Handwritten Homework Assignments - Exploration for MATH 1554

For each assignment, complete the questions by hand on paper/tablet. Write neatly and use complete sentences where necessary. **You must submit original work**, but y'all can share ideas. Handwritten homework is due on Sunday in Gradescope; no late submissions accepted.

Week 1: *Practice sketching in 2D and 3D.* In $\mathbb{R}^2$, using the coordinates x_1, x_2, sketch the following making a new sketch for each part (a)-(b):

(a) sketch two *random* lines in $\mathbb{R}^2$ and label each line with the equation describing it, and draw and label the point of intersection and at least one other point on each line. Verify with a calculation that each point really belongs to each line it is drawn on.

(b) sketch one *random* line and a parallel line and label each line with the equation describing it, and again verify with a calculation that each point really is on the line you say it is on.

In $\mathbb{R}^3$, using the coordinates x_1, x_2, x_3, sketch the following making a new sketch for each part (i)-(v):

(i) use the exact same equations from part (a) above in $\mathbb{R}^2$ above, but with the variables x_1, x_2, x_3 and understanding that the coefficient of x_3 should be understood to be zero; sketch the two **planes** and the **line** of intersection; for each plane/line draw at least two points with different x_3-values; check with calculations that your sketched points are on each object that it belongs on;

(ii) sketch the plane consisting of points satisfying the condition that $x_2 = 0$ (the *side wall*), and sketch at least three non-colinear points;

(iii) sketch the plane $x_2 = 0$ and the plane $x_1 = 0$ on the same axes, and sketch and label the line of intersection of these two planes;

(iv) sketch a line which passes through the origin which is not contained in any of the three coordinate planes, include and label at least three points on the line, you do not have to label the line;

(v) sketch the plane defined by $x + 3y + 2z = 12$, include and label at least four points in this plane no three of which are colinear, and at least one of the points should not lie in any of the coordinate axes or coordinate planes.

Main idea: In each problem, you are practicing drawing an accurate, representative graph of the plane (or line) of points which satisfy the given equation in the variables x_1, x_2, and x_3, and including a few points on the geometric object (plane/line), and verifying with a calculation that your point really belongs to the object on which it has been drawn.

Week 2: *Practice with span, linear combination, and inconsistent systems.* ***(a)*** Choose two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^2$ and a third vector $\vec{b}$ also in $\mathbb{R}^2$, and express b as a linear combination of v, w by finding scalars c_1, c_2 such that $c_1\vec{v}+c_2\vec{w} = \vec{b}$. Sketch the situation in $\mathbb{R}^2$ with an illustration that uses the parallelogram rule and verify with calculations that $\vec{b}$ is really a linear combination of $\vec{v}$ and $\vec{w}$. ***(b)*** Repeat part *(a)* with **new** vectors in $\mathbb{R}^3$ such that the augmented matrix $[\vec{v}\ \vec{w} \mid \vec{b}]$ gives a consistent system, again illustrating by graphing but this time in $\mathbb{R}^3$. ***(c)*** Why is it harder to find a consistent system for part *(b)* compared to part *(a)*? Explain your idea(*) clearly using complete sentences.

*Note(*): we are looking for something different than "it is harder to draw things in three dimensions" here - try to explain **why** finding a consistent system is harder for part 2 compared to part 1.*

Warning!

PLEASE NOTE: Your submissions for this and future exploration assignments need to contain vectors which are *general looking*. Choosing vectors which are scalar multiples of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$, or $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, or have too many zeros or ones, or are otherwise too simple and miss the point of the exploration will receive a deduction of points.

Please, do not ask on Piazza if your vectors are general enough to get full credit. The explorations are assignments which require you to make a *judgement call*, to **explore** a particular concept of the course and NOT to come up with the simplest example which satisfies the minimum requirements of the assignment.

Week 3: *Learn some basics of MATLAB.* See Sal's personal website for links to the MATLAB #1 Exploration. (requires *MATLAB installation* OR MathWorks access to *MATLAB online*)

Week 4: *Propositional logic.* This week we will explore the basics of *propositional logic* in order to help develop some framework to practice True/False questions when studying for the exam. The assignment this week is to complete the three questions **Q1, Q2, Q3** and upload to Gradescope. The additional (non-question) text is just to help you understand the assignment.

