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Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices
2. Orthogonal diagonalization
3. Spectral decomposition

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.
2. Construct a spectral decomposition of a matrix.

Quadratic Forms Ch. 7

12:29 PM Mon Nov 11

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4	9/9 – 9/13	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/16 – 9/20	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/23 – 9/27	2.9	WS2.8,2.9	2.9,3.1	WS3.1	3.2
7	9/30 – 10/4	3.3	WS3.3	4.9	WS4.9	5.1
8	10/7 – 10/11	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	10/14 – 10/18	Break	Break	5.3	WS5.3	5.5
10	10/21 – 10/25	6.1	WS5.5,6.1	6.2	WS6.2	6.3
11	10/28 – 11/1	6.4	WS6.3,6.4	6.5	WS6.5	6.6
12	11/4 – 11/8	6.6, Review	WS6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/11 – 11/15	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/18 – 11/22	7.3	WS7.3	7.4	WS7.4	7.4
15	11/25 – 11/29	7.4, Review	WS7.4, Review	Break	Break	Break
16	12/2 – 12/6	Last lecture	Last Studio	Reading Period		
17	12/5 – 12/12	Final Exams: **MATH 1554 Common Final Exam 6-8:50pm on Tuesday, Dec. 10th**				

last day for MQEs

$x_1^2 + x_2^2 = Q(x_1, x_2)$

$x_1^2 + 2x_1x_2 + x_2^2 = Q_2(x_1, x_2)$

$A^T = A$ (!!)

Symmetric Matrices

Definition

Matrix A is **symmetric** if $A^T = A$.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

$A = [*]$ $\quad B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $\quad C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$

$D = \begin{bmatrix} \star & \star \\ 0 & 0 \end{bmatrix}$ $\quad E = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix}$

$a_{1,4}$ row 1 col 4

$a_{4,1}$ row 4 col 1

$a_{ij} = a_{ji}$ if $i \neq j$

$A^T A$ is Symmetric

A very common example: For **any** matrix A with columns $a_1, \ldots, a_n$,

$$A^T A = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \cdots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \cdots & a_n^T a_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

$$A = \begin{array}{cc} v_1 & v_2 \\ \end{array} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 13 \end{pmatrix}$$

v_1^T $\quad v_1$ $\quad v_1 \cdot v_1$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

$V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$V_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} r = 1 \\ s = 2 \end{pmatrix}$ random lin comb

$V_1 \cdot V_2 = -1 + 0 + 1 = 0$

$\lambda = -1$ $\quad \lambda = 1$

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

$\underline{\lambda_1 = -1}$

$$A + I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

r

$$X = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 1}$

$r \quad s$

$$A - I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

Suppose v_1, v_2 eigenvectors of A, & $A^T = A$.

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$.

WTS $\boxed{v_1 \cdot v_2 = 0}$.

Consider

b/c $A^T = A$

$$\lambda_1 v_1 \cdot v_2 = Av_1 \cdot v_2 = (Av_1)^T v_2 = v_1^T A^T v_2 = v_1^T A v_2 = v_1 \cdot Av_2 = \lambda_2 v_1 \cdot v_2$$

Conclude $\lambda_1 (v_1 \cdot v_2) = \lambda_2 (v_1 \cdot v_2)$

$ab = cb$.

If $\lambda_1 \neq \lambda_2 \Rightarrow \boxed{v_1 \cdot v_2 = 0}$ ✓.

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

Spectral Theorem

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real.
2. The dimension of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity. alg = geo
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is orthogonal.

Proof (if time permits):

orthonormal. by G-S if geo > you can find an orthogonal/orthonormal basis for that λ-eigenspace

$P = [u_1 u_2 u_3]$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$A = PDP^{-1} = PDP^T$

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = [\vec{u}_1 \cdots \vec{u}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank ______.

$P = [u_1 u_2 u_3]$ $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

decompose A as

$A = \lambda_1 \underbrace{u_1 u_1^T}_{} + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$

not the scalar $u_1 \cdot u_1 = u_1^T u_1$ instead it's $\boxed{u_1 u_1^T}$

Example: Find the spectral decomposition of A.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{P = [u_1\ u_2\ u_3]} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T}_{P^T}$$

rank 1.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \boxed{\lambda_1 u_1 u_1^T} + \boxed{\lambda_2 u_2 u_2^T} + \boxed{\lambda_3 u_3 u_3^T}$$

$$= 4 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

each row of $u_1 u_1^T$ is a scalar mult of u_1^T

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{u_1} \underbrace{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{u_2} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

λ_1 λ_2

$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

Note: $\|u\| = 1$.

? $u_1 u_1^T$?

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 4 \boxed{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} + 2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \boxed{4 \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{\text{rank } 1} + 2 \underbrace{\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}_{\text{rank } 1}}$$

ANS: the spectral decomp.

$$\overset{\text{check.}}{=} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A \checkmark$$

Note uu^T always sym.

① b/c $(uu^T)^T = (u^T)^T u^T = uu^T$ ✓

② always rank 1

b/c each row is a scalar of u^T

③ $\boxed{\text{proj}_u(\vec{x}) = uu^T x.}$

7.1 Exercises

Determine which of the matrices in Exercises 1–6 are symmetric.

