
10

Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 – 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 – 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 – 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 – 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank

Topics and Objectives

Topics

1. Dot product of vectors
2. Magnitude of vectors, and distances in $\mathbb{R}^n$
3. Orthogonal vectors and complements
4. Angles between vectors

Learning Objectives

1. Compute (a) dot product of two vectors, (b) length (or magnitude) of a vector, (c) distance between two points in $\mathbb{R}^n$, and (d) angles between vectors.
2. Apply theorems related to orthogonal complements, and their relationships to Row and Null space, to characterize vectors and linear systems.

Motivating Question
For a matrix A, which vectors are orthogonal to all the rows of A? To the columns of A?

The Dot Product

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Example 1: For what values of k is $\vec{u}\cdot\vec{v} = 0$?

$$\vec{u} = \begin{pmatrix} -1 \\ 3 \\ k \\ 2 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} 4 \\ 2 \\ 1 \\ -3 \end{pmatrix}$$

Properties of the Dot Product

The dot product is a special form of matrix multiplication, so it inherits linear properties.

Theorem (Basic Identities of Dot Product)

Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$.

1. (Symmetry) $\vec{u}\cdot\vec{w} =$ ___________
2. (Linear in each vector) $(\vec{v}+\vec{w})\cdot\vec{u} =$ ___________
3. (Scalars) $(c\vec{u})\cdot\vec{w} =$ ___________
4. (Positivity) $\vec{u}\cdot\vec{u} \geq 0$, and the dot product equals ___________

THEOREM 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
b. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$
d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

The Length of a Vector

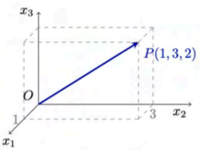
Definition

The **length** of a vector $\vec{u} \in \mathbb{R}^n$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Example: the length of the vector $\overrightarrow{OP}$ is

$$\sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$



Example

Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$ with $\|\vec{u}\| = 5$, $\|\vec{v}\| = \sqrt{3}$, and $\vec{u}\cdot\vec{v} = -1$. Compute the value of $\|\vec{u}+\vec{v}\|$.

$$\boxed{\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}$$

!!

$$= u\cdot u + 2u\cdot v + v\cdot v$$

DEFINITION

The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

Length of Vectors and Unit Vectors

Note: for any vector $\vec{v}$ and scalar c, the length of $c\vec{v}$ is

$$\|c\vec{v}\| =$$

Definition

If $\vec{v} \in \mathbb{R}^n$ has length one, we say that it is a **unit vector.**

Example: Let W be a subspace of $\mathbb{R}^4$ spanned by

$$\vec{v} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

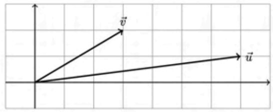
a) Construct a unit vector $\vec{u}$ in the same direction as $\vec{v}$.
b) Construct a basis for W using unit vectors.

Distance in $\mathbb{R}^n$

Definition

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the **distance** between $\vec{u}$ and $\vec{v}$ is given by the formula

Example: Compute the distance from $\vec{u} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



DEFINITION

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as dist$(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

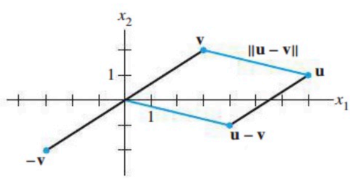


FIGURE 4 The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$.

Orthogonality

Definition (Orthogonal Vectors)

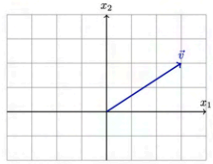
Two vectors $\vec{u}$ and $\vec{w}$ are **orthogonal** if $\vec{u} \cdot \vec{w} = 0$. This is equivalent to:

$$\|\vec{u} + \vec{w}\|^2 =$$

Note: The zero vector is orthogonal to every vector. But we usually only mean non-zero vectors.

Example

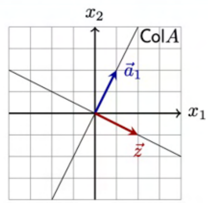
Sketch the subspace spanned by the set of all vectors $\vec{u}$ that are orthogonal to $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



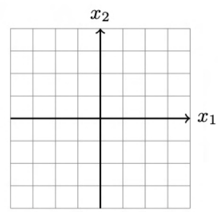
Example

Example: suppose $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

- Col$A$ is the span of $\vec{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- Col$A^\perp$ is the span of $\vec{z} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



Sketch Null$A$ and Null$A^\perp$ on the grid below.



$\perp$ \perp

Orthogonal Compliments

Definitions

Let W be a subspace of $\mathbb{R}^n$. A vector $\vec{z} \in \mathbb{R}^n$ is said to be **orthogonal** to W if $\vec{z}$ is orthogonal to each vector in W.

The set of all vectors orthogonal to W is a subspace, the **orthogonal compliment** of W, or $W^\perp$ or 'W perp.'

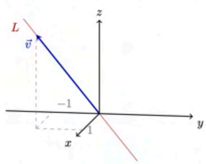
$$W^\perp = \{\vec{z} \in \mathbb{R}^n : \vec{z} \cdot \vec{w} = 0 \quad \}$$

for every $\vec{w} \in W$

careful!

Example

Line L is a subspace of $\mathbb{R}^3$ spanned by $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Then the space $L^\perp$ is a plane. Construct an equation of the plane $L^\perp$.



