LINEAR ALGEBRA

Week 9

Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Motivation: it can be useful to take large powers of matrices, for example A^k, for large k.

But: multiplying two $n \times n$ matrices requires roughly n^3 computations. Is there a more efficient way to compute A^k?

Topics and Objectives

Topics

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

Learning Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course m

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 – 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 – 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 – 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 – 9/15	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2
5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	Exam 2, Review	Cancelled	5.3
9	10/16 – 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 – 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 – 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 – 11/10	6.6	WS6.5,6.6	Exam 3, Review	Cancelled	PageRank
13	11/13 – 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 – 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 – 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 – 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 – 12/15	Final Exams: MATH 1554 Common Final Exam Tuesday, December 12th at 6pm				

https://strawpoll.com/eJnvvpDYanv

Diagonal Matrices

A matrix is **diagonal** if the only non-zero elements, if any, are on the main diagonal.

The following are all diagonal matrices.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad [2], \quad I_n, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll only be working with diagonal square matrices in this course.

Powers of Diagonal Matrices

If A is diagonal, then A^k is easy to compute. For example,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & .25 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3^k & 0 \\ 0 & 0.5^k \end{bmatrix}$$

But what if A is not diagonal?

A^k got a way to do this?

Diagonalization

Why!

Suppose $A \in \mathbb{R}^{n \times n}$. We say that A is **diagonalizable** if it is similar to a diagonal matrix, D. That is, we can write

$$A = PDP^{-1}$$

k times

$$A^k = A \cdot A \cdot A \cdots A$$

$$= (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$

$$= PDP^{-1}PDP^{-1}\cdots P^{-1}PDP^{-1}$$

$$= PD\cdots DP^{-1} = PD^k P^{-1}$$

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Diagonalization

what / how.

Theorem
If A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

Note: the symbol $\Leftrightarrow$ means " if and only if ".

Also note that $A = PDP^{-1}$ if and only if

$$A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix} [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]^{-1}$$

P *D*

where $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors, and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (**in order**).

eigenvalues

$$A = P D P^{-1}$$

$$= [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]^{-1}$$

cols are eigenvectors

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Distinct Eigenvalues

Theorem
If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.

Why does this theorem hold?

eigenvectors w/ different eigenvalues are lin ind.

Is it necessary for an $n \times n$ matrix to have n distinct eigenvalues for it to be diagonalizable?

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Non-Distinct Eigenvalues

For repeated eigenvalues

alg mult > 1

Theorem. Suppose
- A is $n \times n$
- A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \leq n$ $p(\lambda) = (\lambda-1)(\lambda-2)^2$ *alg mult of $\lambda=2$ is 2.*
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity") = *#Free vars of $A - \lambda I$.*

Then
1. $d_i \leq a_i$ for all i *alg $\geq$ geo.*
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n \Leftrightarrow d_i = a_i$ for all i
3. A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

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$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

A has these eigenvectors

has these eigenvalues.

Example 1

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

$A = PDP^{-1}$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$$

↑↑ Find v's and λ's.

Step 1: Find λ's.

$\det(A - \lambda I) =$

$p(\lambda) = \det \begin{bmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 0$

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$= -2 - 2\lambda + \lambda + \lambda^2$

$= \lambda^2 - \lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$

$\boxed{\lambda = 2, -1}$

Example 2

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

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$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}$$

Next:

Step 2

$\lambda_1 = 2$ $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} - 2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\vec{x} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} - (-1)I = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ $\vec{x} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Ex $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)^2 - 1$

$= \lambda^2 - 6\lambda + 8$

$= (\lambda - 4)(\lambda - 2) = 0$

$\lambda_1 = 4$

$A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\boxed{\lambda = 4, 2}$

$\vec{x} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$ $A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\vec{x} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

? ?

Example 3

$p(\lambda) = -(\lambda-1)(\lambda-3)(\lambda-c)$ $c=1$ or 3

The eigenvalues of A are $\lambda = 3, 1$. If possible, construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the nth number in this sequence.

$\lambda_1 = 3$

$$A - 3I = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$s \; t$

$$\vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Can swap.

$$P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^{-1}$$

use same P.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda_2 = 1$

$$A - I = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -3 \\ 2 & 4 & 8 \\ 6 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -3 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & +2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = s \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$\checkmark s$

Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the nth number in this sequence.

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

$$A = PDP^{-1}$$
eigenvalues in D

EXAMPLE 4 Diagonalize the following matrix, if possible.

NOT diag'ble!

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$\lambda_1 = 1 \qquad \lambda_2 = -2.$

$\lambda = 1$

$$A - I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 9 & 9 \\ 0 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

check:

$$A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-4+3 \\ -4+6-3 \\ 3-3+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \checkmark$$

$$\vec{x} = s \begin{bmatrix} +1 \\ -1 \\ 1 \end{bmatrix}$$

If one free var here then A is not diag'ble.

$\lambda_2 = -2.$

$$A + 2I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -7 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

one free ~n

check

$$A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+4 \\ 4-6 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = (-2)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$$\vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

THEOREM 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

THEOREM 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Other way?

If diag'ble $\not\Rightarrow$ n distinct eigenvalues

Basis of Eigenvectors

Express the vector $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ as a linear combination of the vectors

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and find the coordinates of $\vec{x}_0$ in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_0\}$.

