Handwritten Homework Assignments

For each assignment, complete the questions on a separate sheet of paper and put your name on it. Write neatly and use complete sentences where necessary. You must submit original work, but I'm okay with you all working together to share ideas. Handwritten homework assignments are due on Fridays. Please turn in handwritten homework in Gradescope (access the first time through Canvas) every Friday by midnight.

For 1/10 [NOT GRADED - PRACTICE ONLY!] In $\mathbb{R}^3$ (so using three coordinate axes) sketch the following, making a new sketch for each part (i)-(v):

(i) the plane $z = 0$,

(ii) the plane $z = 2$,

(iii) the plane $y = -3$,

(iv) the plane $x + y + z = 0$,

(v) the intersection of the planes $z = 2$ and $x = 0$.

In each case, you are practicing drawing an accurate, representative graph of the plane of points which satisfy the given equation in the variables x, y, and z.

For 1/17: (1) Choose two vectors v_1, v_2 in $\mathbb{R}^2$ and another vector b also in $\mathbb{R}^2$. Find scalars x, y in $\mathbb{R}$ such that $xv_1 + yv_2 = b$ (if this is not possible, pick other v_1, v_2, b vectors). Illustrate the vector equation you just solved by graphing the vectors v_1, v_2, b in the $x-y$-plane, and be sure to illustrate how b is obtained by adding a scalar of one vector to the other. (2) Repeat part 1 with vectors v_1, v_2, b in $\mathbb{R}^3$ that give a consistent system. (3) Why is part 2 more difficult than part 1? Explain clearly using complete sentences.

For 1/24: For each part, if possible, give an example of two sets A and B of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ where the set A is **linearly independent** and the set B is **linearly dependent**, and if it is not possible to do so for either A or B explain why in your own words.

1. One vector in $\mathbb{R}^2$,

2. two vectors in $\mathbb{R}^2$,

3. three vectors in $\mathbb{R}^2$,

4. two vectors in $\mathbb{R}^3$,

5. three vectors in $\mathbb{R}^3$,

6. four vectors in $\mathbb{R}^3$.

For 1/31: For each matrix A below, (0) state the domain and codomain of T_A, (1) find $T_A(e_1), T_A(e_2)$, (2) find $T_A(v), T_A(w)$, (3) describe in a few words what the transformation is doing, and (4) give the matrix an appropriate "name". For the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

1. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

Now, for the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

For 2/7: For full credit, correctly indicate which problem you are solving by writing the statement you are answering (like "$AB = 0$ and $A \neq 0, B \neq 0$"). For grading purposes, please try to write the problems in the same order as listed here. The matrix 0 is the zero matrix and the matrix I is the identity matrix.

For each problem find matrices which satisfy the given conditions. You don't have to justify *how* you found the matrices for each problem, but you *must verify the equality with calculations* in each case.

(a) $AB = BA$ but neither A nor B is 0 nor I, and $A \neq B$.

(b) $AB \neq BA$.

(c) $AB = AC$ but $B \neq C$.

(d) $AB = 0$ but neither A nor B is 0.

(e) $AB = I$ but neither A nor B is I.

For 2/14: **Step 1: Pick a matrix and find nul(A).** Pick a matrix A of size no smaller than 3×5 (to get a good feel for the problem). Choose entries not all positive, and not too many zeros, and your matrix shouldn't be rref (ideally, but it's ok to pick a matrix in rref if you want). Find the null space nul(A) by finding the parametric vector form of the general solution x to $A\mathrm{x} = 0$, and use these vectors to express nul(A) as their span. Call the vectors $v_1, \ldots, v_n$.

Step 2: An example that nul(A) is closed under vector addition. Choose two vectors $\mathrm{w}_1, \mathrm{w}_2$ in the span $\mathrm{nul}(A) = \mathrm{span}\{v_1, \ldots, v_n\}$ from step 1. Do this by taking two or more vectors in the basis from step 1 and adding them to each other using some scalars, *i.e.* chose a random linear combination of the vectors from step 1. Do this twice, once to get w_1 and once to get w_2. Add these vectors together to get $z = \mathrm{w}_1 + \mathrm{w}_2$. Check that z is in the null space of A by verifying $Az = 0$.

Step 3: An example that nul(A) is closed under scalar multiplication. Chose a vector w in the null space of A. Choose a random scalar c. Check that $c\mathrm{w}$ is in the null space of A by verifying $Aw = 0$.

Step 4: The general case. Try to convince yourself that no matter how A is chosen, nul(A) is always closed under scalar multiplication and vector addition. The hint is that $A(x + y) = Ax + Ay$ and $A(cx) = c(Ax)$.

For 2/21: **Part i)** Write down a system of three equations in three variables, whose augmented matrix $[A|b]$ would then be 3×4. Find the determinant of the matrix A whose entries are the coefficients of the system. If the determinant is non-zero find the inverse of the matrix and calculate $A^{-1}b$. Describe how $A^{-1}b$ relates to the system $[A|b]$ in a few words. If your determinant was 0 start over with a more general system.

Part ii) For the matrix A you wrote down in *part (i)*, row reduce the matrix to rref. What is the rref of A? Following the row operations you made to reduce A to rref, state the determinant of each elementary matrix. That is, if I is the rref of A and $A \sim E_1A \sim E_2E_1A \sim \cdots \sim E_n \cdots E_2E_1A = I$ are the matrices you got when you row reduced A to I, then calculate or otherwise find the determinant $\det A$, $\det E_1$, $\det E_2$, $\ldots$, $\det E_n$, and $\det I$.