Can also visualise line and plane with CalcPlot3D: web.monroecc.edu/calcNSF

$W =$ "the floor of $\mathbb{R}^3$" $= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to EVERY $\vec{w} \in W$

$w \in W$ $\quad \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad \vec{z} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ ✓ $\quad \vec{w} \cdot \vec{z} = 0$ ✓

Orthogonal Compliments

Definitions

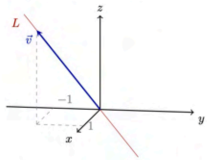
Let W be a subspace of $\mathbb{R}^n$. A vector $\vec{z} \in \mathbb{R}^n$ is said to be **orthogonal** to W if $\vec{z}$ is orthogonal to each vector in W.

The set of all vectors orthogonal to W is a subspace, the **orthogonal compliment** of W, or $W^{\perp}$ or 'W perp.'

$$W^{\perp} = \{\vec{z} \in \mathbb{R}^n \; : \; \vec{z} \cdot \qquad \}$$

Example

Line L is a subspace of $\mathbb{R}^3$ spanned by $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Then the space $L^{\perp}$ is a plane. Construct an equation of the plane $L^{\perp}$.



Can also visualise line and plane with CalcPlot3D: web.monroecc.edu/calcNSF

$L = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$ please give a basis for $L^{\perp}$

$L^{\perp} = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{v} = 0 \text{ for } \underline{\text{every}} \; v \in L\}$

$\iff \vec{v} \cdot \vec{x} = 0$

enough to check $\vec{x} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \iff \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

set $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ want $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \iff \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$\iff v^T * x = 0$

$\underline{\text{Solve.}}$

$\Rightarrow a - b + 2c = 0$ Nul of $A = \begin{bmatrix} 1 & \overset{r}{-1} & \overset{s}{2} \end{bmatrix}$

$L^{\perp} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ $\quad \vec{x} = r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

and a basis is $\boxed{\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}}$

Row$A$

Definition

Row$A$ is the space spanned by the rows of matrix A.

We can show that
- $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$
- a basis for Row$A$ is the pivot rows of A

$\text{rank}(A) = \text{rank}(A^T)$

$A \sim \cdots \sim B$ ← REF of A.

extract the pivot rows from here

Not here.

Example

Describe the Null(A) in terms of an orthogonal subspace.

A vector $\vec{x}$ is in Null A if and only if
1. $A\vec{x} =$
2. This means that $\vec{x}$ is ________ to each row of A.
3. Row A is ________ to Null A.
4. The dimension of Row A plus the dimension of Null A equals ________

basis for L

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

vectors from $L^\perp$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$

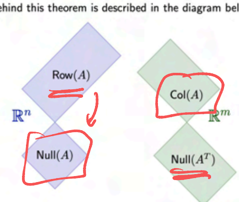
$x \in \text{Nul } A \iff x \in \text{Row} A^\perp$

$\text{Nul } A = (\text{Row} A)^\perp$

Theorem (The Four Subspaces)

For any $A \in \mathbb{R}^{m\times n}$, the orthogonal complement of Row A is Null A, and the orthogonal complement of Col A is Null A^T.

The idea behind this theorem is described in the diagram below.



Additional Example (if time permits)

A has the LU factorization:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Construct a basis for $(\text{Row}A)^\perp$
b) Construct a basis for $(\text{Col}A)^\perp$

Hint: it is not necessary to compute A. Recall that $A^T = U^TL^T$, matrix L^T is invertible, and U^T has a non-empty nullspace.

$(\text{Col } A)^\perp = (\text{Row } A^T)^\perp = \text{Nul } A^T$

THEOREM 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

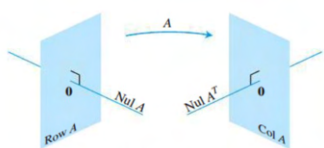


FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

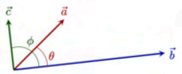
Angles

Theorem

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$. Thus, if $\vec{a}\cdot\vec{b}=0$, then:

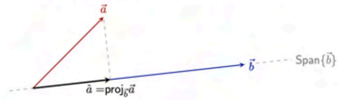
- $\vec{a}$ and/or $\vec{b}$ are zero vectors, or
- $\vec{a}$ and $\vec{b}$ are perpendicular.

For example, consider the vectors below.



Looking Ahead - Projections

Suppose we want to find the closed vector in Span$\{\vec{b}\}$ to $\vec{a}$.



- Later in this Chapter, we will make connections between dot products and **projections**.
- Projections are also used throughout multivariable calculus courses.

6.1 EXERCISES

Compute the quantities in Exercises 1–8 using the vectors

$$\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}4\\6\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix},\quad \mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$$

1. $\mathbf{u}\cdot\mathbf{u}$, $\mathbf{v}\cdot\mathbf{u}$, and $\dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$
2. $\mathbf{w}\cdot\mathbf{w}$, $\mathbf{x}\cdot\mathbf{w}$, and $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$
3. $\dfrac{1}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$
4. $\dfrac{1}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$
5. $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$
6. $\left(\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$
7. $\|\mathbf{w}\|$
8. $\|\mathbf{x}\|$

In Exercises 9–12, find a unit vector in the direction of the given vector.