Let's start with the main definitions:

Definition: A **mathematical statement** (aka, a **proposition**) is a sentence which is either true or false.

Each of the following are statements; they can be true or false, depending on the choice of vectors and/or matrices in each statement.

1. $A\vec{x} = \vec{b}$ is consistent.

2. $[A \mid \vec{b}]$ is row equivalent to $[C \mid \vec{d}]$.

3. $A\vec{x} = \vec{0}$ has a non-trivial solution.

4. The columns of A are linearly dependent.

5. $T(\vec{x}) = A\vec{x}$ is onto.

For example, the first statement (1.) is true if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, but this statement is false if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Q1: *You try it!* Pick **one** of the other statements (2.)-(5.) above and come up with choices for the vectors/matrices that make the statement true, and choices that make the same statement false. Check with calculations that your example works for each.

Note: please write the problem statement of the problem you are solving, to help the grader.

Statements can be combined in a variety of ways to create new statements, using for example AND, OR, or IMPLIES.

For example, each of the following are statements. (connecting word highlighted for emphasis)

6. $[A \mid \vec{b}]$ is row equivalent to $[C \mid \vec{d}]$ **and** $A\vec{x} = \vec{b}$ is consistent.

7. $A\vec{x} = \vec{0}$ has a non-trivial solution **or** $T(\vec{x}) = A\vec{x}$ is onto.

8a. The columns of A are linearly dependent **implies** $A\vec{x} = \vec{0}$ has a non-trivial solution.

8b. **If** the columns of A are linearly dependent, **then** $A\vec{x} = \vec{0}$ has a non-trivial solution.

The most common way that an IMPLIES statement is written is to use **if** and **then**. The statements (8a.) and (8b.) say exactly the same thing using different wording.

An implication statement is true whenever knowing that the "if part" is true forces the "then part" to also be true. So (8a.) is a true implication because whenever the columns of A are linearly dependent there is a free variable in the system $A\vec{x} = \vec{0}$, and assigning a non-zero value to the free variable gives a non-zero $\vec{x}$ which satisfies $A\vec{x} = \vec{0}$.

On the other hand, an implication statement is false if for some choice making the "if part" true, the "then part" is false. For example consider the following false implication.

9. If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, then $A(\vec{x} + \vec{y}) = \vec{b}$.

This implication is false because there is a counterexample for which the first part is true, but the second part is false. For example, choosing $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\vec{b} = \left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right]$ we see that $\vec{x} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\vec{y} = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ both satisfy $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, but $\vec{x} + \vec{y} = \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]$ and so $A(\vec{x} + \vec{y}) = \left[\begin{smallmatrix} 6 \\ 6 \end{smallmatrix}\right]$ (and $\left[\begin{smallmatrix} 6 \\ 6 \end{smallmatrix}\right]$ is not $\vec{b}$).

Q2: *You try it!* Select any one TRUE and any one FALSE true/false question from any of the practice exams that use IMPLIES (aka an if-then statement). For each of the two problems, identify the propositions in the problem (the "if part" and the "then part"). If the implication is false, provide a counter-example with explanation/calculations to show why it works. If the implication is true, give a short *general** explanation using precise and correct terminology from class. (* a short general proof - not an example)

Note: please write the problem statement of the problem you are solving, to help the grader.

Finally, some statements can have one or more **mathematical quantifiers**. There are two kinds of mathematical quantifiers, which are the universal quantifier **for all** (aka **for every**), and the existential quantifier **for some** (aka **there exists**).

For example, each of the following statements has a quantifier. (quantifier highlighted)

10. If $A\vec{x} = \vec{b}$ is consistent **for some** $\vec{b} \in R^m$, then A has a pivot in every row.

11. If A does not have a pivot in every column and $T(\vec{x}) = A\vec{x}$, then **for every** $\vec{x}_1$ and $\vec{x}_2$ with $\vec{x}_1 \neq \vec{x}_2$ we have that $T(\vec{x}_1) = T(\vec{x}_2)$.

Q3: *You try it!* In (10.) and (11.) identify the propositions in each problem. The implications (10.) and (11.) are both false; provide a counterexample to **one** of the implications. Give a short description (one or two short sentences) as to why your counterexample works, and verify any assertions you make with calculations.

Note: please write the problem statement of the problem you are solving, to help the grader.