$[\vec{x}_0]_\mathcal{B} = \begin{bmatrix} 4.5 \\ -.5 \end{bmatrix}$

Let $P = [\vec{v}_1\ \vec{v}_2]$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and find $[A^k \vec{x}_0]_\mathcal{B}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$[A^k \vec{x}_0]_\mathcal{B} = \begin{bmatrix} 4.5 \\ (-1)^k(-.5) \end{bmatrix}$

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$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -1/2 \end{bmatrix}$

idea mult coords by λ's.

```
clc
P=[1 1 ; 1 -1]
% first example
%D=[1 0 ; 0 -1]
% part 2
%D=[1 0 ; 0 -1/2]
% part 3
D=[2 0 ; 0 3/2]
A=P*D*inv(P)
x0=[4;5];
s=10
format bank
for k=0:s
    % convert current index to string and
create xk and coordk strings
    index=string(k);
    s=strcat('x',index,'=');
    c=strcat('[x',index,']_B=');
    % compute xk value
    xk=A^k*x0;
    coordk=inv(P)*xk;
    % display each xk=A^k*x0
    disp(s)
    disp(xk)
    disp(c)
    disp(coordk)
end
```
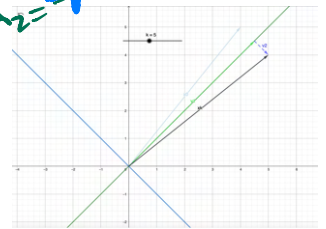
$x_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$-\frac{1}{2}v_2$

$4.5v_1$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 5 \\ 4 \end{bmatrix} = Ax_0 = x_1$

From picture

$X_1 = 4.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{c}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$X_0 = 4.5 \vec{v}_1 - .5 \vec{v}_2$

$A v_1 = v_1 \quad \lambda_1 = 1$
$A v_2 = -v_2 \quad \lambda_2 = -1$

$A X_0 = 4.5 A\vec{v}_1 - .5 A\vec{v}_2$

$= 4.5(1)\vec{v}_1 + 0.5 \cdot (-1)\vec{v}_2 =$

$\boxed{X_1 = 4.5 \vec{v}_1 + .5 v_2}$

$A = PDP^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

https://www.geogebra.org/calculator/bcahjxed

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Basis of Eigenvectors - part 2

Let $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as before.

Again define $P = [\vec{v}_1 \ \vec{v}_2]$ but this time let $D = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$, and now find $[A^k \vec{x}_0]_B$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$

$[A^k \vec{x}_0]_B =$

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$$A = \begin{bmatrix} .25 & .75 \\ .75 & .25 \end{bmatrix}$$

https://www.geogebra.org/calculator/czdnmrgc

$$\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4.5\vec{v}_1 - .5\vec{v}_2$$

$$A\vec{v}_1 = \vec{v}_1$$
$$A\vec{v}_2 = -\frac{1}{2}\vec{v}_2$$

$$A\vec{x}_0 = 4.5 A\vec{v}_1 - .5 A\vec{v}_2$$

$$\vec{x}_1 = 4.5\vec{v}_1 + .25\vec{v}_2 = 4.5\begin{bmatrix} 1 \\ 1 \end{bmatrix} + .25\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4.75 \\ 4.25 \end{bmatrix}.$$

$$A\vec{x}_1 = \vec{x}_2 = 4.5 A\vec{v}_1 + .25 A\vec{v}_2$$
$$= 4.5\vec{v}_1 - .125\vec{v}_2$$

Let $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as before.

Again define $P = [\vec{v}_1 \ \vec{v}_2]$ but this time let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$, and now find $[A^k \vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$$[A^k \vec{x}_0]_{\mathcal{B}} = \begin{bmatrix} 4.5\,(2)^k \\ -.5\,(1.5)^k \end{bmatrix}$$

Section 5.3   Slide 15
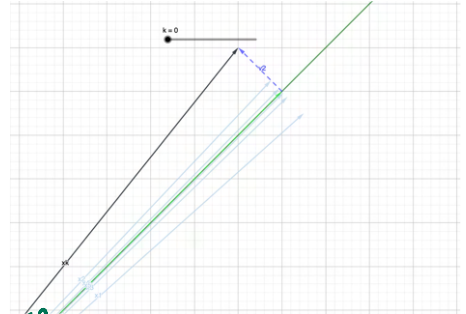
$$A = \begin{bmatrix} 1.75 & .25 \\ .25 & 1.75 \end{bmatrix}$$

https://www.geogebra.org/calculator/ddcanyxh

has two eigenvalues

$\lambda_1 = 2$

$\lambda_2 = 1.5$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
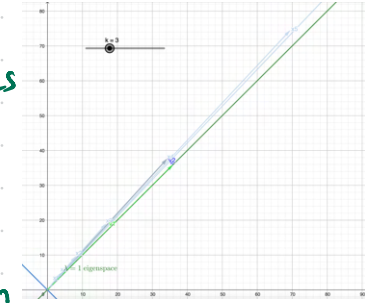
e.g.
$$\vec{x}_8 = A^8 \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1139.19 \\ 1164.81 \end{bmatrix}$$

$$\vec{x}_0 = 4.5\,\vec{v}_1 - 0.5\,\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\vec{e}_1 + 5\vec{e}_2$$

$$\vec{x}_1 = 9\,\vec{v}_1 - .75\,\vec{v}_2 = 4.5(2)\vec{v}_1 - 0.5(1.5)\vec{v}_2 = \begin{bmatrix} 8.25 \\ 9.75 \end{bmatrix} = 8.25\vec{e}_1 + 9.75\vec{e}_2$$

$$\vec{x}_2 = 4.5(2)^2 - 0.5(1.5)^2\,\vec{v}_2 = \begin{bmatrix} 16.88 \\ 19.12 \end{bmatrix} = 16.88\vec{e}_1 + 19.12\vec{e}_2$$

$$\vdots$$

$$\vec{x}_8 = 4.5(2)^8\,\vec{v}_1 - 0.5(1.5)^8\,\vec{v}_2 = \begin{bmatrix} 1139.19 \\ 1164.81 \end{bmatrix} = 1139.19\vec{e}_1 + 1164.81\vec{e}_2$$