1. $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

7. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

10. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

11. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$

12. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix P and a diagonal matrix D. To save you time, the eigenvalues in Exercises 17–22 are the following: (17) $-4, 4, 7$; (18) $-3, -6, 9$; (19) $-2, 7$; (20) $-3, 15$; (21) $1, 5, 9$; (22) $3, 5$.

13. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Then orthogonally diagonalize A.

In Exercises 25–32, mark each statement True or False (**T/F**). Justify each answer.

25. **(T/F)** An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

26. **(T/F)** There are symmetric matrices that are not orthogonally diagonalizable.

27. **(T/F)** An orthogonal matrix is orthogonally diagonalizable.

28. **(T/F)** If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

29. **(T/F)** For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

30. **(T/F)** If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

31. **(T/F)** An $n \times n$ symmetric matrix has n distinct real eigenvalues.

32. **(T/F)** The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

33. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

34. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB, and BB^T are symmetric matrices.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

39. Construct a spectral decomposition of A from Example 2.

40. Construct a spectral decomposition of A from Example 3.

41. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.

a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.

b. Show that B is a symmetric matrix and $B^2 = B$.

c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

42. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

b. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

Orthogonally diagonalize the matrices in Exercises 43–46. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue λ, find an orthonormal basis for $\text{Nul}(A - \lambda I)$, as in Examples 2 and 3.

T **43.** $\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$

T **44.** $\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$

T **45.** $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

T **46.** $\begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$

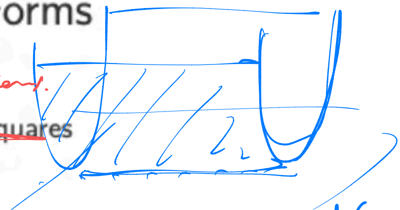
$Q(1,1) = 1^2+1^2=2$

Section 7.2 : Quadratic Forms

Symm. matrices & Quad form.

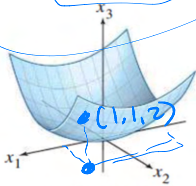
Chapter 7: ~~Orthogonality and Least Squares~~

Math 1554 Linear Algebra



$x_3 = x_1^2+x_2^2$

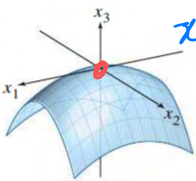
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



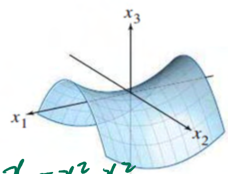
Positive definite

$x_3 = -x_1^2-x_2^2$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



Negative definite



$x_3 = x_1^2-x_2^2$

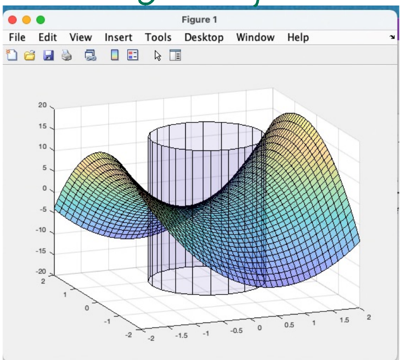
Indefinite

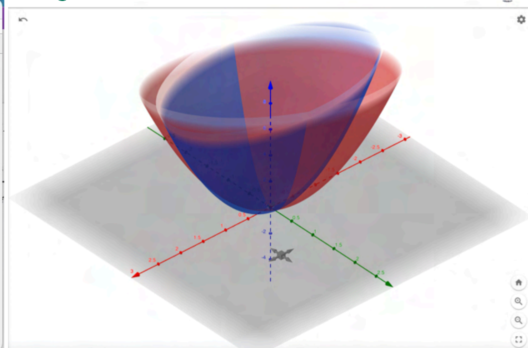
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

positive-semi def.

$x_3 = x_1^2+2x_1x_2+x_2^2$





Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

IF $A^T = A$ (Symmetric)

$\Rightarrow A = PDP^T$

D diagonal, P orthogonal

* λ's all real
* $a_{lg} = geo$
* $\{v_1, \dots, v_n\}$ orthonormal basis for $\mathbb{R}^n$

$Av_i = \lambda_i v_i$

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course m

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	9/18 - 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A \vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question Does this inequality hold for all x, y?

$$x^2 - 6xy + 9y^2 \geq 0$$

Quadratic Forms

Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 - 3x_2 \\ -3x_1 + 9x_2 \end{bmatrix}$$

$$= x_1(x_1 - 3x_2) + x_2(-3x_1 + 9x_2)$$

$$= x_1^2 - 3x_1x_2 - 3x_1x_2 + 9x_2^2 = \boxed{x_1^2 - 6x_1x_2 + 9x_2^2}$$

quadratic form defined by $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$

Example 1

Compute the quadratic form $\vec{x}^T A \vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

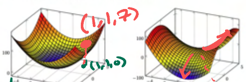
$Q_A(1,1) = 7$

$$Q_A(x_1, x_2) = 4x_1^2 + 3x_2^2$$

$$Q_B(x_1, x_2) = 4x_1^2 + 2x_1x_2 - 3x_2^2$$

Example 1 - Surface Plots

The surfaces for Example 1 are shown below.