Hint: For (10.) a counterexample will be a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent, but A does not have a pivot in every row.

Hint: For (11.) a counterexample will be a matrix A that does not have a pivot in every column, and such that there exist vectors $\vec{x}_1$ and $\vec{x}_1$ such that $\vec{x}_1 \neq \vec{x}_2$ and $T(\vec{x}_1) \neq T(\vec{x}_2)$.

Week 5: *More true/false practice.* Consider the following problems from Exam 1.

1. **1(a)iii T/F (Fall '24)**
   If $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are both consistent linear systems, then $A\vec{x} = \vec{b} + 2\vec{c}$ is also a consistent linear system..
2. **1(a)v T/F (Spring '23)**
   If A is size 3×4 and none of the rows of A consist entirely of zeros, then A has 3 pivots.
3. **1(a)v T/F (Fall '23)**
   If $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of vectors, then $\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$.
4. **1(b)i T/F - modified (Spring '23)** *Note: now T/F and inhomogeneous.*
   If A is an $m \times n$ matrix with a pivot in its last column, then $A\vec{x} = \vec{b}$ is inconsistent for any choice of vector $\vec{b}$.
5. **1(a)ii T/F (Spring '24)**
   If the linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one, then $A\vec{x} = \vec{b}$ has a unique solution.

For each of the true/false and possible/impossible problems above do all of the following (i)-(iii).

(i) Write the problem and provide the answer.

(ii) (*)Complete <u>RULE #1</u>: **do one example** which either demonstrates that the statement is satisfied (an example which shows that the statement *could* be true) or that the statement is not satisfied (a counter-example which shows that the statement *must* be false). For example, for (1.) you would need to find a matrix A, and vectors $\vec{v}, \vec{w}, \vec{b}$ so that $\vec{v}$ and $\vec{w}$ are solutions to $A\vec{x} = \vec{b}$ and then check if $\vec{v} - \vec{w}$ is a solution to $A\vec{x} = \vec{0}$ or not. For your example, say whether your example was a counter-example or not.

(iii) Identify the relevant **definitions** in the statement. For example, for (2.) you need to define *(a)* size of a matrix, and *(b)* what a pivot is. You must use the precise definition from the textbook/lecture, here. Do not summarize or provide some intuitive definition! I want **the textbook definition**.

(*) Note: For part (ii), your example does not have to be a counter-example if the statement is false. Just do any example.

Finally, create TWO NEW problems, Problem (6.) and Problem (7.): for each new problem pick **one** of the five problems above and modify it in some way. For example, you can change the premise of an implication or its conclusion (e.g., change *onto* with *one-to-one*, change *linearly independent* with *linearly dependent*). Another option would be to change the **order** of the implication by swapping the if-then statements, or negating either the if-part, or the then-part, or both.

Repeat steps (i)-(ii) for problems (6.) and (7.) and indicate which of the two problems (1.)-(5.) you are modifying.

Week 6: *Practice with transformations.* For each matrix A below, ***(a)*** state the domain and codomain of $T_A(\vec{x}) = A\vec{x}$, ***(b)*** find $T_A(e_1), T_A(e_2)$, ***(c)*** find $T_A(v), T_A(w)$, ***(d)*** describe in a few words what the transformation is doing, ***(e)*** state whether T is one-to-one and if **not** then find a vector $\vec{x}$ which maps to zero (verify with calculations that your choice maps to zero by plugging in $\vec{x}$, ***(f)*** state whether T is onto and if **not** then find a vector $\vec{b}$ which is not in the range (and verify by showing that $A\vec{x} = \vec{b}$ is inconsistent), ***(g)*** state dimNul$A$ and dimCol$A$, and finally ***(h)*** give the matrix an appropriate "name" (fine to be silly name like, e.g., "the x-zero-er" for projection to y-axis"). For the problems below use

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(1) $A = \begin{bmatrix} 1 & 0 \\ 0 & \text{-}1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(4) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(5) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(6) $A = \begin{bmatrix} 1/\sqrt{2} & \text{-}1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(7) $A = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$

(8) $A = \begin{bmatrix} 1/\sqrt{2} & \text{-}1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$

Next, for the problems below use

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ \text{-}1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 2 \\ \text{-}1 \end{bmatrix}$$

(9) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(10) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(11) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$

(12) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Sal says: For the "name your matrix" this is a bit silly, and that's ok. Just come up with a creative name that makes sense to you, and don't worry about it too much!