5.3 EXERCISES

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

In Exercises 5 and 6, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} =$
$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} =$
$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$

Diagonalize the matrices in Exercises 7–20, if possible. The eigenvalues for Exercises 11–16 are as follows: (11) $\lambda = 1, 2, 3$; (12) $\lambda = 2, 8$; (13) $\lambda = 5, 1$; (14) $\lambda = 5, 4$; (15) $\lambda = 3, 1$; (16) $\lambda = 2, 1$. For Exercise 18, one eigenvalue is $\lambda = 5$ and one eigenvector is $(-2, 1, 2)$.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

18. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 21 and 22, A, B, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

b. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

c. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d. If A is diagonalizable, then A is invertible.

22. a. A is diagonalizable if A has n eigenvectors.

b. If A is diagonalizable, then A has n distinct eigenvalues.

c. If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.

d. If A is invertible, then A is diagonalizable.

23. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

27. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

28. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

29. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix A in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

30. With A and D as in Example 2, find an invertible P_2 unequal to the P in Example 2, such that $A = P_2 D P_2^{-1}$.

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

[M] Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

33. $\begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$ **34.** $\begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$

35. $\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$

Chapter 5 : Eigenvalues and Eigenvectors

5.5 : Complex Eigenvalues

complex

number

crash course.

Chapter 5 : Eigenvalues and Eigenvectors

5.5 : Complex Eigenvalues

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

$\det(A+B) = \det A = 1$

$A+B = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

Topics and Objectives

Topics
1. Complex numbers: addition, multiplication, complex conjugate
2. Diagonalizing matrices with complex eigenvalues
3. Eigenvalue theorems

Learning Objectives
1. Diagonalize 2×2 matrices that have complex eigenvalues.
2. Use eigenvalues to determine identify the rotation and dilation of a linear transform.
3. Apply theorems to characterize matrices with complex eigenvalues.

Motivating Question
What are the eigenvalues of a rotation matrix?

Section 5.5   Slide 245

Imaginary Numbers

Recall: When calculating roots of polynomials, we can encounter square roots of negative numbers. For example:

$$x^2 + 1 = 0$$ root of $\sqrt{-1}$

The roots of this equation are:

$(x-i)(x+i) = 0$

We usually write $\sqrt{-1}$ as i (for "imaginary").

Section 5.5   Slide 246

$\mathbb{Q}, \mathbb{R}$

$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ v · · · $\boxed{\mathbb{C}}$

adding numbers
mult numbers
divide ?

Addition and Multiplication

$\boxed{i^2 = -1}$

The imaginary (or complex) numbers are denoted by C, where

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify C with $\mathbb{R}^2$: $a + bi \leftrightarrow (a, b)$

$a+bi = z \in \mathbb{C}$
↑ real part of z is $a \in \mathbb{R}$
the imaginary part of z is $b \in \mathbb{R}$

We can add and multiply complex numbers as follows:
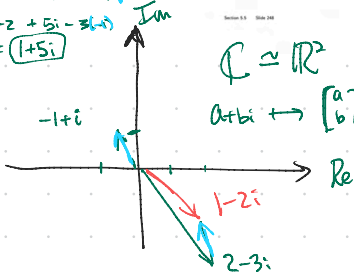
$(2-3) + (-1+4i) = 2-1 + (-3+4)i = 1 - 2i$

$(2-3i)(-1+i) = (2)(-1) + (2)(i) + (-3i)(-1) + (-3i)(i)$
$= -2 + 2i + 3i - 3i^2$
$= -2 + 5i - 3(-1)$
$= \boxed{1 + 5i}$

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$-1+i$

$1-2i$

$2-3i$

$\mathbb{C} \simeq \mathbb{R}^2$

$a+bi \longmapsto \begin{bmatrix} a \\ b \end{bmatrix}$

$T: \mathbb{C} \to \mathbb{R}^2$
ζ 1-1
onto
linear.
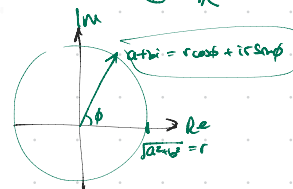
Complex Conjugate, Absolute Value, Polar Form

"bar"

We can **conjugate** complex numbers: $\overline{a+bi} = a - bi$

$\overline{3-2i} = 3+2i$

The **absolute value** of a complex number: $|a+bi| = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2 - b^2 i^2} = \sqrt{a^2+b^2}$

modulus or length

We can write complex numbers in **polar form**: $a + ib = r(\cos \phi + i \sin \phi)$

$\mathbb{C} \cong \mathbb{R}^2$

Im
$a+bi = r\cos\phi + ir\sin\phi$
ϕ
$\sqrt{a^2+b^2} = r$
Re

Section 5.5   Slide 248

Complex Conjugate Properties

If x and y are complex numbers, $\vec{v} \in \mathbb{C}^n$, it can be shown that:
- $\overline{(x+y)} = \bar{x} + \bar{y}$
- $\overline{A\vec{v}} = A\bar{v}$
- $\operatorname{Im}(x\bar{x}) = 0$.

