



LINEAR

ALGEBRA

Week 1

Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Itempool



Section 1.1 Systems of Linear Equations

Topics

We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations
Math 1554 Linear Algebra

Section 1.1 Systems of Linear Equations

Topics

- We will cover these topics in this section.
 - Systems of Linear Equations
 - Matrix Notation
 - Elementary Row Operations
 - Questions of Existence and Uniqueness of Solutions

Objectives

- For the topics covered in this section, students are expected to be able to do the following:
- Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
 - Apply elementary row operations to solve linear systems of equations.
 - Express a set of linear equations as an augmented matrix.

Course Schedule

Calculations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 - 9/15	2.1	WS1.9,2.1	Exam 1 Review	Cancelled	2.2
5	9/18 - 9/22	2.2,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 - 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 - 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 - 10/13	Break	WS4.1	Exam 2 Review	Cancelled	5.3
9	10/16 - 10/20	5.3	WS5.3	5.5	WS5.5	6.1
10	10/23 - 10/27	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	10/30 - 11/3	6.4	WS6.3,6.4	6.4,6.5	WS6.4,6.5	6.5
12	11/6 - 11/10	6.6	WS6.5,6.6	Exam 3 Review	Cancelled	Paperback
13	11/13 - 11/17	7.1	WS7.1	7.2	WS7.2	7.3
14	11/20 - 11/24	7.3,7.4	WS7.2,7.3	Break	Break	Break
15	11/27 - 12/1	7.4	WS7.3,7.4	7.4	WS7.4	7.4
16	12/4 - 12/8	Last Lecture	Last Studio	Reading Period		
17	12/11 - 12/15	Final Exam: MATH 1554 Common Final Exam Tuesday, December 12th at 6pm				

A Single Linear Equation

A linear equation has the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where a_1, \dots, a_n , and b are the coefficients. x_1, \dots, x_n are the variables or unknowns, and n is the dimension, or number of variables.

For example,

- $x_1 + 2x_2 = 4$ is a line in two dimensions
- $2x_1 + 2x_2 + x_3 = 6$ is a plane in three dimensions

$x_1=1$
 $x_2=1$
 $x_3=1$
is a solution.
 $3(1) + 2(1) + (1) = 6$ ✓

Systems of Linear Equations

When we have more than one linear equation, we have a linear system of equations. For example, a linear system with two equations is

$$\begin{cases} x_1 + 1.5x_2 + 2x_3 = 4 \\ 5x_1 + 0x_2 + 7x_3 = 5 \end{cases}$$

$x_1=1$
 $x_2=2$
 $x_3=0$
only solution?

Definition: Solution to a Linear System

The set of all possible values of x_1, x_2, \dots, x_n that satisfy all equations in the solution to the system.

A system can have a unique solution, no solution, or an infinite number of solutions.



The Book!

Two Variables

Consider the following systems. How are they different from each other?

$$\begin{cases} 3 - 2(2) = -1 \\ x_1 - 2x_2 = -1 \\ x_1 + 3x_2 = 3 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_1 + 3x_2 = 3 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$

$$\begin{cases} -3 = 3(2) = 3 \end{cases}$$

non-parallel lines
parallel lines
identical lines

① Unique soln
② NO solution
③ NO-many solutions

at least one solution
Consistent system has at least one solution

Inconsistent system, it has NO soln.

Three-Dimensional Case

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in \mathbb{R}^3 . The solution to a system of three equations is the set of intersections of the planes.

solution set sketch number of solutions

line ∞ -many solns.

point Unique soln.

empty no solution.

Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?

We can manipulate equations in a linear system using **row operations**.

- (Replacement/Addition) Add a multiple of one row to another.
- (Interchange) Interchange two rows.
- (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$$

mult by $\times -5$
 \leftarrow add to Row 3

Example 1

Identify the solution to the linear system.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 0x_1 + 2x_2 - 8x_3 = 8 \\ 5x_1 + 0x_2 - 5x_3 = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array} \right]$$

Coefficient Matrix + augmented column
 augmented matrix.