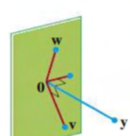
9. $\begin{bmatrix}-30\\40\end{bmatrix}$
10. $\begin{bmatrix}-6\\4\\-3\end{bmatrix}$
11. $\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}$
12. $\begin{bmatrix}8/3\\2\end{bmatrix}$
13. Find the distance between $\mathbf{x}=\begin{bmatrix}10\\-3\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\-5\end{bmatrix}$.
14. Find the distance between $\mathbf{u}=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$.

Determine which pairs of vectors in Exercises 15–18 are orthogonal.

15. $\mathbf{a}=\begin{bmatrix}8\\-5\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-2\\-3\end{bmatrix}$
16. $\mathbf{u}=\begin{bmatrix}12\\3\\-5\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}2\\-3\\3\end{bmatrix}$
17. $\mathbf{u}=\begin{bmatrix}3\\2\\-5\\0\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}-4\\1\\-2\\6\end{bmatrix}$
18. $\mathbf{y}=\begin{bmatrix}-3\\7\\4\\0\end{bmatrix}$, $\mathbf{z}=\begin{bmatrix}1\\-8\\15\\-7\end{bmatrix}$

In Exercises 19 and 20, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

19. a. $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$.
    b. For any scalar c, $\mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$.
    c. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
    d. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.
    e. If vectors $\mathbf{v}_1,\dots,\mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j=1,\dots,p$, then $\mathbf{x}$ is in $W^\perp$.
20. a. $\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}=0$.
    b. For any scalar c, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.
    c. If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^\perp$.
    d. If $\|\mathbf{u}\|^2+\|\mathbf{v}\|^2=\|\mathbf{u}+\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
    e. For an $m\times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.
21. Use the transpose definition of the inner product to verify parts (b) and (c) of Theorem 1. Mention the appropriate facts from Chapter 2.
22. Let $\mathbf{u}=(u_1,u_2,u_3)$. Explain why $\mathbf{u}\cdot\mathbf{u}\geq 0$. When is $\mathbf{u}\cdot\mathbf{u}=0$?
23. Let $\mathbf{u}=\begin{bmatrix}2\\-5\\-1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}-7\\-4\\6\end{bmatrix}$. Compute and compare $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|^2$, $\|\mathbf{v}\|^2$, and $\|\mathbf{u}+\mathbf{v}\|^2$. Do not use the Pythagorean Theorem.
24. Verify the *parallelogram law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:
$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$$
25. Let $\mathbf{v}=\begin{bmatrix}a\\b\end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix}x\\y\end{bmatrix}$ that are orthogonal to $\mathbf{v}$. [*Hint:* Consider $\mathbf{v}=\mathbf{0}$ and $\mathbf{v}\neq\mathbf{0}$.]
26. Let $\mathbf{u}=\begin{bmatrix}5\\-6\\7\end{bmatrix}$, and let W be the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{u}\cdot\mathbf{x}=0$. What theorem in Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.
27. Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u}+\mathbf{v}$.
28. Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span $\{\mathbf{u},\mathbf{v}\}$. [*Hint:* An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u},\mathbf{v}\}$ has the form $\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to such a vector $\mathbf{w}$.]
29. Let $W=\text{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, for $1\leq j\leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.



Section 6.2 : Orthogonal Sets

Chapter 6 : Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 6.2 : Orthogonal Sets

Chapter 6 : Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Orthogonal Sets of Vectors
2. Orthogonal Bases and Projections.

Learning Objectives

1. Apply the concepts of orthogonality to
   a) compute orthogonal projections and distances,
   b) express a vector as a linear combination of orthogonal vectors,
   c) characterize bases for subspaces of $\mathbb{R}^n$, and
   d) construct orthonormal bases.

Motivating Question
What are the special properties of this basis for $\mathbb{R}^3$?

$$\begin{bmatrix}3\\1\\1\end{bmatrix}/\sqrt{11},\quad \begin{bmatrix}-1\\2\\1\end{bmatrix}/\sqrt{6},\quad \begin{bmatrix}-1\\-4\\7\end{bmatrix}/\sqrt{66}$$

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through cour

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Orthogonal Vector Sets

Definition
A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ are an **orthogonal set** of vectors if for each $j \neq k$, $\vec{u}_j \perp \vec{u}_k$.

Example: Fill in the missing entries to make $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ an orthogonal set of vectors.

nonzero

$$\vec{u}_1 = \begin{bmatrix}4\\1\\1\end{bmatrix},\quad \vec{u}_2 = \begin{bmatrix}-2\\1\\7\end{bmatrix},\quad \vec{u}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$u_3 = \begin{pmatrix}1\\-5\\1\end{pmatrix}$$

$$\left\{\begin{bmatrix}4\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\7\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$ is an orthogonal set of vectors.

Linear Independence

Theorem (Linear Independence for Orthogonal Sets)
Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal set of vectors. Then, for scalars $c_1, \dots, c_p$,

$$\|c_1\vec{u}_1 + \cdots + c_p\vec{u}_p\|^2 = c_1^2\|\vec{u}_1\|^2 + \cdots + c_p^2\|\vec{u}_p\|^2.$$

In particular, if all the vectors $\vec{u}_r$ are non-zero, the set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ are linearly independent.

$$\operatorname{Nul} A = \operatorname{Span}\left\{\begin{pmatrix}1\\-5\\1\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\7\end{pmatrix}, \begin{pmatrix}1\\-5\\1\end{pmatrix}\right\}$$ orthogonal set in $\mathbb{R}^3$.