Example True or false: if x and y are complex numbers, then

$$\overline{(xy)} = \bar{x}\,\bar{y}$$

$$\overline{\begin{bmatrix} 2-i \\ 3+2i \end{bmatrix}} = \begin{bmatrix} 2+i \\ 3-2i \end{bmatrix}$$

$x \in \mathbb{C}$ $x = a+bi$
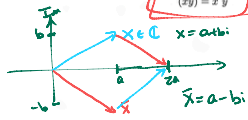
$\bar{x} = a - bi$

$x + \bar{x} = (a+bi) + (a-bi)$
$= 2a + bi - bi$
$= 2a$ ✓

Polar Form and the Complex Conjugate

Conjugation reflects points across the real axis.

$z = x + iy$

$\bar{z} = x - iy$
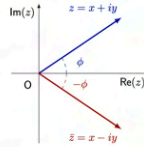
$X = 3 + 2i$
$y = 1 - 4i$

$\overline{Xy} = \overline{(3+2i)(1-4i)}$
$= \overline{(3 - 12i + 2i - 8i^2)}$
$= \overline{(11 - 10i)}$
$= 11 + 10i$

$\bar{X}\,\bar{y} = \overline{(3+2i)}\,\overline{(1-4i)}$
$= (3 - 2i)(1 + 4i)$
$= 3 + 12i - 2i - 8i^2$
$= 3 + 10i + 8$
$= 11 + 10i$ ✓

Euler's Formula

Suppose z_1 has angle ϕ_1, and z_2 has angle ϕ_2.

$z_1 + z_2 = z_3$

$1 + 2i = z_2$
$3 + 2i = z_1$
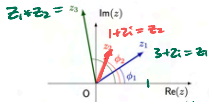
The product $z_1 z_2$ has angle $\phi_1 + \phi_2$ and modulus. Easy to remember using Euler's formula.

call $|z| = r$

$z = |z| e^{i\phi} = r e^{i\theta} = r(\cos\theta + i\sin\theta)$

The product $z_1 z_2$ is:

$z_3 = z_1 z_2 = (|z_1| e^{i\phi_1})(|z_2| e^{i\phi_2}) = |z_1||z_2| e^{i(\phi_1 + \phi_2)}$

Complex Numbers and Polynomials

$p(\lambda) = \det(A - \lambda I)$

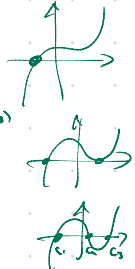if A is non real matrix then $p(\lambda)$ is deg n real poly.

Theorem: Fundamental Theorem of Algebra
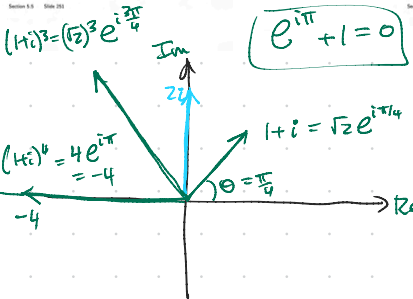Every polynomial of degree n has exactly n complex roots, counting multiplicity.

Theorem
1. If $\lambda \in \mathbb{C}$ is a root of a real polynomial $p(x)$, then the conjugate $\bar{\lambda}$ is also a root of $p(x)$.
2. If λ is an eigenvalue of real matrix A with eigenvector $\vec{v}$, then $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{\vec{v}}$.

$f(x) = x^3 - 3x^2 + 2x - 1$
$= (x - c_1)(x - c_2)(x - c_3)$

$(1+i)^3 = (\sqrt{2})^3 e^{i\frac{3\pi}{4}}$

$e^{i\pi} + 1 = 0$

$(1+i)^2 = 2e^{i\frac{\pi}{2}} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
$= 2i$

$(1+i)^4 = 4 e^{i\pi} = -4$

$1 + i = \sqrt{2} e^{i\pi/4}$

$\theta = \frac{\pi}{4}$

$(1+i)(1+i)$
$= 1 + i + i + i^2$
$= 1 + 2i - 1 = 2i$ ✓

7 factors total

other roots are $4-i, -4+i, -i$

Four of the eigenvalues of a 7×7 matrix are $-2, 4+i, -4-i,$ and i. What are the other eigenvalues?

$P(\lambda) = -(\lambda+2)(\lambda - (4+i))(\lambda - (-4-i))(\lambda - i)(\lambda - c_1)(\lambda - c_2)(\lambda - c_3)$

$= -\lambda^9 + a_8\lambda^6 + a_2\lambda^5 - \cdots + a_1\lambda + a_0$

root of $a\lambda^2 + b\lambda + c$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\sqrt{-4} = \sqrt{-1} * \sqrt{4} = i * 2$

The matrix that rotates vectors by $\phi = \pi/4$ radians about the origin, and then scales (or dilates) vectors by $r = \sqrt{2}$, is

$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

What are the eigenvalues of A? Express them in polar form.

$p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$

$\lambda_1 = 1+i \qquad \lambda_2 = 1-i$

$= \lambda^2 - 2\lambda + 2 = (\lambda - (1+i))(\lambda - (1-i))$

$\lambda_{1,2} = \dfrac{2 \pm \sqrt{4 - 4\cdot 2}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= \dfrac{2}{2} \pm \dfrac{2i}{2} = \boxed{1 \pm i}$

Eigenvectors? $A - \lambda I$ row reduce!