Itempool



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 10x_2 - 10x_3 = 10 \end{cases}$$

mult by $\times -5$
 \leftarrow add to Row 3

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array} \right]$$

$$\sim -5R_2 + R_3 \rightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 30x_3 = -30 \end{cases}$$

$$\sim -5R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{array} \right]$$

$$\begin{cases} x_1 - 7x_3 = 8 \\ 2x_2 - 8x_3 = 8 \\ x_3 = -1 \end{cases}$$

$R_2 + R_1 \rightarrow$
 $\frac{1}{30} R_3 \rightarrow$

$$\begin{cases} R_2 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array} \right] \\ \frac{1}{30} R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array} \right] \end{cases}$$

Row echelon form!

$$\sim \frac{1}{2} R_2 \rightarrow \begin{cases} x_1 - 7x_3 = 8 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{cases}$$

$$\sim \frac{1}{2} R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{cases} 7R_3 + R_1 \rightarrow x_1 = 1 \\ 4R_3 + R_2 \rightarrow x_2 = 0 \\ x_3 = -1 \end{cases}$$

$$\begin{cases} 7R_3 + R_1 \\ 4R_3 + R_2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

simplest possible.

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 0x_1 + 2x_2 - 8x_3 = 8 \\ 5x_1 + 0x_2 - 5x_3 = 10 \end{cases}$$

can be written as the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array} \right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables $x_1, x_2,$ and x_3 .

augmented matrix

coefficient matrix

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one **solution**.
No solution means INCONSISTENT

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of **row operations** transforms one matrix into the other.

Note: If the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

Itempool



The "augmentation bar" does NOT affect any properties of the matrix

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{rcl} x_1 & -2x_2 & +x_3 = 0 \\ & 2x_2 & -8x_3 = 8 \\ 5x_1 & & -5x_3 = 10 \end{array}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array} \right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables $x_1, x_2,$ and x_3 .

Section 1.1 Slide 9

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one _____.

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of _____ transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

Section 1.1 Slide 10

Fundamental Questions

Two questions that we will revisit many times throughout our course.

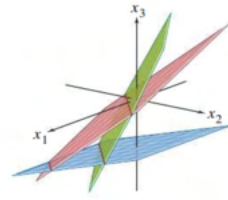
1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

Section 1.1 Slide 11

EXAMPLE 3 Determine if the following system is consistent:

$$\begin{aligned}x_2 - 4x_3 &= 8 \\2x_1 - 3x_2 + 2x_3 &= 1 \\4x_1 - 8x_2 + 12x_3 &= 1\end{aligned}$$

(5)



The system is inconsistent because there is no point that lies on all three planes.

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array} \right]$$

$$\sim \begin{array}{l} -2R_1 + R_2 \rightarrow \\ \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{array} \right] \sim \end{array} \begin{array}{l} 2R_2 + R_3 \rightarrow \\ \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right] \end{array}$$

$$\begin{aligned}2x_1 - 3x_2 + 2x_3 &= 1 \\x_2 - 4x_3 &= 8\end{aligned}$$

$$0x_1 + 0x_2 + 0x_3 = 15 \quad ??$$

↑ Since third equation can not be satisfied for any choice of x_1, x_2, x_3 .

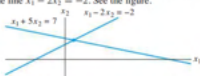
The system is inconsistent
(no soln)

1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $x_1 + 5x_2 = 7$ 2. $2x_1 + 4x_2 = -4$
 $-2x_1 - 7x_2 = -5$ $5x_1 + 7x_2 = 11$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and on the line $x_1 - 2x_2 = -2$. See the figure.



4. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $\begin{bmatrix} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{bmatrix}$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

Solve the systems in Exercises 11–14.