Nul A

$$A = \begin{pmatrix}4 & 1 & 1\\-2 & 1 & 7\end{pmatrix} = \begin{bmatrix}u_1^T\\u_2^T\end{bmatrix}$$

$$\sim \begin{bmatrix}-2 & 1 & 7\\0 & 3 & 15\end{bmatrix} \sim \begin{bmatrix}-2 & 1 & 7\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}-2 & 0 & 2\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 5\end{bmatrix}$$

$$x = r\begin{pmatrix}1\\-5\\1\end{pmatrix}$$

T/F if $\{u_1, u_2, u_3\}$ is orthogonal set then it is linearly independent

$$\vec{u}_3 = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$u_1 \cdot u_3 = 0 \iff \begin{pmatrix}4\\1\\1\end{pmatrix} \cdot \begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \iff 4a + b + c = 0$$

$$u_2 \cdot u_3 = 0 \iff \begin{pmatrix}-2\\1\\7\end{pmatrix} \cdot \begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \iff -2a + b + 7c = 0$$

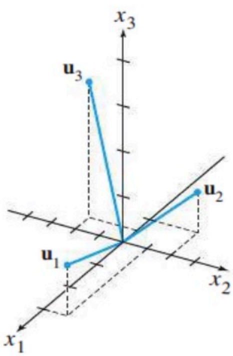


FIGURE 1

EXAMPLE 1 Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where

$$\mathbf{u}_1 = \begin{bmatrix}3\\1\\1\end{bmatrix},\quad \mathbf{u}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix},\quad \mathbf{u}_3 = \begin{bmatrix}-1/2\\-2\\7/2\end{bmatrix}$$

Defn: An orthogonal basis for a subspace H in $\mathbb{R}^n$ is a basis of H consisting of an orthogonal set of vectors.

Orthogonal Bases

Theorem (Expansion in Orthogonal Basis)

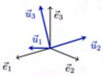
Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then, for any vector $\vec{w} \in W$,

$$\vec{w} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p.$$

Above, the scalars are $c_q = \dfrac{\vec{w}\cdot\vec{u}_q}{\vec{u}_q\cdot\vec{u}_q}$.

$[x]_B$

For example, any vector $\vec{w} \in \mathbb{R}^3$ can be written as a linear combination of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, or some other orthogonal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.



$[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\quad B = \{u_1, u_2\}$

$c_1 = \dfrac{x\cdot u_1}{u_1\cdot u_1} \qquad c_2 = \dfrac{x\cdot u_2}{u_2\cdot u_2}$

Example

$$\vec{x} = \begin{pmatrix}1\\1\\1\end{pmatrix},\quad \vec{u} = \begin{pmatrix}1\\-2\\1\end{pmatrix},\quad \vec{v} = \begin{pmatrix}-1\\0\\1\end{pmatrix},\quad \boxed{\vec{s} = \begin{pmatrix}3\\-4\\1\end{pmatrix}}$$

$s \in \operatorname{Span}\{\vec{u}, \vec{v}\} = W$

Let W be the subspace of $\mathbb{R}^3$ that is orthogonal to $\vec{x}$.

a) Check that an orthogonal basis for W is given by $\vec{u}$ and $\vec{v}$.

b) Compute the expansion of $\vec{s}$ in basis W.

Find $[s]_{\{u,v\}}$

Recall; want c_1, c_2

$\begin{bmatrix}3\\-4\\1\end{bmatrix} = c_1\begin{bmatrix}1\\-2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$c_1 = \dfrac{s\cdot u}{u\cdot u} = \dfrac{\begin{bmatrix}3\\-4\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}}{\begin{bmatrix}1\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}} = \dfrac{12}{6} = 2$

Notice $u\cdot v = 0$

$\begin{bmatrix}1\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix} = 0$

$c_2 = \dfrac{s\cdot v}{v\cdot v} = \dfrac{\begin{bmatrix}3\\-4\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix}} = \dfrac{-2}{2} = -1$

$[s]_B = \begin{bmatrix}2\\-1\end{bmatrix}$

$\vec{s}$ ✓

Check $2\begin{bmatrix}1\\-2\\1\end{bmatrix} - \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\-4\\1\end{bmatrix}$

THEOREM 4 If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

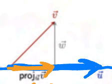
Projections

Let $\vec{u}$ be a non-zero vector, and let $\vec{v}$ be some other vector. The **orthogonal projection of $\vec{v}$ onto the direction of $\vec{u}$** is the vector in the span of $\vec{u}$ that is closest to $\vec{v}$.

$$\boxed{\operatorname{proj}_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}.}$$

The vector $\vec{w} = \vec{v} - \operatorname{proj}_{\vec{u}}\vec{v}$ is orthogonal to $\vec{u}$, so that

$$\vec{v} = \operatorname{proj}_{\vec{u}}\vec{v} + \vec{w}$$
$$\|\vec{v}\|^2 = \|\operatorname{proj}_{\vec{u}}\vec{v}\|^2 + \|\vec{w}\|^2$$



$L = \operatorname{span}\{\vec{u}\}$

$\operatorname{proj}_u(v)$ $\quad u$

The projection of v onto the line spanned by u.