$\lambda = 1+i$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - (1+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix}$

$= \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \underset{iR_1 + R_2 \to}{\sim} \begin{bmatrix} 1 & -i \\ 0 & \boxed{0} \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i-1 \\ i+1 \end{bmatrix} = (1+i)\begin{bmatrix} i \\ 1 \end{bmatrix}$?

$x_1 - ix_2 = 0 \qquad x_1 = is$

$x_2 = \text{free} \qquad x_2 = s \text{ free} \qquad \boxed{\vec{x} = s\begin{bmatrix} i \\ 1 \end{bmatrix}}$

$i(-i) - 1$
$= -i^2 - 1 = -(-1) - 1$

$= \begin{bmatrix} i + i^2 \\ 1 + i \end{bmatrix}$

$= \begin{bmatrix} i - 1 \\ 1+i \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
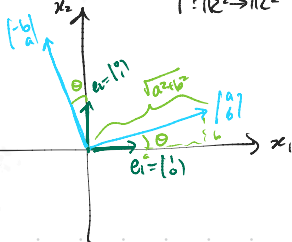
$= \begin{bmatrix} i-1 \\ 1+i \end{bmatrix}$

rotation-dilation

The matrix in the previous example is a special case of this matrix:

$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$Ce_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

$Ce_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$

Calculate the eigenvalues of C and express them in polar form.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Theorem

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector $\vec{v}$. Then we may construct the diagonalization

$A = PCP^{-1}$

where

$P = (\text{Re}\,\vec{v} \quad \text{Im}\,\vec{v})$ and $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = A.$

Note the following.

- C is referred to as a **rotation dilation** matrix, because it is the composition of a rotation by ϕ and dilation by r.
- The proof for why the columns of P are always linearly independent is a bit long, it goes beyond the scope of this course.

$P \quad C \quad P^{-1} = A$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

x_2 axis:

$\begin{bmatrix} -b \\ a \end{bmatrix}$

$\theta \quad a = \begin{bmatrix} a \\ b \end{bmatrix}$

$\sqrt{a^2 + b^2}$ $\begin{bmatrix} a \\ b \end{bmatrix}$

$\theta \quad \begin{bmatrix} a \\ b \end{bmatrix}$ ... b

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\tan\theta = b/a$

$\theta = \tan^{-1}(b/a)$

$r = \sqrt{a^2 + b^2}$

$C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$C^k = \begin{bmatrix} r^k & 0 \\ 0 & r^k \end{bmatrix}\begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$

e.g.

$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \lambda_1 = 1+i$

$\lambda_2 = 1-i$

$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$p(\lambda) = \det(C - \lambda I)$

$= \det\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} = (a-\lambda)^2 + b^2$

$= \lambda^2 - 2a\lambda + a^2 + b^2$

$\sqrt{-4b^2} = \sqrt{-4}\sqrt{b^2} = 2ib$

$\lambda_{1,2} = \dfrac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = \dfrac{a \pm \sqrt{-4b^2}}{2} = \boxed{a \pm bi}$

Find the complex eigenvalues and an associated complex eigenvector for each eigenvalue for the matrix.

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

5.5 EXERCISES

Let each matrix in Exercises 1–6 act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

1. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

In Exercises 7–12, use Example 6 to list the eigenvalues of A. In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \leq \pi$, and give the scale factor r.

7. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$

9. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

10. $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$

11. $\begin{bmatrix} .1 & .1 \\ -.1 & .1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & .3 \\ -.3 & 0 \end{bmatrix}$

In Exercises 13–20, find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$. For Exercises 13–16, use information from Exercises 1–4.

13. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 \\ .4 & .6 \end{bmatrix}$

19. $\begin{bmatrix} 1.52 & -.7 \\ .56 & .4 \end{bmatrix}$

20. $\begin{bmatrix} -1.64 & -2.4 \\ 1.92 & 2.2 \end{bmatrix}$

21. In Example 2, solve the first equation in (2) for x_2 in terms of x_1, and from that produce the eigenvector $\mathbf{y} = \begin{bmatrix} 2 \\ -1 + 2i \end{bmatrix}$ for the matrix A. Show that this $\mathbf{y}$ is a (complex) multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Let A be a complex (or real) $n \times n$ matrix, and let $\mathbf{x}$ in $\mathbb{C}^n$ be an eigenvector corresponding to an eigenvalue λ in $\mathbb{C}$. Show that for each nonzero complex scalar μ, the vector $\mu\mathbf{x}$ is an eigenvector of A.

Chapter 7 will focus on matrices A with the property that $A^T = A$. Exercises 23 and 24 show that every eigenvalue of such a matrix is necessarily real.

23. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\overline{q} = q$. Give a reason for each step.

$$\overline{q} = \overline{\overline{\mathbf{x}}^T A\mathbf{x}} = \mathbf{x}^T \overline{A}\overline{\mathbf{x}} = \mathbf{x}^T A\overline{\mathbf{x}} = (\mathbf{x}^T A\overline{\mathbf{x}})^T = \overline{\mathbf{x}}^T A^T \mathbf{x} = q$$
$$\quad\text{(a)}\qquad\text{(b)}\qquad\text{(c)}\qquad\text{(d)}\qquad\text{(e)}$$

Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Dot product of vectors
2. Magnitude of vectors, and distances in $\mathbb{R}^n$
3. Orthogonal vectors and complements
4. Angles between vectors

$W^\perp$

Learning Objectives

1. Compute (a) dot product of two vectors, (b) length (or magnitude) of a vector, (c) distance between two points in $\mathbb{R}^n$, and (d) angles between vectors.
2. Apply theorems related to orthogonal complements, and their relationships to Row and Null space, to characterize vectors and linear systems.