11. $x_1 + 4x_2 = -5$
 $x_1 + 3x_2 + 5x_3 = -2$
 $3x_1 + 7x_2 + 7x_3 = 6$

23. a. Every elementary row operation is reversible.
 b. A 5×6 matrix has six rows.
 c. The solution set of a linear system involving variables x_1, \dots, x_n is a list of numbers (s_1, \dots, s_n) that makes each equation in the system a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively.
 d. Two fundamental questions about a linear system involve existence and uniqueness.
24. a. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.
 b. Two matrices are row equivalent if they have the same number of rows.
 c. An inconsistent system has more than one solution.
 d. Two linear systems are equivalent if they have the same solution set.
25. Find an equation involving g , h , and k that makes this augmented matrix correspond to a consistent system:

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array} \right]$$

26. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2, x_2 = 1, x_3 = 0$.

12. $x_1 - 3x_2 + 4x_3 = -4$
 $3x_1 - 7x_2 + 7x_3 = -8$
 $-4x_1 + 6x_2 - x_3 = 7$

13. $x_1 - 3x_2 = 8$
 $2x_1 + 2x_2 + 9x_3 = 7$
 $x_2 + 5x_3 = -2$

14. $x_1 - 3x_2 = 5$
 $-x_1 + x_2 + 5x_3 = 2$
 $x_2 + x_3 = 0$

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. $x_1 + 3x_2 = 2$
 $x_2 - 3x_3 = 3$
 $-2x_2 + 3x_3 + 2x_4 = 1$
 $3x_1 + 7x_4 = -5$

16. $x_1 - 2x_2 = -3$
 $2x_2 + 2x_3 = 0$
 $x_3 + 3x_4 = 1$
 $-2x_1 + 3x_2 + 2x_3 + x_4 = 5$

17. Do the three lines $x_2 - 4x_3 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

18. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$

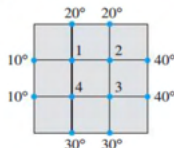
21. $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$

22. $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, \dots, T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.² For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



33. Write a system of four equations whose solution gives estimates for the temperatures T_1, \dots, T_4 .
34. Solve the system of equations from Exercise 33. [Hint: To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]

² See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics

We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Section 1.2 : Row Reductions and Echelon Forms

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations
Math 1354 Linear Algebra

Topics

We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives

For the topics covered in this section, students are expected to be able to do the following:

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Week Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
- 1 8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
- 2 8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
- 3 9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
- 4 9/11 - 9/15	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2

Section 1.2 Slide 12

Section 1.2 Slide 12

Definition: Echelon Form

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. Below a leading entry (if any), all entries are zero.

A matrix in echelon form is in **row reduced echelon form (RREF)** if

1. The leading entry in each row is equal to 1.
2. Each leading 1 is the only nonzero entry in that column.

Example of a Matrix in Echelon Form

■ = non-zero number, * = any number

$$\begin{array}{cccccccc}
 0 & \blacksquare & * & 0 & * & * & * & * \\
 0 & 0 & * & 0 & * & * & * & * \\
 0 & 0 & 0 & * & 0 & 0 & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$

Staircase

Definitions are KEY!

Section 1.2 Slide 14

Section 1.2 Slide 15

REF / RREF / Neither

Example 1

Which of the following are in RREF?

- a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ REF
- b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ RREF
- c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ Neither
- d) $\begin{bmatrix} 0 & 6 & 3 & 0 \end{bmatrix}$ REF
- e) $\begin{bmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ RREF?

Vacuously true for (2)-(5)

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A .

A **pivot column** is a column of A that contains a pivot position.

Example 2. Express the matrix in row reduced echelon form and identify the pivot columns.

$$\begin{bmatrix} 1 & -3 & -6 & 4 \\ -1 & -3 & -1 & 3 \\ -2 & -3 & 0 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ 0 & 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & -3 & -6 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

Definition: Echelon Form

A rectangular matrix is in **echelon form** if

- All zero rows (if any are present) are at the bottom.
- The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
- Below a leading entry (if any), all entries are zero.

