LINEAR ALGEBRA

week 4

Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, ...)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

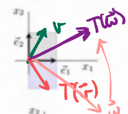
$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = [T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n)]$$

The matrix A is the **standard matrix** for a linear transformation.

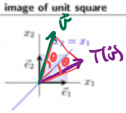
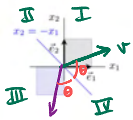
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection through x_2-axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

onto ? yes

1-1 ? yes.

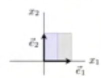
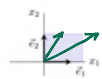
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through $x_2 = x_1$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

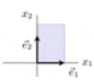
onto? yes

1-1 ? yes.

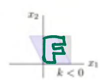
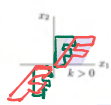
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $\lvert k\rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

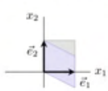
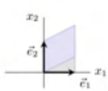
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\cancel{\lvert k\rvert < 1}$ $0<\lvert k\rvert<1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

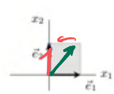
Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
Horizontal Shear(left)		$\boxed{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, k < 0}$
Horizontal Shear(right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$

Two Dimensional Examples: Projections

transformation	image of unit square	standard matrix
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

not onto!

not 1-1!

range is codomain.

Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why.**

a) A is a 2×3 standard matrix for a one-to-one linear transform.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

NP at most 2 pivots.

b) B is a 3×2 standard matrix for an onto linear transform.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$$

NP at most 2 pivots.

c) C is a 3×3 standard matrix of a linear transform that is one-to-one and onto.

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

need 3 pivots.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto.
2. The matrix A has columns which span $\mathbb{R}^m$. ← eqn.
3. The matrix A has m pivotal columns.

every row has a pivot.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^3$$

range of $T(\vec{x}) = C\vec{x}$

Example 2

Define a linear transformation by

$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Is this one-to-one? Is it onto?

yes

No.

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

$T(\vec{x}) = A\vec{x}$

$$A = \left[T(\vec{e}_1) \;\; T(\vec{e}_2) \right]$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & * \\ 0 & * \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & * \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Additional Example (if time permits)

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 8 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Is the transformation onto? Is it one-to-one?

1.9 EXERCISES

In Exercises 1–10, assume that T is a linear transformation. Find the standard matrix of T.

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (counterclockwise).

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $-\pi/4$ radians (clockwise). [*Hint:* $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$.]

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 2\mathbf{e}_2$ but leaves the vector $\mathbf{e}_2$ unchanged.

6. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a horizontal shear transformation that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 3\mathbf{e}_1$.

7. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal x_1-axis. [*Hint:* $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$.]

8. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

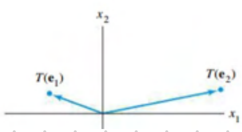
9. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

10. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2-axis and then rotates points $\pi/2$ radians.

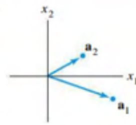
11. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?

12. Show that the transformation in Exercise 8 is merely a rotation about the origin. What is the angle of the rotation?

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2, 1)$.



14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = [\mathbf{a}_1 \;\; \mathbf{a}_2]$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are shown in the figure. Using the figure, draw the image of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation T.



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

15. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

16. $\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$

In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

17. $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

18. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

19. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

20. $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4 \qquad (T : \mathbb{R}^4 \to \mathbb{R})$

21. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (3, 8)$.

22. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (-1, 4, 9)$.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

    b. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle φ, then T is a linear transformation.

    c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

    d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

    e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

24. a. Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

    b. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

NOTES FROM $\boxed{\text{FALL 2023}}$

In-Class Midterm 1 Review, Math 1554

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	8/21 – 8/25	1.1	WS1.1	1.2	WS1.2	1.3
2	8/28 – 9/1	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	9/4 – 9/8	Break	WS1.7	1.8	WS1.8	1.9
4	9/11 – 9/15	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2