Example

Let L be spanned by $(1,1,1,1)$ in $\mathbb{R}^4$.

1. Find the projection of $\vec{v} = (-3,5,6,-4)$ onto the line L.
2. How close is $\vec{v}$ to the line L?

"v hat"

$\hat{v} = \operatorname{proj}_{\vec{u}}(\vec{v})$

$= \dfrac{\begin{bmatrix}-3\\5\\6\\-4\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$= \dfrac{4}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

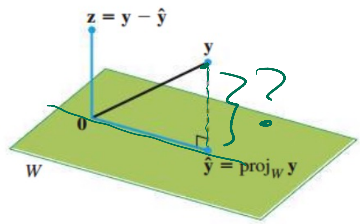


FIGURE 2

Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

v

$\operatorname{dist}\left(\begin{bmatrix}-3\\5\\6\\-4\end{bmatrix}, L\right) = \|\vec{v} - \hat{v}\|$

$= \left\|\begin{bmatrix}-3\\5\\6\\-4\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\| = \left\|\begin{bmatrix}-4\\4\\5\\-5\end{bmatrix}\right\|$

$= \sqrt{16+16+25+25}$

$= \sqrt{82}$

EXAMPLE 3 Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

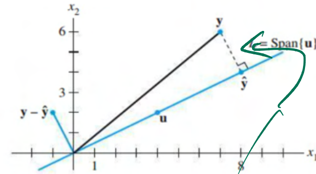


FIGURE 3 The orthogonal projection of $\mathbf{y}$ onto a line L through the origin.

$$\hat{y} = \text{proj}_{\vec{u}}(\vec{y}) = \frac{y \cdot u}{u \cdot u} u$$

$$\hat{y} = \frac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \hat{y}$$

$$\|\vec{y} - \hat{y}\|$$

distance from vector to line span $\{\vec{u}\}$

Definition

Definition (Orthonormal Basis)

An **orthonormal basis** for a subspace W is an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_p\}$ in which every vector $\vec{u}_q$ has unit length. In this case, for each $\vec{w} \in W$,

$$\vec{w} = [(\vec{w}) \cdot \vec{u}_1]\vec{u}_1 + \cdots + [(\vec{w}) \cdot \vec{u}_p]\vec{u}_p$$

$$\|\vec{w}\| = \sqrt{[(\vec{w}) \cdot \vec{u}_1]^2 + \cdots + [(\vec{w}) \cdot \vec{u}_p]^2}$$

Example

in W

also in W and orth. to $\vec{v}_1$

The subspace W is a subspace of $\mathbb{R}^3$ perpendicular to $(1, 1, 1)$. Calculate the missing coefficients in the orthonormal basis for W.

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} / \sqrt{\ }$$

$$\boxed{\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}}$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\left\| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \sqrt{1+1} = \sqrt{2}$$

compute Nul A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\vec{x} = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\|\vec{v}_2\| = \sqrt{6} \qquad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \stackrel{?}{=} 1 \times$$

didn't work

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \checkmark \quad \text{So } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \in W$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0 \checkmark \quad \text{So } v_2 \cdot v_1 = 0.$$

Orthogonal Matrices

An **orthogonal matrix** is a square matrix whose columns are orthonormal.

> **Theorem**
> An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Note that this theorem does not apply when $n > m$. Why?

Theorem

> **Theorem (Mapping Properties of Orthogonal Matrices)**
> Assume $m \times m$ matrix U has orthonormal columns. Then
> 1. (Preserves length) $\|U\vec{x}\|^2 = \boxed{\|x\|^2}$
> 2. (Preserves angles) $(U\vec{x}) \cdot (U\vec{y}) = \boxed{x \cdot y}$
> 3. (Preserves orthogonality) IF $x \cdot y = 0 \iff Ux \cdot Uy = 0$.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\|U\vec{x}\|^2 = U\vec{x} \cdot U\vec{x}$$
$$= (U\vec{x})^T U\vec{x}$$
$$= \vec{x}^T U^T U \vec{x} = x^T I x$$
$$= \vec{x}^T \vec{x} = \vec{x} \cdot \vec{x}$$
$$= \|\vec{x}\|^2$$

Example

Compute the length of the vector below.

$$\begin{bmatrix} 1/2 & 2/\sqrt{14} \\ 1/2 & 1/\sqrt{14} \\ 1/2 & -3/\sqrt{14} \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ -3 \end{bmatrix}$$

Additional Example (if time permits)

A 4×4 ortho~~normal~~gonal. matrix is below. It's columns are orthonormal.

$$A = \begin{bmatrix} 1/2 & 2/\sqrt{10} & 1/2 & 1/\sqrt{10} \\ 1/2 & 1/\sqrt{10} & -1/2 & -2/\sqrt{10} \\ 1/2 & -1/\sqrt{10} & -1/2 & 2/\sqrt{10} \\ 1/2 & -2/\sqrt{10} & 1/2 & -1/\sqrt{10} \end{bmatrix}$$

Verify that the rows also form an orthonormal basis.