Part iii) Compute the determinant of each of the intermediate matrices E_1A, E_2E_1A, $\ldots$, $E_n \cdots E_2E_1A$. Compare your result to what you did in *part (ii)*. Try to state the relationship between A and the intermediate row equivalent matrices, in terms of the determinants of the E_i's. Formulate a relationship between det(A) and the product of the $\det(E_i)$'s.

For 2/28 First, find a 2×2 matrix A which has **no real eigenvalues** and show that your answer is correct by finding the characteristic polynomial and explaining why it has no real roots. Then, find an eigenvector/eigenvalue pair for each matrix below **without calculations** by thinking it out using the linear transformation's geometric interpretation. Write a few words (like 5) in each case explaining why your eigenvector/eigenvalue pair works.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A = id$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A =$projection onto x-axis.

(c) $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, $T_A =$rotation by θ.
(and state the values of θ for which A has eigenvectors)

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T_A =$reflect about the line "$y = x$".

(e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A =$stretch in x-direction.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $T_A =$shear.

For 3/6: **Part 1:** Show that there is no relationship between any kind of row operation and the eigenvalues of the matrices involved as follows. For each of the three types of row operation $cR_i + R_j \to R_j$, $cR_{\to}R_j$, and $R_i \leftrightarrow R_j$ which are *adding two rows*, *multiplying a row by a scalar*, and *switching two rows*: Find four matrices A, B, C and D such that $A \sim B$ and $C \sim D$, the matrix A is row equivalent to B and also C is row equivalent to D, and such that A and B have the exact same eigenvalues but C and D have different eigenvalues. Conclude that there is no relationship whatsoever between two matrices "being row equivalent" and "have the same eigenvalues".

Part 2: Let λ be a real eigenvalue of a matrix with real entries A. Show that the set $V_\lambda = \{x \ : \ Ax = \lambda x\}$ is a subspace of $\mathbb{R}^n$. If you reduce your solution to a question about null spaces, be sure to include prove that null spaces are subspaces (but that's fine if you want to do it that way so long as your argument is clear, and correct of course).

Part 3: Why is the set of eigenvectors NOT a subspace? Why does this not contradict what you were asked to do in Part 2?

For 3/27: (For 1pt) Write a paragraph which explains how eigenvectors/eigenvalues or some other topic from the course are used in a field which interests you. Be specific and put some thought into this. (For 2pt) If your paragraph is not essentially the thing on wikipedia about how bridges have something to do with eigenvalues, but actually give some details or original content then you get 2pts instead of 1pt. (For 3pt) If you also support your research with real math or alternately something creative. This has to include *some mathematical content* but can take *any form whatsoever*. Last year's submissions included a poem, several posters, a few slide-show presentations like using PowerPoint, etc., and a very few of them actually were pretty decent research project results that I was quite impressed by, but I remember the poem the best; it was funny and it used the right math ideas about linear algebra to *be* funny, which essentially forces that the person *understood* the concepts. It was brilliant.

Please understand the point of this exercise, should you choose to do it: Pick any scientific discipline. I mean ANY. If it is scientific it's ok: how to build a bridge, how do design a new chemical, how to solve some hard algorithmic problem using computers (like how many stars the Netflix algorithm should predict for your enjoyment of the 1977 original version of Disney's Pete's Dragon, for example). Take 5 steps into your chosen scientific field and you will bump into linear algebra. That's the exercise. 1pt is essentially "write down in your own words what wiki has to say about it", 2pts is essentially "do something a little better but without any real math content", and then 3pts is "a pretty good job explaining how math is used in a scientific field you are interested in", where I will collect and grade these myself so it is up to my subjective expert opinion if what you say is a good job with the math explaining.

For 4/3 *This exercise will ask you to explore the equality* $Null(A^T) = (Col(A))^\perp$. Pick a vector u in $\mathbb{R}^4$ and a vector v such that u and v are orthogonal; then find a vector w which is orthogonal to both u and v. Finally, find a vector z which is orthogonal to u, v, w. For the last two steps, you should realize that you are solving a system of linear equations, and if you write them down that the system is represented by A^T where the columns of A are the vectors you are trying to be orthogonal to. Now, verify with computations that the set $\{u, v, w, z\}$ is linearly independent. If any of the vectors u, v, w, z are **scalars of the standard basis vectors** e_1, e_2, e_3, e_4 then start over. Set the matrix $P = [u\ v\ w\ z]$ and compute *without calculations* the vectors $P^{-1}u, P^{-1}v, P^{-1}w$, and $P^{-1}z$.

For 4/10 Pick any three vectors u, v, w in $\mathbb{R}^4$ which are *linearly independent* but **not orthogonal** and a vector b which is **not** in the span of u, v, w. If any of your vectors u, v, w are scalars of the standard basis vectors e_1, e_2, e_3, e_4 then start over. Let $W = \text{span}\{u, v, w\}$. Compute the orthogonal projection $\hat{b}$ of b onto the subspace W in two ways: (1) using the basis $\{u, v, w\}$ for W, and (2) using an orthogonal basis $\{u', v', w'\}$ obtained from $\{u, v, w\}$ via the Gram-Schmidt process. Finally, explain in a few words why the two answers differ, and explain why only ONE answer is correct.