Motivating Question

For a matrix A, which vectors are orthogonal to all the rows of A? To the columns of A?

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The Dot Product

$$\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 1: For what values of k is $\vec{u} \cdot \vec{v} = 0$?

$$\vec{u} = \begin{pmatrix} -1 \\ \frac{3}{k} \\ 2 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} 4 \\ 2 \\ 1 \\ -3 \end{pmatrix}$$

$\vec{u} \cdot \vec{v} = -4 + 6 + k - 6$
$= -4 + k = 0$
$\Rightarrow k = 4$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = (1)(1) + (2)(0) + (3)(-1) = 1 - 3 = \boxed{-2}$$

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$$(1 \quad 2 \quad 3) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 3 = \boxed{-2}$$

$\underset{1\times 3}{} \quad \underset{3\times 1}{}$

Properties of the Dot Product

The dot product is a special form of matrix multiplication, so $\mathbb{R}$ inherits linear properties.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{?}{=} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$xa + yb + zc.$

Theorem (Basic Identities of Dot Product)

Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$.

1. (Symmetry) $\vec{u} \cdot \vec{w} = \vec{w} \cdot \vec{u}$
2. (Linear in each vector) $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$
3. (Scalars) $(c\vec{u}) \cdot \vec{w} = c(\vec{u} \cdot \vec{w})$
4. (Positivity) $\vec{u} \cdot \vec{u} \geq 0$, and the dot product equals $\|\vec{u}\|^2$

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$$(\vec{v} + \vec{w}) \cdot (\vec{u} + \vec{z})$$

$$= \vec{v} \cdot (\vec{u} + \vec{z}) + \vec{w} \cdot (\vec{u} + \vec{z})$$

$$= \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{z} + \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{z}$$

WE'VE GOT THAT FRIDAY FEELING

THEOREM 1

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$$\vec{u} \cdot \vec{u} = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{a^2 + b^2 + c^2} = 0$$
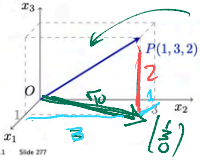
$$\Rightarrow a = b = c = 0.$$

Definition

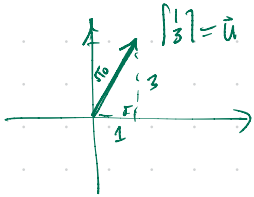The **length** of a vector $\vec{u} \in \mathbb{R}^n$ is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Example: the length of the vector $\overrightarrow{OP}$ is

$$\sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$P(1, 3, 2)$

$\sqrt{\sqrt{10}^2 + 2^2} = \sqrt{10 + 4} = \sqrt{14}$

$= \sqrt{(\sqrt{1^2 + 3^2})^2 + 2^2} = \sqrt{1^2 + 3^2 + 2^2}$

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$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{u}$ $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{1^2 + 3^2}$

$= \sqrt{10}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 3   1

Key Idea $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$

$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$ with $\|\vec{u}\| = 5$, $\|\vec{v}\| = \sqrt{3}$, and $\vec{u} \cdot \vec{v} = -1$. Compute the value of $\|\vec{u} + \vec{v}\|$.

$|-3|$ large   Compute

$\left\|\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\|$

$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$

$\overset{?}{=} \vec{u} \cdot \vec{u} + 2 \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$

$\uparrow \sqrt{5}$
$\uparrow 5$
$\uparrow 25$

$= (5)^2 + 2(-1) + (\sqrt{3})^2$

$= 25 - 2 + 3 = \boxed{26}$

$\boxed{\|\vec{u} + \vec{v}\| = \sqrt{26}}$???

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$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

DEFINITION

The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

$\|2v\| = ?$

$\|5v\| = 3$ v 25

Note: for any vector $\vec{v}$ and scalar c, the length of $c\vec{v}$ is

$$\|c\vec{v}\| = c\,\|\vec{v}\|$$

Definition

If $\vec{v} \in \mathbb{R}^n$ has length one, we say that it is a **unit vector**.

Example: Let W be a subspace of $\mathbb{R}^4$ spanned by

$$\vec{v} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 1 \end{bmatrix} \qquad \vec{u} = \frac{1}{\|\vec{v}\|}\,\vec{v}$$

a) Construct a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

b) Construct a basis for W using unit vectors.

$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$ prob vect.

$\frac{1}{\sqrt{5}} \vec{v}$

$\sqrt{5}$ v

$\|\vec{v}\| = \sqrt{1^2 + 3^2 + 2^2 + 1^2}$
$= \sqrt{1+9+4+1} = \sqrt{15}$

Definition

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the **distance** between $\vec{u}$ and $\vec{v}$ is given by the formula

$$\|\vec{u} - \vec{v}\| \quad \text{or} \quad \|\vec{v} - \vec{u}\| \quad \text{both ok.}$$

Example: Compute the distance from $\vec{u} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$\vec{v} + \vec{x} = \vec{u}$

$\vec{x} = \vec{u} - \vec{v}$

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{v}$ $\vec{x}$ $\|\vec{x}\| = \sqrt{17}$
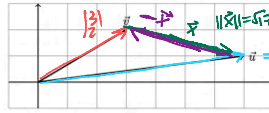
$\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

So, $\|\vec{x}\| = \|\vec{u} - \vec{v}\|$

$$\left\| \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

DEFINITION   For **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as dist(**u**, **v**), is the length of the vector **u** − **v**. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$
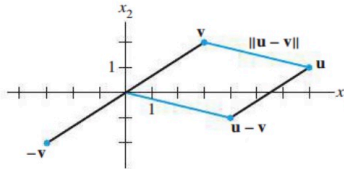
FIGURE 4 The distance between **u** and **v** is the length of **u** − **v**.