Exam 1 today
@ 6:30

1. Consider the matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

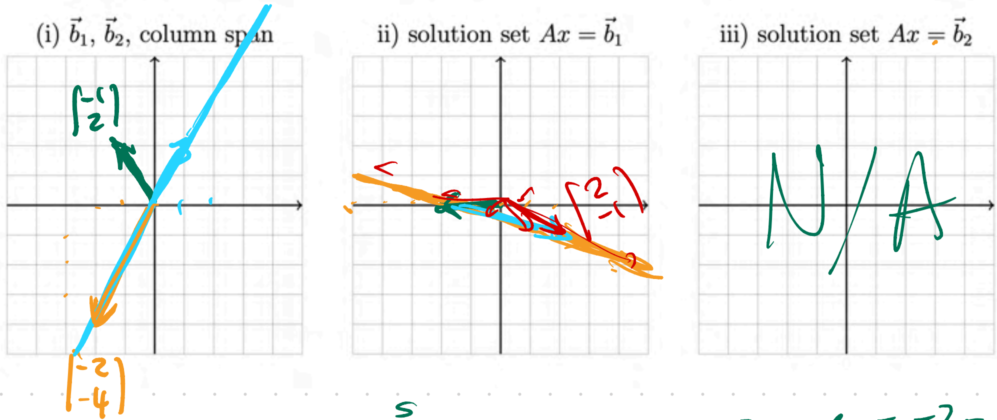
If possible, on the grids below, draw

(i) the two vectors and the span of the columns of A,

(ii) the solution set of $A\vec{x} = \vec{b}_1$.

(iii) the solution set of $A\vec{x} = \vec{b}_2$.

(i) $\vec{b}_1$, $\vec{b}_2$, column span | ii) solution set $Ax = \vec{b}_1$ | iii) solution set $Ax = \vec{b}_2$



$$\left[\begin{array}{cc|c} 1 & 4 & -2 \\ 2 & 8 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \overset{s}{4} & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 4s = -2 \\ x_2 = s \end{cases} \qquad \begin{cases} x_1 = -2 - 4s \\ x_2 = s \end{cases}$$

$$X = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ a soln to

$$Ax = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

soln. yes

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \checkmark$$

$\boxed{x_2 = -1}$ Chose.

$$X = \begin{bmatrix} -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

2. Indicate **true** if the statement is true, otherwise, indicate **false**. For the statements that are false, give a counterexample.

$M\{\underbrace{\begin{bmatrix} \vdots & \cdots & \\ \vdots & & \end{bmatrix}}^{N}$

	true	false	counterexample
a) If $A \in \mathbb{R}^{M\times N}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^M$.	○	●	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix}$
b) If there are some vectors $\vec{b} \in \mathbb{R}^M$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every row of A.	●	○	there is a b s.t. $Ax=b$ is inconsistent
c) If the transform $\vec{x} \to A\vec{x}$ projects points in $\mathbb{R}^2$ onto a line that passes through the origin, then the transform cannot be one-to-one.	●	○	$[A\|b] \sim \begin{bmatrix} \cdots & \vdots \\ 0\ 0 \cdots 0 & * \end{bmatrix}$

T is not onto.

Some row of A doesn't have a pivot.

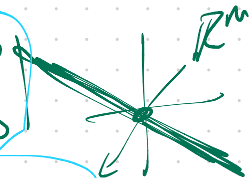
in $\mathbb{R}^m$

$\begin{bmatrix} * & * \\ \vdots & \vdots \\ * & * \end{bmatrix}$ $\begin{bmatrix} * \\ * \end{bmatrix} = c\vec{v}$

scalar mult of some v.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$c_1v_1 + c_2v_2$



3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*.

(a) A linear system that is homogeneous and has no solutions.

could $A\vec{x} = \vec{0}$ have no solutions? No.

$\vec{x} = \vec{0}$ always a soln to $A\vec{x} = \vec{0}$

(b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

pivot in every col.

RREF?

$4\{\overbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}^{3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and vector b.

(c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

4. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(a) Express the augmented matrix $(\,A \,|\, \vec{b}\,)$ in RREF.

(b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 0 & 7 & 0 & -5 & 1 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \sim \begin{array}{cccccc} & & & s & t & \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -5 & -13 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\begin{cases} x_1 - 5t = -13 \\ x_2 + 3t = -2 \\ x_3 = 2 \\ x_4 = s \text{ (free)} \\ x_5 = t \text{ (free)} \end{cases} \rightarrow \begin{cases} x_1 = -13 + 5t \\ x_2 = -2 - 3t \\ x_3 = 2 \\ x_4 = s \text{ (free)} \\ x_5 = t \text{ (free)} \end{cases}$$

$$X = \begin{bmatrix} -13 + 5t \\ -2 - 3t \\ 2 \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -13 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\text{Sols to } Ax=b} + \underbrace{s \overset{v_1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + t \overset{v_2}{\begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}}}_{\text{Solve to } Ax=0}$$

$$T : \mathbb{R}^5 \to \mathbb{R}^3 \qquad T(\vec{x}) = \vec{0}$$

$\mathbb{R}^3$

Q1: codomain, range image.
50%
Q2: onto vs. 1-1.
20%
(non-zero)
Q3: $AC = BC$ $A \neq B$ (poss/imp)
20%
Q4: Clockwise rotation by θ in $\mathbb{R}^2$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$T(\vec{x}) = Ax$ is rotate CCW by θ.

$$B = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$S(\vec{x}) = Bx$ rotate CW by θ.

