Math 1554 Linear Algebra   Spring   2023

Midterm 3

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: Key ____________________ GTID Number: ____________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ____________________

Circle your instructor:

Prof Kim   Prof Barone   Prof Schroeder   Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

11:26

12:01

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You do not need to justify your reasoning for questions on this page.

1. (a) (6 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
●	○	If the projection of vector $\vec{y}$ onto subspace W is equal to the zero vector, then $\vec{y} \in W^{\perp}$.
●	○	If A and B are $n \times n$ orthogonal matrices, then AB is also $n \times n$ and orthogonal.
●	○	If $\{\vec{u}, \vec{v}\}$ is an orthonormal set in $\mathbb{R}^n$, then $\|\vec{u} + \vec{v}\| = \sqrt{2}$.
○	●	If A is row equivalent to a diagonalizable matrix B, then A is diagonalizable.
●	○	For any rectangular $m \times n$ matrix A, $(\text{Row}A)^{\perp} = (\text{Row}A^TA)^{\perp}$.
●	○	If A has the QR factorization $A = QR$, then $\text{Col}A = \text{Col}Q$.

$\hat{y} \in W$, $z \in W^{\perp}$

$y = \hat{y} + z$, & $\hat{y} = 0 \Rightarrow z = y$

$(AB)^T AB = B^T A^T A B = B^T B = I$ ✓

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 = 1 + 1 = 2$ ✓

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\text{Nul } A = \text{Nul } A^TA$ ✓

(b) (2 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	●	A 2×2 real matrix A with eigenvalues $1 + i$ and $-1 - i$.
○	●	A diagonalizable matrix A that is similar to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

needs $\lambda_2 = \overline{\lambda_1}$

not diag'ble

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You do not need to justify your reasoning for questions on this page.

(c) (2 points) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for subspace V. Fill in circles next to orthogonal bases for V; leave the other circles empty.

- ○ $\{\vec{v}_1, \vec{v}_2\}$ needs 3 vectors
- ● $\{\vec{v}_3, 4\vec{v}_1, 3\vec{v}_2\}$ scalar multiples
- ● $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2, \vec{v}_3\}$ $(v_1+v_2)\cdot(v_1-v_2) = v_1\cdot v_1 - v_2\cdot v_2 = 1-1=0$ ✓
- ○ $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ $(v_1+v_2)\cdot v_3 = v_1\cdot v_3 + v_2\cdot v_3 = 0+0=0$ ✓

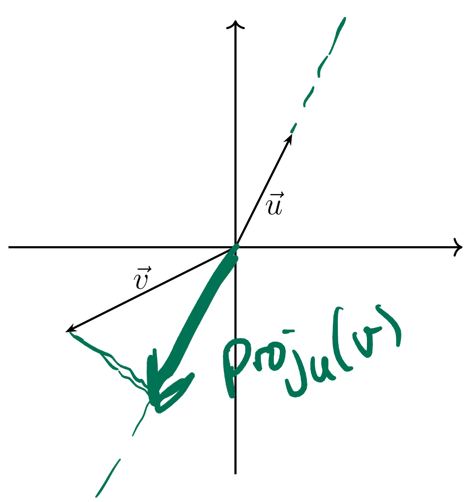
$(v_1-v_2)\cdot v_3 = v_1\cdot v_3 - v_2\cdot v_3 = 0-0=0$ ✓

$v_1\cdot(v_1+v_2)$
$= v_1\cdot v_1 + v_1\cdot v_2 \neq 0$

(d) (2 points) Let W be a 4-dimensional subspace of $\mathbb{R}^5$ and let A be the standard matrix for the orthogonal projection onto W. The following situations are either possible or impossible. Fill in the circles next to the **possible** situations; leave the other circles empty.

- ○ A is invertible. ✗ A not invertible $\overset{\text{IMT}}{\Longleftrightarrow}$ $\lambda=0$ is an eigenvalue
- ● A has eigenvalue zero. ✓
- ○ Null$A = W$. Null $A = W^\perp$ & W and $W^\perp$ only share zero vector
- ● $Av = v$, for some vector $v \in \mathbb{R}^5$. true for any $v \in W$

2. (1 point) In the graph below, sketch $\text{proj}_{\vec{u}}(\vec{v})$.