A matrix in echelon form is in **row reduced echelon form (RREF)** if

- The leading entry in each row is equal to 1.
- Each leading 1 is the only nonzero entry in that column.

Example of a Matrix in Echelon Form

■ = non-zero number, * = any number



Itempool



Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

- Step 1a: Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row
 - Step 1b: Scale the 1st row so that its leading entry is equal to 1
 - Step 1c: Use row replacement so all entries above and below this 1 are 0
 - Step 2a: Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in the 2nd row; uncover 1st row
- etc.

Basic And Free Variables

Consider the augmented matrix



$$\begin{aligned} x_1 + 3x_2 + 7x_4 &= 4 \\ x_3 + 4x_4 &= 5 \\ x_5 &= 6 \end{aligned}$$

The leading one's are in first, third, and fifth columns. So:

- Its pivot variables are $x_1, x_3,$ and x_5 .
- The free variables are x_2 and x_4 . **Any choice** of the free variables leads to a solution of the system.

parametric equation form

e.g. Free choices

$$\begin{aligned} x_1 &= 4 - 3x_2 - 7x_4 \\ x_2 &= 1 \\ x_3 &= 5 - 4x_4 \\ x_4 &= 2 \\ x_5 &= 6 \end{aligned}$$

$$\begin{aligned} x_1 &= 4 - 3x_2 - 7x_4 \\ x_2 &= x_2 \text{ (Free)} \\ x_3 &= 5 - 4x_4 \\ x_4 &= x_4 \text{ (Free)} \\ x_5 &= 6 \end{aligned}$$

$$\begin{aligned} x_1 &= 4 - 3r - 7t \\ x_2 &= r \text{ (Free)} \\ x_3 &= 5 - 4t \\ x_4 &= t \text{ (Free)} \\ x_5 &= 6 \end{aligned}$$

one particular solution

$$\begin{aligned} x_1 &= -13 \\ x_2 &= 1 \\ x_3 &= -3 \\ x_4 &= 2 \\ x_5 &= 6 \end{aligned}$$

Existence and Uniqueness

Theorem

A linear system is consistent if and only if (exactly when) the last column of the augmented matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$[0 \ 0 \ 0 \ \dots \ 0 \ 1]$$

in augmented column.

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no *Free vars.*
2. *∞-many* many solutions that are parameterized by free variables.

META

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

22 CHAPTER 1 Linear Equations in Linear Algebra

2. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols \blacksquare , $*$, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3×2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix}$ 10. $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

15. a. $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$

16. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

In Exercises 17 and 18, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

17. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$ 18. $\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$

In Exercises 19 and 20, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19. $x_1 + hx_2 = 2$ 20. $x_1 + 3x_2 = 2$
 $4x_1 + 8x_2 = k$ $3x_1 + hx_2 = k$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.⁴

- a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.
 b. The row reduction algorithm applies only to augmented matrices for a linear system.
 c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.
 d. Finding a parametric description of the solution set of a linear system is the same as *solving* the system.
 e. If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 5 \ 0]$, then the associated linear system is inconsistent.
- a. The echelon form of a matrix is unique.
 b. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.
 c. Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.
 d. Whenever a system has free variables, the solution set contains many solutions.
 e. A general solution of a system is an explicit description of all solutions of the system.
- Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?
- Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in \mathbb{R}^n , and their basic properties
2. Linear combinations of vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in \mathbb{R}^n to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

1.3: Vector Equations

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations
Math 1554 Linear Algebra

Topics

We will cover these topics in this section.

1. Vectors in \mathbb{R}^n , and their basic properties
2. Linear combinations of vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following:

1. Apply geometric and algebraic properties of vectors in \mathbb{R}^n to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Week Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1 8/21 - 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2 8/28 - 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3 9/4 - 9/8	Break	WS1.7	1.8	WS1.8	1.9
4 9/11 - 9/15	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$$\begin{aligned}x - 3y &= -3 \\ 2x + y &= 8\end{aligned}$$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n -dimensional space \mathbb{R}^n , and **vectors** inside it.