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

$$\binom{4}{2} = \frac{4 \cdot 3}{2} = \frac{12}{2} = 6.$$

$v_1 \cdot v_2 = 0$
$v_1 \cdot v_3 = 0$
$v_1 \cdot v_4 = 0$
$v_2 \cdot v_3 = 0$
$v_2 \cdot v_4 = 0$
$v_3 \cdot v_4 = 0$

v_1^T, v_2^T, v_3^T, v_4^T; $v_1\ v_2\ v_3\ v_4$

$$A^T A = \begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{pmatrix} \begin{pmatrix} | & | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & -1 & 2 \\ 1 & -2 & 1 & -1 \end{bmatrix}$$

has orthogonal columns ✓

$$\boxed{v_1 \cdot v_1 = \|v_1\|^2 = 1.}$$

$$= v_1^T v_1 = \|v_1\|^2$$

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2$$
$$= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = 1$$

6.2 EXERCISES

In Exercises 1–6, determine which sets of vectors are orthogonal.

1. $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

4. $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

5. $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$

6. $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

In Exercises 7–10, show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ or $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

7. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

[1] A better name might be *orthonormal matrix*, and this term is found in some statistics texts. However, *orthogonal matrix* is the standard term in linear algebra.

8. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

9. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

10. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

11. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

12. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.

13. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

14. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span $\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

15. Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

16. Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

In Exercises 17–22, determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

17. $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

18. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

19. $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \begin{bmatrix} .8 \\ .6 \end{bmatrix}$

20. $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

21. $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

22. $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

In Exercises 23 and 24, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

b. If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d. A matrix with orthonormal columns is an orthogonal matrix.

e. If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

24. a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

c. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

d. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e. An orthogonal matrix is invertible.

24. a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

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e. An orthogonal matrix is invertible.

25. Prove Theorem 7. [*Hint:* For (a), compute $\|U\mathbf{x}\|^2$, or prove (b) first.]

26. Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

27. Let U be a square matrix with orthonormal columns. Explain why U is invertible. (Mention the theorems you use.)

28. Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

29. Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix. [That is, explain why UV is invertible and its inverse is $(UV)^T$.]

30. Let U be an orthogonal matrix, and construct V by interchanging some of the columns of U. Explain why V is an orthogonal matrix.

31. Show that the orthogonal projection of a vector $\mathbf{y}$ onto a line L through the origin in $\mathbb{R}^2$ does not depend on the choice of the nonzero $\mathbf{u}$ in L used in the formula for $\hat{\mathbf{y}}$. To do this, suppose $\mathbf{y}$ and $\mathbf{u}$ are given and $\hat{\mathbf{y}}$ has been computed by formula (2) in this section. Replace $\mathbf{u}$ in that formula by $c\mathbf{u}$, where c is an unspecified nonzero scalar. Show that the new formula gives the same $\hat{\mathbf{y}}$.

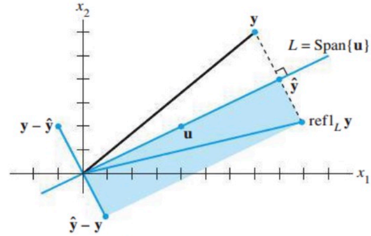
32. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthogonal set of nonzero vectors, and let c_1, c_2 be any nonzero scalars. Show that $\{c_1\mathbf{v}_1, c_2\mathbf{v}_2\}$ is also an orthogonal set. Since orthogonality of a set is defined in terms of pairs of vectors, this shows that if the vectors in an orthogonal set are normalized, the new set will still be orthogonal.

33. Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\,\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \text{proj}_L\,\mathbf{x}$ is a linear transformation.

34. Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\,\{\mathbf{u}\}$. For $\mathbf{y}$ in $\mathbb{R}^n$, the **reflection of y in** L is the point $\text{refl}_L\,\mathbf{y}$ defined by

$$\text{refl}_L\,\mathbf{y} = 2 \cdot \text{proj}_L\,\mathbf{y} - \mathbf{y}$$

See the figure, which shows that $\text{refl}_L\,\mathbf{y}$ is the sum of $\hat{\mathbf{y}} = \text{proj}_L\,\mathbf{y}$ and $\hat{\mathbf{y}} - \mathbf{y}$. Show that the mapping $\mathbf{y} \mapsto \text{refl}_L\,\mathbf{y}$ is a linear transformation.



The reflection of $\mathbf{y}$ in a line through the origin.

35. [M] Show that the columns of the matrix A are orthogonal by making an appropriate matrix calculation. State the calculation you use.

$$A = \begin{bmatrix} -6 & -3 & 6 & 1 \\ -1 & 2 & 1 & -6 \\ 3 & 6 & 3 & -2 \\ 6 & -3 & 6 & -1 \\ 2 & -1 & 2 & 3 \\ -3 & 6 & 3 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

36. [M] In parts (a)–(d), let U be the matrix formed by normalizing each column of the matrix A in Exercise 35.

a. Compute U^TU and UU^T. How do they differ?

b. Generate a random vector $\mathbf{y}$ in $\mathbb{R}^8$, and compute $\mathbf{p} = UU^T\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \mathbf{p}$. Explain why $\mathbf{p}$ is in Col A. Verify that $\mathbf{z}$ is orthogonal to $\mathbf{p}$.