Definition (Orthogonal Vectors)

Two vectors $\vec{u}$ and $\vec{w}$ are **orthogonal** if $\vec{u} \cdot \vec{w} = 0$. This is equivalent to:
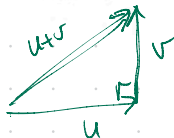
$$\|\vec{u} + \vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{w}\|^2$$

Note: The zero vector is orthogonal to every vector. But we usually only mean non-zero vectors.
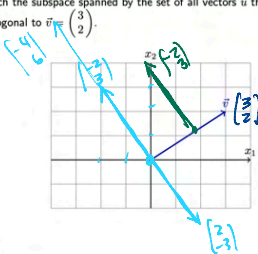
$\vec{u}^2$

Assume $\vec{u} \cdot \vec{w} = 0$

$$\| \vec{u} + \vec{w} \|^2 = (\vec{u} + \vec{w}) \cdot (\vec{u} + \vec{w}) \qquad \text{cross term is zero.}$$
$$= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{w} + \vec{w} \cdot \vec{w} \qquad (\vec{u} \cdot \vec{w} = 0)$$
$$= \vec{u} \cdot \vec{u} + \vec{w} \cdot \vec{w}$$
$$= \|\vec{u}\|^2 + \|\vec{w}\|^2$$

$u + v$ v u

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Example

Sketch the subspace spanned by the set of all vectors $\vec{u}$ that are orthogonal to $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

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What vectors are orthogonal to $\vec{v}$?

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0 + 0 = 0 \checkmark$$

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = -6 + 6 = 0 \checkmark$$

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 6 - 6 = 0 \checkmark$$

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = -12 + 12 = 0 \checkmark$$

$$c \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2c \\ 3c \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = (-6c) + 6c = 0$$

for any c

The set of vectors orthogonal to v

$H \leftarrow \infty$-many.

Suppose $\vec{x} \in H$ $\vec{y} \in H$

$$\vec{x} \cdot \vec{v} = 0$$
$$\vec{y} \cdot \vec{v} = 0$$

$\vec{x} + \vec{y}$ also in H.

$$(\vec{x} + \vec{y}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v}$$
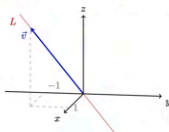$$= 0 + 0 = 0$$

Definitions

Let W be a subspace of $\mathbb{R}^n$. A vector $\vec{z} \in \mathbb{R}^n$ is said to be **orthogonal** to W if $\vec{z}$ is orthogonal to each vector in W.

The set of all vectors orthogonal to W is a subspace, the **orthogonal compliment** of W, or $W^\perp$ or 'W perp.'

$$W^\perp = \{\vec{z} \in \mathbb{R}^n : \vec{z} \cdot \vec{w} \text{ for all } \vec{w} \text{ in } W\}$$
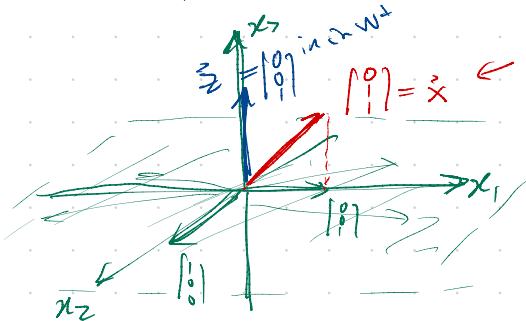
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Example

Line L is a subspace of $\mathbb{R}^3$ spanned by $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Then the space $L^\perp$ is a plane. Construct an equation of the plane $L^\perp$.

Can also visualise line and plane with CalcPlot3D: web.monroecc.edu/calcNSF

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$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. The floor of $\mathbb{R}^3$.

$\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R} \cdot W^\perp$

$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \vec{x}$ ← orthogonal + some $\vec{w} \in W$ but not all

Definition

Row A is the space spanned by the rows of matrix A.

We can show that
- $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$
- a basis for Row A is the pivot rows of A

Describe the Null(A) in terms of an orthogonal subspace.

A vector $\vec{z}$ is in Null(A) if and only if

1. $A\vec{z} = \vec{0}$ (defn!)

2. This means that $\vec{z}$ is **orthogonal** to each row of A.

3. Row A is **the orthogonal complement** of Null A.

4. The dimension of Row A plus the dimension of Null A equals **# cols of A.**

$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix} \right) = \vec{0}$ ✓

$\checkmark$

Need

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

same $\rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$
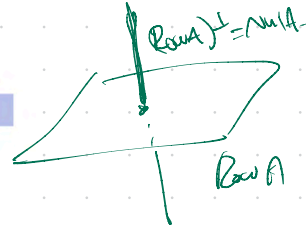
$a + b + c = 0$

$-a + b + 0c = 0$

$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$

Row $A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Find $\vec{x}$
$\vec{x} \cdot \vec{v} = 0$ for any $\vec{v} \in$ Row A

want

$\vec{x} \cdot \left(c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = 0$ for any $c_1, c_2 \in \mathbb{R}$

$\Rightarrow c_1 \left(\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) + c_2 \left(\vec{x} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = 0$

iff $\vec{x} \cdot \vec{v}_1 = 0$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{x} \cdot \vec{v}_2 = 0$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$(\text{Row } A)^{\perp} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$

$(\text{Row } A)^{\perp} = \text{Null } A$

Four subspaces

Theorem (The Four Subspaces)

For any $A \in \mathbb{R}^{m \times n}$, the orthogonal complement of Row A is Null A, and the orthogonal complement of Col A is Null A^T.