Section 1.3 Slide 28

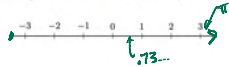
\mathbb{R}^n

Recall that \mathbb{R} denotes the collection of all real numbers.

Let n be a positive whole number. We define

\mathbb{R}^n = all ordered n -tuples of real numbers $(x_1, x_2, x_3, \dots, x_n)$.

When $n = 1$, we get \mathbb{R} back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.



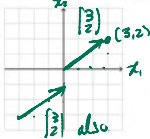
Section 1.3 Slide 28

\mathbb{R}^2

Note that:

- when $n = 2$, we can think of \mathbb{R}^2 as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x - and y -coordinates

Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



Vectors

In the previous slides, we were thinking of elements of \mathbb{R}^n as **points**: in the line, plane, space, etc.

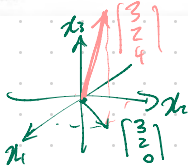
We can also think of them as **vectors**: arrows with a given length and direction.



For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ points **horizontally** in the amount of its x -coordinate, and **vertically** in the amount of its y -coordinate.

Section 1.3 Slide 28

Section 1.3 Slide 28



Vector Algebra



When we think of an element of \mathbb{R}^n as a vector, we write it as a matrix with n rows and one column:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad 4\vec{v} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Vectors have the following properties.

1. **Scalar Multiple:**

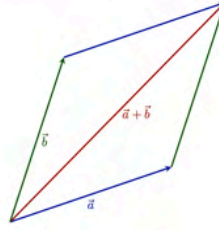
$$c\vec{u} =$$

2. **Vector Addition:**

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

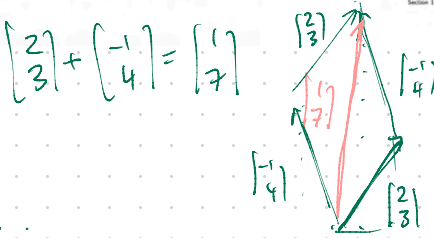
Note that vectors in higher dimensions have the same properties.

Parallelogram Rule for Vector Addition



Section 1.3 Slide 27

Section 1.3 Slide 28



$$4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} -1 & 0 \\ -2 & 2 \end{pmatrix} =$$

Linear Combinations and Span

Definition

- Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$, and scalars c_1, c_2, \dots, c_p , the vector below

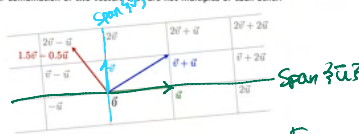
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

is called a **linear combination** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ with **weights** c_1, c_2, \dots, c_p .

- The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Geometric Interpretation of Linear Combinations

Note that any two vectors in \mathbb{R}^2 that are not scalar multiples of each other, span \mathbb{R}^2 . In other words, any vector in \mathbb{R}^2 can be represented as a linear combination of two vectors that are not multiples of each other.



Example:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Section 1.3 Slide 29

Section 1.3 Slide 30

$$v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

e.g. $\vec{v}_1 + \vec{v}_2, 3\vec{v}_1 - \vec{v}_2, c_1\vec{v}_1 + c_2\vec{v}_2$

some linear combinations of v_1 & v_2

$$\text{Span} \{v_1, v_2\} = \left\{ \vec{x} \mid \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2, c_1, c_2 \in \mathbb{R} \right\}$$

The span of v_1 and v_2

such that \vec{x} is a linear combination of v_1 and v_2

\vec{x} is a linear combination of v_1 and v_2

Example $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

List some vectors in $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ is in the span of } \vec{v}_1, \vec{v}_2 \quad \checkmark$$

$$0\vec{v}_1 + 0\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ "ditto" } \quad \checkmark$$

$$-\vec{v}_1 + 0\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -2 \end{bmatrix} \text{ "ditto" } \quad \checkmark$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is NOT in the span of \vec{v}_1, \vec{v}_2 ?

