1 = 3$

$$A - 3I = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

can swap

$$P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^{-1}$$

use same P.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda_2 = 1$

$$A - I = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -3 \\ 2 & 4 & 8 \\ 6 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -3 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \begin{bmatrix} 1 & 0 & +2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \vec{x} = s\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the n^{th} number in this sequence.

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE 4 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

THEOREM 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

THEOREM 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

5.3 EXERCISES

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

In Exercises 5 and 6, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} =$
$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} =$
$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$

Diagonalize the matrices in Exercises 7–20, if possible. The eigenvalues for Exercises 11–16 are as follows: (11) $\lambda = 1, 2, 3$; (12) $\lambda = 2, 8$; (13) $\lambda = 5, 1$; (14) $\lambda = 5, 4$; (15) $\lambda = 3, 1$; (16) $\lambda = 2, 1$. For Exercise 18, one eigenvalue is $\lambda = 5$ and one eigenvector is $(-2, 1, 2)$.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

18. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 21 and 22, A, B, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.
    b. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
    c. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
    d. If A is diagonalizable, then A is invertible.

22. a. A is diagonalizable if A has n eigenvectors.
    b. If A is diagonalizable, then A has n distinct eigenvalues.
    c. If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.
    d. If A is invertible, then A is diagonalizable.

23. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

27. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

28. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

29. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix A in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

30. With A and D as in Example 2, find an invertible P_2 unequal to the P in Example 2, such that $A = P_2 D P_2^{-1}$.

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

[M] Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

33. $\begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$ **34.** $\begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$

35. $\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$