$Q_A(x_1, x_2) = x_3$ $\quad$ $Q_B(x_1, x_2) = x_3$

Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

$Q_B(1,1) = 3$

$Q_B(0,1) = -3$

$Q_B(1,0) = 4$

Q: why must A be symmetric?

A: it doesn't? sort of. Only if A is symmetric does it help to decide questions about $Q_A(\vec{x}) = \vec{x}^T A \vec{x}$.

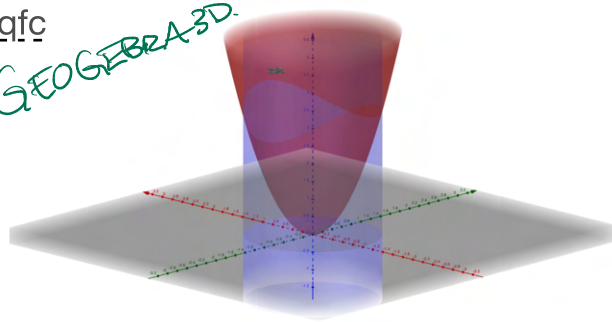
$$A' = \begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix} \quad B' = \begin{bmatrix} 4 & 2 \\ 0 & -3 \end{bmatrix}$$

also worked to get

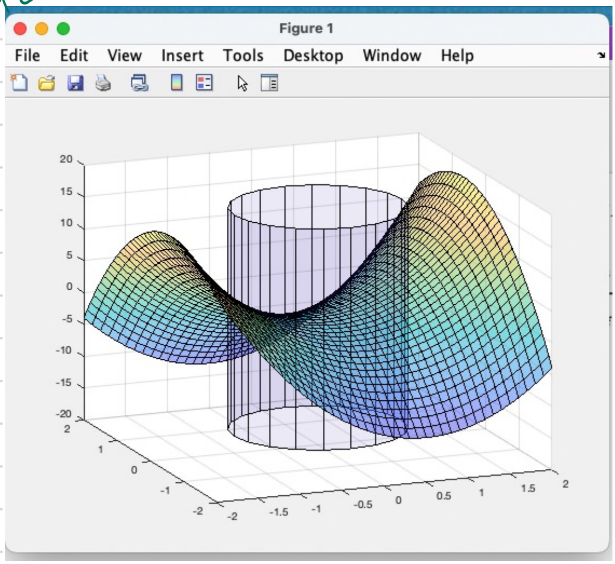
$Q_A = Q_{A'}$ $\quad$ $Q_B = Q_{B'}$

https://www.geogebra.org/m/pbzpeqfc

GEOGEBRA 3D



MATLAB




```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```

Example 2

Write Q in the form $\vec{x}^T A\vec{x}$ for $\vec{x}\in\mathbb{R}^3$

$$Q(x) = 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3 + 0x_1x_2$$

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}$$

$a_{1,3}$ $a_{3,1}$

$$\underset{1\times 3}{x^TAx = [x_1\ x_2\ x_3]} \underset{3\times 3}{\begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$$= \underset{1\times 3}{[x_1\ x_2\ x_3]} \underset{3\times 1}{\begin{bmatrix} 5x_1 + 3x_3 \\ -x_2 - 6x_3 \\ 3x_1 - 6x_2 + 3x_3 \end{bmatrix}}$$

$$= 5x_1^2 + 3x_1x_3 - x_2^2 - 6x_2x_3 + 3x_1x_3 - 6x_2x_3 + 3x_3^2$$

$$= 5x_1^2 - x_2^2 + 3x_3^2 + 6x_1x_3 - 12x_2x_3 \ \cancel{\overset{?}{\geq}}\ 0$$

$Q_A(1,0,0) = 5$

$Q_A(0,1,0) = -1$

$Q_A(0,0,1) = 3$

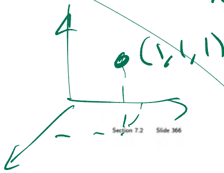
Change of Variable

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A\vec{x}$ becomes:

$$Q_A(1,1,1) = 5 - 1 + 3 + 6 - 12$$
$$= 1.$$



Q:

rewrite

$$Q_A(x_1, x_2) = x_1^2 - 6x_1x_2 + 9x_2^2 \geq 0$$

$$Q_D(y_1, y_2) = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

easy to analyze.

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$Q_A(\vec{x}) = x^TAx \overset{?}{=} x^TPDP^Tx$$
$$= \underbrace{(P^Tx)^T}_{y^T} D \underbrace{(P^Tx)}_{y}$$
$$= y^TDy = Q_D(\vec{y})$$

$$\boxed{P^Tx = y}$$
$$\Rightarrow PP^Tx = Py$$
$$\Rightarrow \boxed{x = Py}$$

Q1: Find $Q_A(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 6x_2^2$

Q2: Find $Q_D(y_1, y_2) = 2y_1^2 + 7y_2^2$