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You do not need to justify your reasoning for questions on this page.

3. (9 points) Fill in the blanks.

$(\mathrm{Col}\,A)^{\perp} = \mathrm{Nul}\,A^T$

(a) If A is 20×25 and $\dim(\mathrm{Col}(A)^{\perp}) = 10$, the rank of A is $\boxed{10}$.

25×20 10 free $\Rightarrow$ 10 pivots

(b) If A is an $n \times n$ orthogonal matrix, A^TA is equal to $\boxed{I_n}$.

(c) The distance between vector $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and subspace $W = \mathrm{Span}(\vec{v})$,

where $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is equal to $\boxed{3}$.

(d) If u and v are orthogonal vectors in $\mathbb{R}^n$ and the columns of $A \in \mathbb{R}^{n \times n}$ are orthonormal, then $(Au) \cdot (Av)$ is equal to $\boxed{0}$ (a number).

$Au \cdot Av \overset{?}{=} u \cdot v$ if $A^TA = I$.

$= (Au)^T Av = u^T A^T A v =$

(e) If $\vec{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $W = \mathrm{Span}(\vec{u})$, then $\mathrm{proj}_{\vec{u}}\vec{b}$ is the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\frac{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(f) If $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $W = \mathrm{Span}(\vec{v})$, a basis for $W^{\perp}$ is the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(g) If A is 32×9 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{32}$, then the number of pivot columns in A is $\boxed{9}$.

Cols of A are linearly independent

(h) If A is 30×8 and $\dim(\mathrm{Row}(A)^{\perp}) = 1$, then rank A is equal to $\boxed{7}$.

$\dim \mathrm{Nul}\,A = 1$

(i) If $W = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, then $\dim W^{\perp} = \boxed{2}$.

You do not need to justify your reasoning for questions on this page.

4. (4 points) If possible, give examples of the following. If it is not possible, write `NP`.

   (a) A 4×5 non-zero matrix, A, in RREF, that satisfies $\dim\left((\text{Row}(A))^{\perp}\right) = 3$.

   $$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   $\dim \text{Nul} A = 3$

   (b) A 2×3 matrix such that $\text{Col}(A)^{\perp}$ is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   $$A = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix}$$

   (c) A non-zero vector, $\vec{w}$, whose projection onto the space spanned by $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the zero vector.

   $$\vec{w} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

   (d) A 3×2 matrix A that is in echelon form, has QR factorization $A = QR$, and $A = Q$.

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

5. (2 points) W is the subspace spanned by $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$. Give a basis for $W^{\perp}$.

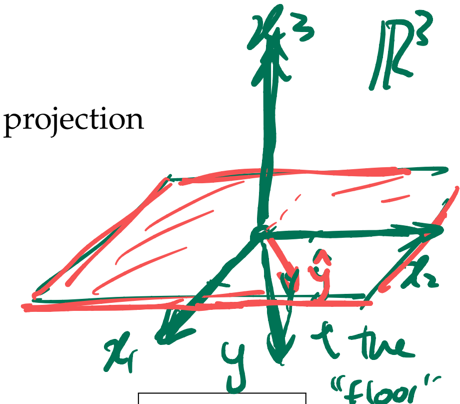
$$\text{Nul}\begin{bmatrix} 1 & \overset{s}{0} & \overset{t}{-4} \end{bmatrix}$$

$$x = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

alt → Method #2

$S = \text{span}\{u_1, u_2\}$ is "the floor"



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6. (2 points) Given that $\vec{u}_1$ and $\vec{u}_2$ form an orthogonal set, compute the projection of $\vec{y}$ onto $S = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

Method #1

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$$

$$\hat{y} = \text{proj}_S(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}}{\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$= \frac{30}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \frac{-15}{25} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 90+60 \\ 120-45 \\ 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 150 \\ 75 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$\text{proj}_S(\vec{y}) = \boxed{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$

7. (3 points) If $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and $\lambda_1 = 2 + i$, find λ_2, v_1, v_2.