The idea behind this theorem is described in the diagram below.

$\mathbb{R}^n$ — ② Row(A)   ③ Col(A) — $\mathbb{R}^m$
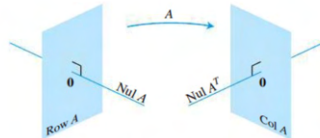
① Null(A)   ④ Null(A^T)

Additional Example (if time permits)

A has the LU factorization:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Construct a basis for $(\text{Row } A)^{\perp}$
b) Construct a basis for $(\text{Col } A)^{\perp}$

Hint: it is not necessary to compute A. Recall that $A^T = U^T L^T$, matrix L^T is invertible, and U^T has a non-empty nullspace.

(a) $(\text{Row } A)^{\perp} = \text{Nul } A$ $\leftarrow$ works for any A.

$\Rightarrow (\text{Row}(A^T))^{\perp} = \text{Nul}(A^T)$

(b) $(\text{Col } A)^{\perp} = \text{Nul}(A^T)$ $\leftarrow$

Row A

THEOREM 3   Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$

FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.
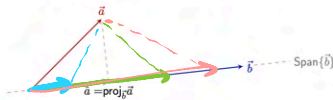
Angles

Theorem

$\vec{a} \cdot \vec{b} = |\vec{a}| \, |\vec{b}| \cos \theta$. Thus, if $\vec{a} \cdot \vec{b} = 0$, then:

- $\vec{a}$ and/or $\vec{b}$ are ⓪ vectors, or
- $\vec{a}$ and $\vec{b}$ are *perpendicular.*

For example, consider the vectors below.

Looking Ahead - Projections
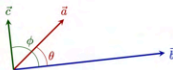
Suppose we want to find the closed vector in Span$\{\vec{b}\}$ to $\vec{a}$.

$\vec{a} = \text{proj}_{\vec{b}} \vec{a}$

Span$\{\vec{b}\}$

- Later in this Chapter, we will make connections between dot products and **projections**.
- Projections are also used throughout multivariable calculus courses.

6.1 EXERCISES

Compute the quantities in Exercises 1–8 using the vectors

$$u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

1. $u \cdot u, v \cdot u,$ and $\dfrac{v \cdot u}{u \cdot u}$

2. $w \cdot w, x \cdot w,$ and $\dfrac{x \cdot w}{w \cdot w}$

3. $\dfrac{1}{w \cdot w} w$

4. $\dfrac{1}{u \cdot u} u$

5. $\left(\dfrac{u \cdot v}{v \cdot v} \right) v$

6. $\left(\dfrac{x \cdot w}{x \cdot x} \right) x$

7. $\|w\|$

8. $\|x\|$

In Exercises 9–12, find a unit vector in the direction of the given vector.

9. $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

10. $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

11. $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

13. Find the distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

14. Find the distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Determine which pairs of vectors in Exercises 15–18 are orthogonal.

15. $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

16. $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

17. $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

18. $y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

In Exercises 19 and 20, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

19. a. $v \cdot v = \|v\|^2$.

b. For any scalar c, $u \cdot (cv) = c(u \cdot v)$.

c. If the distance from u to v equals the distance from u to $-v$, then u and v are orthogonal.

d. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

e. If vectors $v_1, \ldots, v_p$ span a subspace W and if x is orthogonal to each v_j for $j = 1, \ldots, p$, then x is in $W^\perp$.

20. a. $u \cdot v - v \cdot u = 0$.

b. For any scalar c, $\|cv\| = c\|v\|$.

c. If x is orthogonal to every vector in a subspace W, then x is in $W^\perp$.

d. If $\|u\|^2 + \|v\|^2 = \|u + v\|^2$, then u and v are orthogonal.

e. For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.

21. Use the transpose definition of the inner product to verify parts (b) and (c) of Theorem 1. Mention the appropriate facts from Chapter 2.

22. Let $u = (u_1, u_2, u_3)$. Explain why $u \cdot u \geq 0$. When is $u \cdot u = 0$?

23. Let $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. Compute and compare $u \cdot v, \|u\|^2, \|v\|^2,$ and $\|u + v\|^2$. Do not use the Pythagorean Theorem.

24. Verify the *parallelogram law* for vectors u and v in $\mathbb{R}^n$:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

25. Let $v = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to v. [*Hint:* Consider $v = 0$ and $v \neq 0$.]

26. Let $u = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let W be the set of all x in $\mathbb{R}^3$ such that $u \cdot x = 0$. What theorem in Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.

27. Suppose a vector y is orthogonal to vectors u and v. Show that y is orthogonal to the vector $u + v$.

28. Suppose y is orthogonal to u and v. Show that y is orthogonal to every w in Span $\{u, v\}$. [*Hint:* An arbitrary w in Span $\{u, v\}$ has the form $w = c_1 u + c_2 v$. Show that y is orthogonal to such a vector w.]

29. Let $W = \text{Span} \{v_1, \ldots, v_p\}$. Show that if x is orthogonal to each v_j, for $1 \leq j \leq p$, then x is orthogonal to every vector in W.