Q3: want some way to go from $(x_1, x_2) \longleftrightarrow (y_1, y_2)$

Eg. $\boxed{Q_D(0,1) = 7}$

$Q_A(?,?) = 7$

$$\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{x} \overset{?}{=} Py$$

$$\vec{x} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$Q_A\left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}\right) = 3\left(\tfrac{1}{\sqrt{5}}\right)^2 + 4\left(\tfrac{1}{\sqrt{5}}\right)\left(\tfrac{2}{\sqrt{5}}\right) + 6\left(\tfrac{2}{\sqrt{5}}\right)^2$$

$$= \frac{3}{5} + \frac{8}{5} + \frac{24}{5} = \frac{35}{5} = \boxed{7}$$

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$Q_A(\vec{x}) = x^T A x \stackrel{?}{=} x^T P D P^T x$$

$$= \underbrace{(P^T x)^T}_{y^T} D \underbrace{(P^T x)}_{y}$$

$$= y^T D y = Q_D(\vec{y})$$

$$\boxed{P^T x = y}$$

$$\Rightarrow PP^T x = Py$$

$$\Rightarrow \boxed{x = Py}$$

Q1: Find $Q_A(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 6x_2^2$

Q2: Find $Q_D(y_1, y_2) = 2y_1^2 + 7y_2^2$

same

Q3: want some way to go from $(x_1, x_2) \longleftrightarrow (y_1, y_2)$

Eg. $\boxed{Q_D(0,1) = 7}$

$Q_A(?,?) = 7$

$$\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \boxed{\vec{x} \stackrel{?}{=} Py}$$

$$\vec{x} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$Q_A\left(1/\sqrt{5}, 2/\sqrt{5}\right) = 3\left(1/\sqrt{5}\right)^2 + 4\left(1/\sqrt{5}\right)\left(2/\sqrt{5}\right) + 6\left(2/\sqrt{5}\right)^2$$

$$= \frac{3}{5} + \frac{8}{5} + \frac{24}{5} = \frac{35}{5} = \boxed{7}$$

$$Q_D(y_1, y_2) = 2y_1^2 + 7y_2^2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T \vec{x} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}}x_1 - \frac{1}{\sqrt{5}}x_2 \\ 1/\sqrt{5}\, x_1 + \frac{2}{\sqrt{5}}x_2 \end{bmatrix}$$

$$= 2\left(\frac{2}{\sqrt{5}}x_1 - \frac{1}{\sqrt{5}}x_2\right)^2 + 7\left(\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2\right)^2$$

$$\begin{cases} y_1 = \frac{2}{\sqrt{5}}x_1 - \frac{1}{\sqrt{5}}x_2 \\ y_2 = \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \end{cases}$$

$$= 2\left(\frac{4}{5}x_1^2 - \frac{4}{5}x_1x_2 + \frac{1}{5}x_2^2\right)$$

$$+ 7\left(\frac{1}{5}x_1^2 + \frac{4}{5}x_1x_2 + \frac{4}{5}x_2^2\right)$$