Are there numbers c_1, c_2 such that?

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \checkmark ??$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = 1 \\ 2c_1 + 4c_2 = 1 \end{cases}$$

System of linear eqns

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \end{array} \right] \quad \text{--- } -2R_1 + R_2 \quad \text{--- } \textcircled{-1}$$

Example

Is \vec{g} in the span of vectors \vec{v}_1 and \vec{v}_2 ?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{g} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -3 & 6 & 15 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & -18 \\ 0 & 12 & 36 \end{array} \right] \begin{array}{l} 2c_1 + 2c_2 \\ 3c_1 + 6c_2 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]$$

The Span of Two Vectors in \mathbb{R}^3

In the previous example, did we find that \vec{g} is in the span of \vec{v}_1 and \vec{v}_2 ?

In general: Any two non-parallel vectors in \mathbb{R}^3 span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.

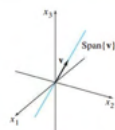
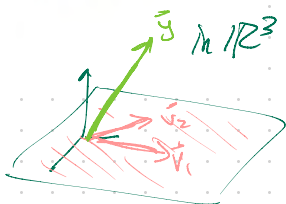


FIGURE 10 Span $\{v\}$ is a line through the origin.

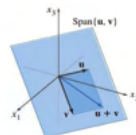


FIGURE 11 Span $\{u, v\}$ is a plane through the origin.

1.3 EXERCISES

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy -graph: \mathbf{u} , \mathbf{v} , $-\mathbf{v}$, $-\mathbf{u}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and $-\mathbf{v}$.

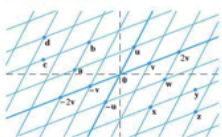
3. \mathbf{u} and \mathbf{v} as in Exercise 1 4. \mathbf{u} and \mathbf{v} as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$

6. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of \mathbf{u} and \mathbf{v} . Is every vector in \mathbb{R}^2 a linear combination of \mathbf{u} and \mathbf{v} ?



7. Vectors \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{d}

8. Vectors \mathbf{w} , \mathbf{x} , \mathbf{y} , and \mathbf{z}

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. $x_2 + 5x_3 = 0$ 10. $4x_1 + x_2 + 3x_3 = 9$

$4x_1 + 6x_2 - x_3 = 0$ $x_1 - 7x_2 - 2x_3 = 2$

$-x_1 + 3x_2 - 8x_3 = 0$ $8x_1 + 6x_2 - 5x_3 = 15$

In Exercises 11 and 12, determine if \mathbf{b} is a linear combination of \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 .

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

In Exercises 13 and 14, determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 15 and 16, list five vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on \mathbf{v}_1 and \mathbf{v}_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

16. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

17. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. For what

value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?

18. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what

value(s) of h is \mathbf{y} in the plane generated by \mathbf{v}_1 and \mathbf{v}_2 ?

19. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors

$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 16.

21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ for all h and k .

22. Construct a 3×3 matrix A , with nonzero entries, and a vector \mathbf{b} in \mathbb{R}^3 such that \mathbf{b} is *not* in the set spanned by the columns of A .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \ 3]$.
 b. The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.
 c. An example of a linear combination of vectors \mathbf{v}_1 and \mathbf{v}_2 is the vector $\frac{1}{2}\mathbf{v}_1$.
 d. The solution set of the linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.
 e. The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.
24. a. Any list of five real numbers is a vector in \mathbb{R}^5 .
 b. The vector \mathbf{u} results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector \mathbf{v} .
 c. The weights c_1, \dots, c_p in a linear combination $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ cannot all be zero.
 d. When \mathbf{u} and \mathbf{v} are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line through \mathbf{u} and the origin.
 e. Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ has a solution amounts to asking whether \mathbf{b} is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.