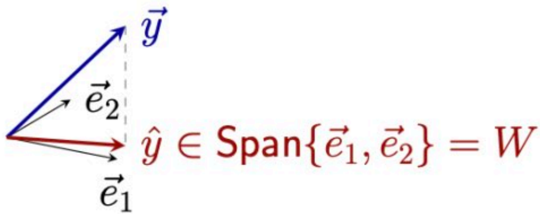
c. Verify that $\mathbf{z}$ is orthogonal to each column of U.

d. Notice that $\mathbf{y} = \mathbf{p} + \mathbf{z}$, with $\mathbf{p}$ in Col A. Explain why $\mathbf{z}$ is in $(\text{Col}\,A)^\perp$. (The significance of this decomposition of $\mathbf{y}$ will be explained in the next section.)

Section 6.3 : Orthogonal Projections

Chapter 6 : Orthogonality and Least Squares

Math 1554 Linear Algebra



Vectors $\vec{e}_1$ and $\vec{e}_2$ form an orthonormal basis for subspace W.
Vector $\vec{y}$ is not in W.
The orthogonal projection of $\vec{y}$ onto $W =$Span$\{\vec{e}_1, \vec{e}_2\}$ is $\hat{y}$.

Section 6.3 : Orthogonal Projections

Chapter 6 : Orthogonality and Least Squares

Math 1554 Linear Algebra

Vectors $\vec{e}_1$ and $\vec{e}_2$ form an orthonormal basis for subspace W.
Vector $\vec{y}$ is not in W.
The orthogonal projection of $\vec{y}$ onto W =Span$\{\vec{e}_1, \vec{e}_2\}$ is $\hat{y}$.

Topics and Objectives

Topics

1. Orthogonal projections and their basic properties
2. Best approximations

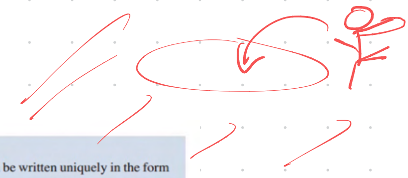
Learning Objectives

1. Apply concepts of orthogonality and projections to
   a) compute orthogonal projections and distances,
   b) express a vector as a linear combination of orthogonal vectors,
   c) construct vector approximations using projections,
   d) characterize bases for subspaces of $\mathbb{R}^n$, and
   e) construct orthonormal bases.

Motivating Question For the matrix A and vector $\vec{b}$, which vector $\hat{b}$ in column space of A, is closest to $\vec{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -4 & -2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
10/16 – 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10/23 – 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
10/30 – 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
11/6 – 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank



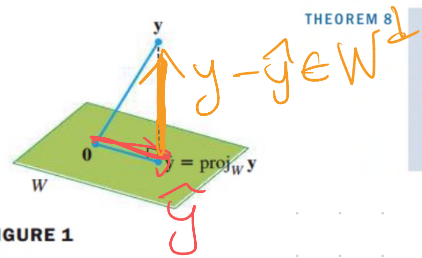


FIGURE 1

THEOREM 8 **The Orthogonal Decomposition Theorem**

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \tag{2}$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

$\{u_1, \ldots, u_5\}$ basis of $\mathbb{R}^5$ (orthonormal basis of $\mathbb{R}^n$), $y \in \mathbb{R}^n$
w/ orthogonal u_i's.

$$y = \underbrace{\frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 + \frac{y\cdot u_3}{u_3\cdot u_3}u_3}_{\hat{y} = \text{proj}_W(\vec{y})} + \underbrace{\frac{y\cdot u_4}{u_4\cdot u_4}u_4 + \frac{y\cdot u_5}{u_5\cdot u_5}u_5}_{y - \hat{y}}$$

Suppose.

$\text{span}\{u_1, u_2, u_3\} = W \qquad W^{\perp} = \text{span}\{u_4, u_5\}$

Ex. $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ Q: Let $W = \text{Span}\{u_1, u_2\}$
Find $\hat{y}$ & $\vec{y} - \hat{y} = \vec{z}$

Notice $u_1 \cdot u_2 = 0$ which we are doing FIRST.

$$\hat{y} = \text{proj}_W(\vec{y}) = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2 \qquad y = \hat{y} + z$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \hat{y}} \in W$$

$$\boxed{\vec{z} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}} \in W^{\perp}$$

Example 1

Let $\vec{u}_1, \dots, \vec{u}_5$ be an orthonormal basis for $\mathbb{R}^5$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$. For a vector $\vec{y} \in \mathbb{R}^5$, write $\vec{y} = \hat{y} + w^{\perp}$, where $\hat{y} \in W$ and $w^{\perp} \in W^{\perp}$.

Orthogonal Decomposition Theorem

Theorem

Let W be a subspace of $\mathbb{R}^n$. Then, each vector $\vec{y} \in \mathbb{R}^n$ has the **unique** decomposition

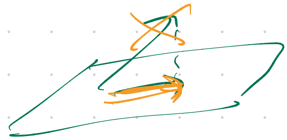
$$\vec{y} = \hat{y} + w^{\perp}, \quad \hat{y} \in W, \quad w^{\perp} \in W^{\perp}.$$

And, if $\vec{u}_1, \dots, \vec{u}_p$ is any orthogonal basis for W,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p.$$

We say that $\hat{y}$ is the **orthogonal projection of** $\vec{y}$ **onto** W.

If time permits, we will prove this theorem on the next slide.