$$= \frac{8}{5}x_1^2 + \frac{7}{5}x_1^2 - \frac{8}{5}x_1x_2 + \frac{28}{5}x_1x_2 + \frac{2}{5}x_2^2 + \frac{28}{5}x_2^2$$

$$= \boxed{3x_1^2 + 4x_1x_2 + 6x_2^2} = Q_A(x_1, x_2)$$

Principle Axes Theorem

Theorem

If A is a Symmetric matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{y}^T D\vec{y}$ with no cross-product terms.

§7.2

* $Q(\vec{x}) = x^T A x \leftrightarrow$ construct A given $Q(\vec{x})$.
* use orth. diag'n
  $Q_A(\vec{x}) = Q_D(\vec{y})$
* change of vars
  $P^T x = y$ or $Py = x$

(*) PD/ND/PSD/NSD
positive/negative (semi-) definite

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

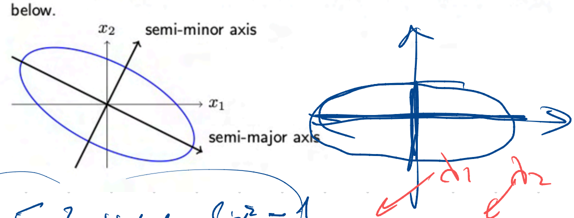
$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$Q_A(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 6x_2^2$$

$$Q_D(y_1, y_2) = 2y_1^2 + 7y_2^2$$

Example 5

Compute the quadratic form $Q = \vec{x}^T A\vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.



$$5x_1^2 + 4x_1x_2 + 8x_2^2 = 1 \leftrightarrow ax_1^2 + bx_2^2 = c$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad ??$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

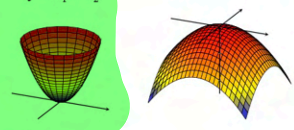
$$x^T A x = x^T PDP^T x = (P^T x)^T D \underbrace{(P^T x)}_{y}$$

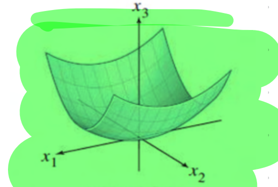
$$y = P^T x$$

$$\Rightarrow Py = \underbrace{PP^T}_{I} x = x$$

Classifying Quadratic Forms

$Q = x_1^2 + x_2^2$ $\qquad$ $Q = -x_1^2 - x_2^2$



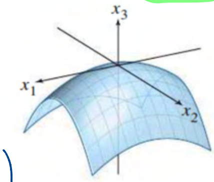


Positive definite

Definition

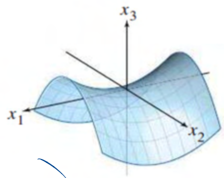
A quadratic form Q is

1. **positive definite** if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$. PD
2. **negative definite** if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$. ND
3. **positive semidefinite** if $Q(\vec{x}) \geq 0$ for all $\vec{x}$. PSD
4. **negative semidefinite** if $Q(\vec{x}) \leq 0$ for all $\vec{x}$. NSD
5. **indefinite** if none of the above.



Negative definite

$$Q(\vec{x}) = (x_1 - x_2)^2$$
$$= \underbrace{x_1^2 - 2x_1x_2 + x_2^2}_{} \geq 0 \checkmark$$



Indefinite

$Q(1,-1) = 0$

$Q(-2,2) = 0$

NOTE: minus sign $\not\Rightarrow$ indefinite

Quadratic Forms and Eigenvalues

Theorem