Hint: For part **(11)** above, an appropriate name could be 'the total mystery' or the 'crazy transformation', or maybe 'some kind of weird shear embedding'. It's not always easy to see what the matrix is doing! Don't overthink this one, please.

Week 7: *Transformations and the determinant.*

Step 1: Sketch a parallelogram somewhere in $\mathbb{R}^2$ such that none of the vertices of the parallelogram lie on the origin. Label the vertices of the parallelogram a, b, c, d and also label the coordinates (x_1, x_2) for each of the four points. Label the parallelogram S for *shape*.

Step 2b: Showing your work, compute the area area(S) using the content of Section 3.3, by finding a pair of vectors $\vec{v}_1, \vec{v}_2$ which determine the parallelogram from *Step 1*. Go back and label the vectors $\vec{v}_1, \vec{v}_2$ in your sketch from *Step 1*.

Step 2: Choose a (somewhat random) linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which can be anything but must satisfy:

(i) the transformation should not have a diagonal standard matrix,

(ii) the transformation should be invertible (one-to-one and onto),

(iii) the transformation should not be a single rotation or a single reflection*.

(*) it is ok for the transformation you pick to be a rotation followed by a reflection, for example. However, if your transformation is 'two rotations' or 'two reflections' but can be represented by a single rotation or single reflection, then you will lose points.

Step 3: Transform your shape S from *Step 1* using your transformation T from *Step 2*. That is, sketch the **image of** S in $\mathbb{R}^2$. Make a new sketch for this part and label the images $T(a), T(b), T(c), T(d)$ and give the new coordinates for all four points.

Step 3b: Showing all work, compute the area $\text{Vol}(T(S))$ using the same method as *Step 2b* by finding the images of $\vec{v}_1, \vec{v}_2$ under the transformation T. Label the images $T(\vec{v}_1), T(\vec{v}_2)$ in your sketch from *Step 3*.

Step 4: Compute $\det A$ where A is the standard matrix of $T(\vec{x}) = A\vec{x}$, and compare the areas $\text{Vol}(S)$ and $\text{Vol}(T(S))$ with the value of $\det A$. Write a general formula which relates these three quantities.

Week 8: *Eigenvectors and eigenvalues exploration.* For each of the statements (1.)-(6.) below do the following:

(a) Write the statement and indicate if the statement is true or false.

(b) If the statement is true, then give an example. If the statement is false and impossible, then give a counter-example. If the statement is false but possible, then give both an example and a counter-example. Note: Your statements can be as simple as possible (the usual rules about avoiding trivial examples does not apply this week) but with the following restrictions: neither λ nor μ can be 0 or 1, and neither $\vec{v}_1$ nor $\vec{v}_2$ can be scalar multiples of the standard basis vectors $\vec{e}_1, \vec{e}_2$. You don't have to explain how you came up with your examples, but you must justify that your examples meet the conditions of the statement with calculations.

(c) Finally, rephrase the statement using common language. (An example of this is done for you for the first statement (1.) below.)

Suppose $A\vec{v}_1 = \lambda\vec{v}_1$ and $B\vec{v}_2 = \mu\vec{v}_2$.

1. If $A = B$ and $\lambda = \mu$, then $\vec{v}_1 = \vec{v}_2$. (*)

2. If $A = B$ and $\vec{v}_1 = \vec{v}_2$, then $\lambda = \mu$.

3. If $A = B$ and $\lambda \neq \mu$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

4. If $A \neq B$ and $\vec{v}_1 = \vec{v}_2$, then $\vec{v}_1$ is an eigenvector of AB.

5. If $A \neq B$ and $\lambda \neq \mu$, then $\vec{v}_1 \neq \vec{v}_2$.

6. If $A = B$ and $\lambda = \mu \neq 0$ and $\vec{v}_1 \neq -\vec{v}_2$, then $\vec{v}_1 + \vec{v}_2$ is an eigenvector of A with eigenvalue 2λ.

(*) Part (c) rephrase could be: If $\vec{v}_1, \vec{v}_2$ are eigenvectors of A with the same eigenvalue, then $\vec{v}_1$ and $\vec{v}_2$ are the same vector.

or alternately: There can't be two different eigenvectors of A with the same eigenvalue.