Proof (if time permits)

We can write

$\hat{y} =$

Then, $w^{\perp} = \vec{y} - \hat{y}$ is in $W^{\perp}$ because

Uniqueness:

Example 2a

$$\vec{y} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

Construct the decomposition $\vec{y} = \hat{y} + w^{\perp}$, where $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto the subspace $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

Let $W = \text{span}\{u_1, u_2\}$

Q_1: Find $\hat{y} = \text{proj}_{\{u_1, u_2\}}(y)$

Q_2: Find $\vec{z} \in W^{\perp}$ s.t. $\vec{y} = \hat{y} + \vec{z}$.

STEP!!

(0) $u_1 \cdot u_2 \overset{?}{=} 0$ $\quad \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 3 + 1 - 4 = 0$ ✓

Soln.

(1) $$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{7}{14} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \frac{-15}{6} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \frac{5}{2} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \frac{1}{2} \left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = \hat{y}$$

$y = \hat{y}$

(2) $$z = y - \hat{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} - \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Best Approximation Theorem

Theorem

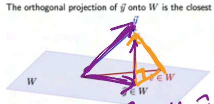
Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$, and $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W. Then for **any** $\vec{w} \neq \hat{y} \in W$, we have

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{w}\|$$

That is, $\hat{y}$ is the unique vector in W that is closest to $\vec{y}$.

Proof (if time permits)

The orthogonal projection of $\vec{y}$ onto W is the closest point in W to $\vec{y}$.



$\hat{y}$ $\quad y - \hat{y} = z$

$\vec{y} = \hat{y} + (y - \hat{y})$

$= \vec{y}$ ✓

Example 2b

$$\vec{y} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

What is the distance between $\vec{y}$ and subspace $W = \mathrm{Span}\{\vec{u}_1, \vec{u}_2\}$? Note that these vectors are the same vectors that we used in Example 2a.

Additional Example (if time permits)

Indicate whether each statement is true or false. If true, explain why in one or two sentences. If false, give a counter example or explain why in one or two sentences.

a) If $\vec{x}$ is orthogonal to $\vec{v}$ and $\vec{w}$, then $\vec{x}$ is also orthogonal to $\vec{v} - \vec{w}$.

b) If $\mathrm{proj}_W \vec{y} = \vec{y}$, then $\vec{y} \in W$.

c) If $\vec{y} = \vec{u}_1 + \vec{v}_1$, where $\vec{u}_1 \in W$ and $\vec{v}_1 \in W^{\perp}$, then $\vec{u}_1$ is the orthogonal projection of $\vec{y}$ onto W.

6.3 EXERCISES

In Exercises 1 and 2, you may assume that $\{\mathbf{u}_1, \dots, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

1. $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$,

$\mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$. Write $\mathbf{x}$ as the sum of two vectors, one in Span $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in Span $\{\mathbf{u}_4\}$.

2. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$,

$\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. Write $\mathbf{v}$ as the sum of two vectors, one in Span $\{\mathbf{u}_1\}$ and the other in Span $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

In Exercises 3–6, verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

4. $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

5. $\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

6. $\mathbf{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–10, let W be the subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

7. $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

8. $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

9. $\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

10. $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

In Exercises 11 and 12, find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

11. $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

12. $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

In Exercises 13 and 14, find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

13. $\mathbf{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

14. $\mathbf{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$

15. Let $\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Find the distance from $\mathbf{y}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

16. Let $\mathbf{y}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ be as in Exercise 12. Find the distance from $\mathbf{y}$ to the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

17. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, and $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.

a. Let $U = [\,\mathbf{u}_1 \;\; \mathbf{u}_2\,]$. Compute U^TU and UU^T.

b. Compute $\text{proj}_W\,\mathbf{y}$ and $(UU^T)\mathbf{y}$.

18. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, and $W = \text{Span}\,\{\mathbf{u}_1\}$.

a. Let U be the 2×1 matrix whose only column is $\mathbf{u}_1$. Compute U^TU and UU^T.

b. Compute $\text{proj}_W\,\mathbf{y}$ and $(UU^T)\mathbf{y}$.

19. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal but that $\mathbf{u}_3$ is not orthogonal to $\mathbf{u}_1$ or $\mathbf{u}_2$. It can be shown that $\mathbf{u}_3$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

20. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be as in Exercise 19, and let $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{u}_4$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

In Exercises 21 and 22, all vectors and subspaces are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$.

b. For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$ is orthogonal to W.

c. The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.

d. If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

e. If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of U.

22. a. If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

b. In the Orthogonal Decomposition Theorem, each term in formula (2) for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

c. If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

d. The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$.

e. If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

23. Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p}$ is in Row A and $\mathbf{u}$ is in Nul A. Also, show that if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then there is a unique $\mathbf{p}$ in Row A such that $A\mathbf{p} = \mathbf{b}$.

24. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that $\dim W + \dim W^\perp = n$.

25. **[M]** Let U be the 8×4 matrix in Exercise 36 in Section 6.2. Find the closest point to $\mathbf{y} = (1, 1, 1, 1, 1, 1, 1, 1)$ in Col U. Write the keystrokes or commands you use to solve this problem.

26. **[M]** Let U be the matrix in Exercise 25. Find the distance from $\mathbf{b} = (1, 1, 1, 1, -1, -1, -1, -1)$ to Col U.