If A is a symmetric matrix with eigenvalues λ_i, then $Q = \vec{x}^T A \vec{x}$ is

1. **positive definite** iff $\lambda_i > 0$
2. **negative definite** iff $\lambda_i < 0$
3. **indefinite** iff $\lambda_i > 0$ & some $\lambda_i < 0$

PSD iff $\lambda_i \geq 0$

NDS iff $\lambda_i \leq 0$

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y?

$$\underbrace{\boxed{x^2 - 6xy + 9y^2}}_{\text{PSD?}} \geq 0$$

yes.

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

$\lambda_i \geq 0$?

orthonormal

$v_1 \quad v_2$

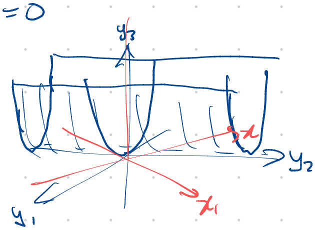
$$P = \begin{bmatrix} \vdots & \vdots \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

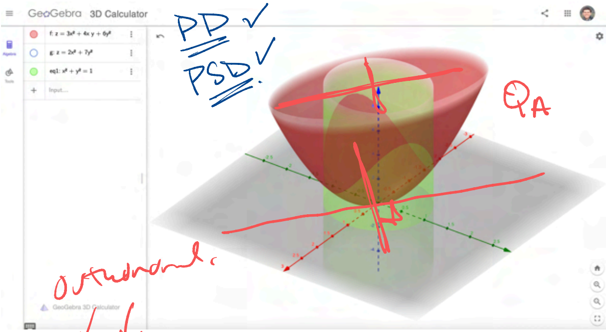
$\lambda_1 \quad \lambda_2$

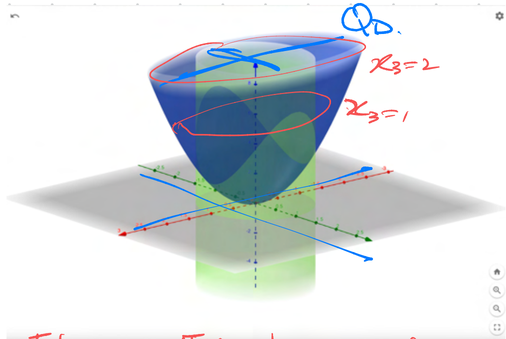
Step 1: $p(\lambda) = \lambda^2 - 10\lambda = (\lambda - 10)\lambda = 0$

$\lambda_1 = 10, \ \lambda_2 = 0$

Step 2: $Q_D(y_1, y_2) = 10y_1^2 = y_3$





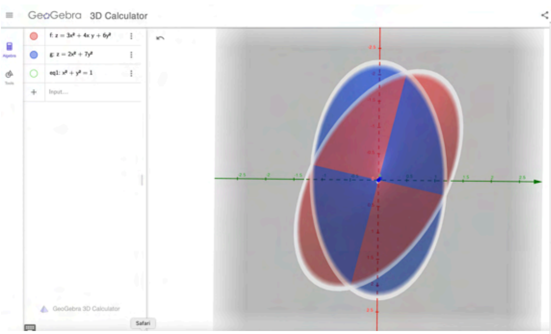


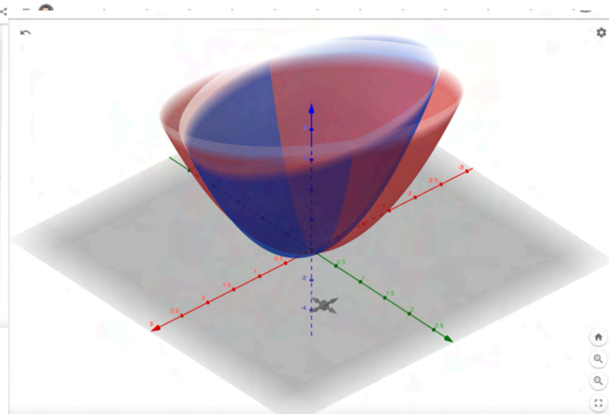
orthonormal.

$P=[v_1 \; v_2]$

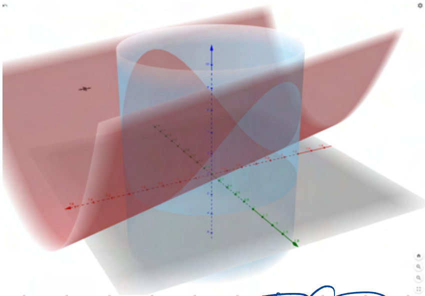
$$Q_A(\vec{v}_1) = v_1^T A v_1 = v_1^T d_1 v_1 = d_1 \underbrace{v_1^T v_1}_{1.} = d_1.$$

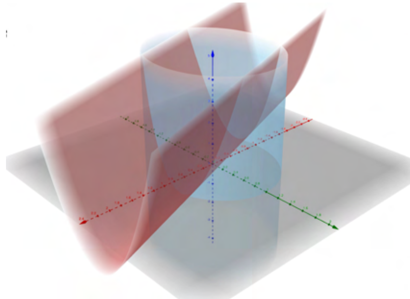
$$Q_D(e_i) = e_i^T D e_i = d_i$$





https://www.geogebra.org/m/c6yg2agh





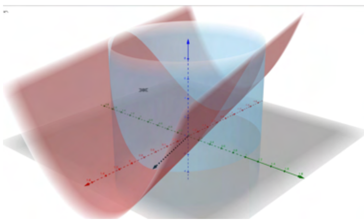
f: $z = x^2 - 6x\,y + 9y^2$

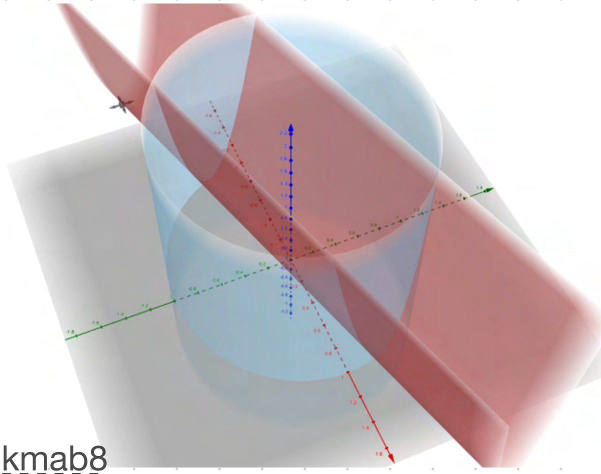
eq1: $x^2 + y^2 = 1$

$u = \text{Vector}\left(\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right)$

$= \begin{pmatrix} 0.95 \\ 0.32 \end{pmatrix}$

PSD
and NOT PD.





https://www.geogebra.org/m/akbkmab8

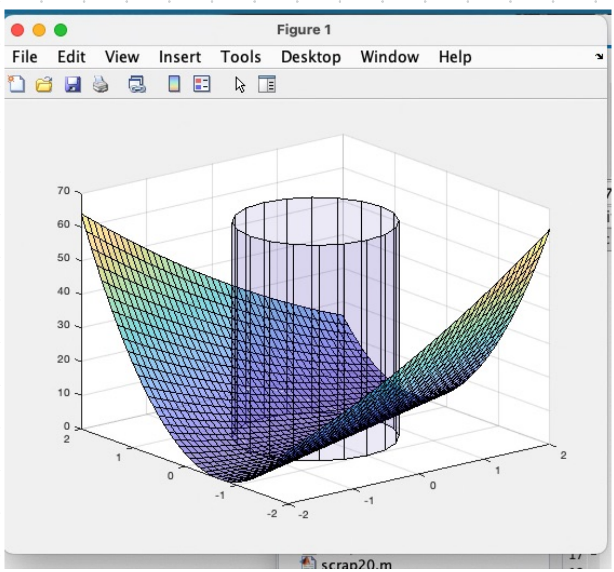
```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and
   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and
   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $3x_1^2 - 4x_1x_2 + 5x_2^2$ b. $3x_1^2 + 2x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $5x_1^2 + 16x_1x_2 - 5x_2^2$ b. $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$
   b. $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $3x_1^2 - 2x_2^2 + 5x_3^2 + 4x_1x_2 - 6x_1x_3$
   b. $4x_3^2 - 2x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

8. Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

   It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct P using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$
10. $2x_1^2 + 6x_1x_2 - 6x_2^2$
11. $2x_1^2 - 4x_1x_2 - x_2^2$
12. $-x_1^2 - 2x_1x_2 - x_2^2$
13. $x_1^2 - 6x_1x_2 + 9x_2^2$
14. $3x_1^2 + 4x_1x_2$
15. [M] $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$
16. [M] $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$
17. [M] $11x_1^2 + 11x_2^2 + 11x_3^2 + 11x_4^2 + 16x_1x_2 - 12x_1x_4 + 12x_2x_3 + 16x_3x_4$
18. [M] $2x_1^2 + 2x_2^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.
   b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.
   c. The principal axes of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are eigenvectors of A.
   d. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
   e. If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.
   f. A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is not a quadratic form.
   b. If A is symmetric and P is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term.
   c. If A is a 2×2 symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola.
   d. An indefinite quadratic form is neither positive semidefinite nor negative semidefinite.
   e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all positive.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of A.

23. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.
   a. Q is positive definite if $\det A > 0$ and $a > 0$.
   b. Q is negative definite if $\det A > 0$ and $a < 0$.
   c. Q is indefinite if $\det A < 0$.

25. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

26. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.]

B works.]

27. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let A be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

SG Mastering: Diagonalization and Quadratic Forms 7-7

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

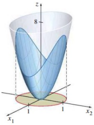


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

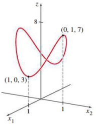
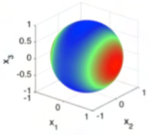


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.



13	11/13 - 11/17	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exams: **MATH 1554 Common Final Exam Tuesday, December 12th at 6pm**				

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

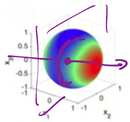
Example 1

The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = \|\vec{x}\|^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$



Find the largest and smallest values of Q on the surface of the sphere.

$Q(1,0,0) = 9(1)^2 + 4(0)^2 + 3(0)^2 = 9$ ✓

$Q(0,1,0) = 9(0)^2 + 4(1)^2 + 3(0)^2 = 4$ ✓

$Q(0,0,1) = 9(0)^2 + 4(0)^2 + 3(1)^2 = 3$ ✓

$$3 \le Q(\vec{x}) \le 9$$

$Q\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = 9\left(\frac{1}{\sqrt{2}}\right)^2 + 4\left(\frac{1}{\sqrt{2}}\right)^2 + 0 = \frac{9+4}{2} = 6.5$ ✓

$Q\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 9\left(\frac{1}{\sqrt{3}}\right)^2 + 4\left(\frac{1}{\sqrt{3}}\right)^2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{9+4+3}{3} = \frac{16}{3} = 5.\overline{33}$ ✓

$Q\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = 9\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^2 = \frac{9 + 4(4) + 3(4)}{9} = \frac{37}{9} \approx 4.\overline{11}$ ✓

If $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$ and $\|\vec{v}\| = 1$

Then $Q(\vec{v})$ is a weighted average of $a, b, \& c$.

Ex. Find the largest output z-value with restricted input ||x||=1 where z is given by:

$$z = 3x_1^2 + 7x_2^2.$$

$Q(0,1) = 7$

$Q(1,0) = 3$

$Q\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 3\left(\frac{1}{5}\right) + 7\left(\frac{4}{5}\right)$

$= 6.2$

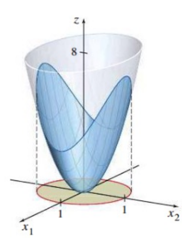


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

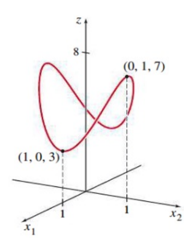


FIGURE 2 The intersection of

EXAMPLE 3 Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of the quadratic form $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, and find a unit vector at which this maximum value is attained.

TBD....

SOLUTION By Theorem 6, the desired maximum value is the greatest eigenvalue of A. The characteristic equation turns out to be

$$0 = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 6)(\lambda - 3)(\lambda - 1)$$

The greatest eigenvalue is 6.

?

Sol's $(\lambda_1 \geq \lambda_2 \geq \lambda_3)$

Q_1: Find max $\vec{x}$ s.t.

$$Q_A(x_1,x_2,x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$$

attains max value w/ unit length inputs.

Q_2: What is this max value? $\boxed{6}$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix}$$

Soln $Q_D(y_1,y_2,y_3) = 6y_1^2 + 3y_2^2 + y_3^2$

$$\vec{x} = Pe_1 = v_1$$

a unit length eigenvector in $\lambda = 6$ eigenspace.

$$Q_D(1,0,0) = 6$$

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$$

Today Find v_1 unit length eigenvector $\lambda = 6$.

$$A - 6I = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Check $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$

Notice

$$x^TAx = x^T 6x = 6x^Tx = 6\|x\|^2$$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boxed{Q(\vec{x}) = 18} \quad \checkmark \qquad v = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\left\|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\|^2 = (1^2 + 1^2 + 1^2) = 3$$

EXAMPLE 5 Let A be the matrix in Example 3 and let $\mathbf{u}_1$ be a unit eigenvector corresponding to the greatest eigenvalue of A. Find the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the conditions

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T\mathbf{u}_1 = 0 \tag{4}$$

u_1

$$Q_A\left(\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}\right) = 6$$

$\|x\| = 1$ & $x \cdot u_1 = 0$

$$Q(\vec{x}) = 3 = \lambda_2$$

$\vec{u}_2 = \begin{bmatrix} \vdots \end{bmatrix}$ unit length e-vector in $\lambda = 3$ eigenspace.