$\lambda_1 = 2+i$

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} - (2+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{bmatrix}$$

$$= \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \underset{iR_1+R_2 \to}{\sim} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

s

$$X = s\begin{bmatrix} i \\ 1 \end{bmatrix} \qquad V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = \overline{\lambda_1} = 2 - i \qquad V_2 = \overline{v_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\lambda_2 = \boxed{2-i}$

$\vec{v}_1 = \boxed{\begin{bmatrix} i \\ 1 \end{bmatrix}}$

$\vec{v}_2 = \boxed{\begin{bmatrix} -i \\ 1 \end{bmatrix}}$

8. (4 points) **Show all work for problems on this page.** If $A = QR = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, determine the least-squares solution to $A\hat{x} = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$. You do not need to determine A.

$\hat{x} = \boxed{\begin{bmatrix} -7/2 \\ 3 \end{bmatrix}}$

idea

$A^T A \hat{x} = A^T b$

$\Rightarrow (QR)^T QR\hat{x} = (QR)^T b$

$\Rightarrow R^T \underbrace{\cancel{Q^T} Q}_{I} R\hat{x} = R^T Q^T b$

$\Rightarrow R^T R\hat{x} = R^T Q^T b$

R^T invertible

$\Rightarrow R\hat{x} = Q^T b$

Solve $R\hat{x} = Q^T b$

$R = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$

$Q^T b = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1-2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$[R \mid Q^T b] = \left[\begin{array}{cc|c} 2 & 2 & -1 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1/2 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -7/2 \\ 0 & 1 & 3 \end{array}\right]$

$\hat{x} = \begin{bmatrix} -7/2 \\ 3 \end{bmatrix}$

9. (5 points) **Show all work for problems on this page.** Four points in $\mathbb{R}^3$ with coordinates (x, y, z) are $(2, 0, 1)$, $(0, -2, 1)$, $(1, -1, 3)$, and $(1, 1, 3)$. Determine the coefficients c_1 and c_2 for the plane $z = c_1 x + c_2 y$ that best fits the points using the method of least-squares.

$c_1 = \boxed{4/3}$ $\qquad c_2 = \boxed{-1/3}$

Least squares problem

$c_1 x + c_2 y = z$ w/ data $\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}$

plug in data into model

$$\begin{cases} c_1 \cdot 2 + c_2 \cdot 0 = 1 \\ c_1 \cdot 0 + c_2 \cdot (-2) = 1 \\ c_1 \cdot 1 + c_2 (-1) = 3 \\ c_1 \cdot 1 + c_2 \cdot 1 = 3 \end{cases}$$

$\longrightarrow$ system of equations $Ax = b$ w/

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$$

Set up normal equations

$$A^T A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

Solve normal equations $\qquad 8/6 = 4/3$

$$\left[\begin{array}{cc|c} 6 & 0 & 8 \\ 0 & 6 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \end{array}\right]$$

Soln

$$\begin{cases} c_1 = 4/3 \\ c_2 = -1/3 \end{cases}$$

10. (4 points) **Show all work for problems on this page.** Write $\vec{y}$ as the sum of a vector $\hat{y}$ in Span$\{\vec{u}\}$ and a vector $\vec{z}$ orthogonal to $\vec{u}$.

$$y = \underbrace{\hat{y}}_{\text{in } W} + \underbrace{z}_{\text{in } W^\perp}$$

$$\vec{y} = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 1 \end{pmatrix}, \qquad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \hat{y} = \boxed{\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}} \qquad \vec{z} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}}$$

$$\hat{y} = \text{proj}_W(y) = \frac{y \cdot u}{u \cdot u} u = \frac{\begin{bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{12}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

11. (4 points) Find matrices P and D such that $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 1, 2$$

$\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad x = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \boxed{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}$$

$$D = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}$$

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