Q_1: Find max $\vec{x}$ s.t.

$$Q_A(x_1,x_2,x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$$

EXAMPLE 5 Let A be the matrix in Example 3 and let $\mathbf{u}_1$ be a unit eigenvector corresponding to the greatest eigenvalue of A. Find the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the conditions

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T\mathbf{u}_1 = 0 \qquad (4)$$

$Q_A\left(\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right) = 6$ $\leftarrow u_1$

$\|x\|=1$ & $x \cdot u_1 = 0$

$Q(\vec{x}) = 3 = \lambda_2$

$\vec{u}_2 = \begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix}$ unit length e-vector in $\lambda=3$ eigenspace.

Q_1: Find max $\vec{x}$ s.t.

$Q_A(x_1,x_2,x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

Lets find $\vec{u}_2$, $\lambda=3$ eigenspace

$$A - 3I = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = r\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

$(r=2)$ $\quad \vec{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$

$u_2 = \frac{1}{\|\vec{x}\|}\vec{x}$

$= \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ ✓

$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{x} \cdot \vec{u}_1 = 0$ ✓

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

subject to

$$||\vec{x}|| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$
$$M = \max\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $||\vec{x}|| = 1$,

- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm\vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm\vec{u}_n$.

Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

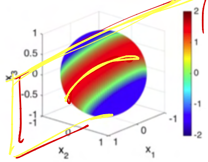
Q_0: Find A symmetric s.t. $Q(\vec{x}) = x^T A x$.

Q_1: Find $\lambda_1 \geq \lambda_2 \geq \lambda_3$

Q_2: Find u_1 unit length e-vector for λ_1.

Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.



$x_2 - x_3 = 0$

or $x_2 = x_3$

$\lambda = 1$ eigenspace

$x_2 = x_3$

$A\begin{bmatrix} a \\ b \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ b \end{bmatrix}$.

Soln.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\lambda_1 = 1 \quad v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = -1 \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$u_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ ✓

$u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ✓

} any lin comb of unit length.

$u_3 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Check $u_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$Q(u_1) = (0)^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{2} = 1 \quad ✓$$

Q_4: Find MAX of $Q(\vec{x})$ subject to $||x|| = 1$ & $x \cdot u_1 = 0$

ANS MAX still $\lambda_2 = 1$

but $u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \ge \lambda_2 \ldots \ge \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_2$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$ and to $\vec{x} \cdot \vec{u}_1 = 0$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Example 4 (if time permits)

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

$$Q(cx_1, cx_2)$$
$$= (cx_1^2) + 2(cx_2)(cx_3)$$
$$= c^2x_1^2 + c^2 2x_2x_3$$
$$= c^2(x_1^2 + x_2x_3)$$
$$= c^2 Q(x_1, x_2).$$

Find r s.t.

$$Q\left(\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right) = 1$$

$$\vec{x} = r\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad ||\vec{x}|| = 5$$

$(r=5)$

Try $\vec{x} = \begin{pmatrix} 0 \\ 5/\sqrt{2} \\ 5/\sqrt{2} \end{pmatrix}$

$$||x|| = \sqrt{\frac{25}{2} + \frac{25}{2}} = \sqrt{25} = 5 \checkmark$$

$$Q\left(\begin{bmatrix} 0 \\ 5/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}\right) \overset{?}{=} 0^2 + 2\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = 2 \cdot \frac{25}{2} = \boxed{25}$$

? not 5.

$$Q(c\vec{x}) = c^2 Q(\vec{x}) \; ?$$

$$Q(cx) = (cx)^T A cx$$
$$= c^2 x^T A x$$
$$= c^2 Q(\vec{x}).$$

NOTE

$$Q(-\vec{x}) = Q(\vec{x})$$

EVEN

7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2+6x_2^2+7x_3^2+4x_1x_2-4x_2x_3=9y_1^2+6y_2^2+3y_3^2$

2. $3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3=7y_1^2+4y_2^2$

Hint: $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for y_3^2.

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$ (See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3$ (See Exercise 2.)

5. $Q(\mathbf{x}) = x_1^2 + x_2^2 - 10x_1x_2$

6. $Q(\mathbf{x}) = 3x_1^2 + 9x_2^2 + 8x_1x_2$

7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are $2, -1$, and -4.]

8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 9 and -3.]

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^T A\mathbf{x}$?

12. Let λ be any eigenvalue of a symmetric matrix A. Justify the statement made in this section that $m \le \lambda \le M$, where m and M are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^T A\mathbf{x}$.]

13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T\mathbf{x} = 1$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

[M] In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$

15. $4x_1^2-6x_1x_2-10x_1x_3-10x_1x_4-6x_2x_3-6x_2x_4-2x_3x_4$

16. $-6x_1^2-10x_2^2-13x_3^2-13x_4^2-4x_1x_2-4x_1x_3-4x_1x_4+6x_3x_4$

17. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$