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\cos\theta = \cos(-\theta)$

Q1: codomain, range image.

Q2: onto vs. 1-1.

20

Q3: $AC = BC \quad A \neq B$ (non-zero) (poss/imp)

20

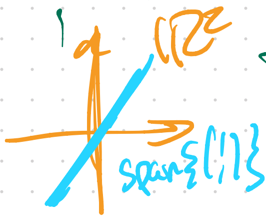
Q4: Clockwise rotation by θ in $\mathbb{R}^2$

$\boxed{T: \mathbb{R}^2 \to \mathbb{R}^2}$

$f(x) = y$ input image/output

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Q: Is b in the range of $T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x) = Ax$?



range is span$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$

$x = \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}5\\-1\end{bmatrix}, \begin{bmatrix}0\\3/2\end{bmatrix}$

$$A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$$

$$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{pmatrix}2\\2\end{pmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$$

$$AC = BC$$

but $A \neq B$.

different

$$\begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} \qquad = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$$

same

c. The standard matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where a and d are ± 1.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17

26. The transformation in Exercise 2

27. The transformation in Exercise 19

28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation T. Use the notation of Example 1 in Section 1.2.

29. $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

30. $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T is one-to-one if and only if A has ____ pivot columns." Explain why the statement is true. [*Hint:* Look in the exercises for Section 1.7.]

32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has ____ pivot columns." Find some theorems that explain why the statement is true.

33. Verify the uniqueness of A in Theorem 10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B. Show that if A is the standard matrix for T, then $A = B$. [*Hint:* Show that A and B have the same columns.]

34. Why is the question "Is the linear transformation T onto?" an existence question?

35. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$, can you give a relation between m and n? If T is one-to-one, what can you say about m and n?

36. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation (from $\mathbb{R}^p$ to $\mathbb{R}^m$). [*Hint:* Compute $T(S(c\mathbf{u} + d\mathbf{v}))$ for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^p$ and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.]

[**M**] In Exercises 37–40, let T be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if T is a one-to-one mapping. In Exercises 39 and 40, decide if T maps $\mathbb{R}^5$ onto $\mathbb{R}^5$. Justify your answers.

37. $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$

39. $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$

Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Itempool

Topics and Objectives

Topics

We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 - 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 - 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

Covered on Wednesday's exam @ 6:30 pm.

Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

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Definitions: Zero and Identity Matrices

$\begin{bmatrix}0\\0\\0\end{bmatrix} \in \mathbb{R}^3$

$\begin{bmatrix}0\\0\end{bmatrix} \in \mathbb{R}^2$

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times3} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad 0_{2\times1} = \begin{bmatrix}0\\0\end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

Sums and Scalar Multiples

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A + B$ are $a_{i,j} + b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

For example, if

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} + c\begin{bmatrix}7 & 4 & 7\\0 & 0 & k\end{bmatrix} = \begin{bmatrix}15 & 10 & 17\\4 & 5 & 16\end{bmatrix}$$

What are the values of c and k?

$1 + 7c = 15 \Rightarrow c = 2$

$6 + k\cdot(2) = 16 \Rightarrow k = 5$

$c=2$

Facts that you believed anyway.

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

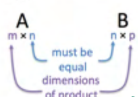
1. $A + 0_{m\times n} = A$
2. $(A + B) + C = A + (B + C)$
3. $r(A + B) = rA + rB$
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

Matrix Multiplication

Definition
Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A\left[\vec{b}_1 \ \cdots \ \vec{b}_p\right] = \left[A\vec{b}_1 \ \cdots \ A\vec{b}_p\right]$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



$(1)(0)+(2)(1)+(1)(1)$

A, B

$$\begin{bmatrix}1 & 2 & 1\\-1 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\2 & 1\\3 & 1\end{bmatrix} = \begin{bmatrix}8 & 3\\-1 & -1\end{bmatrix} \leftarrow A*B$$

$3\times2 \quad 2\times3 \quad 3\times3$

$$BA = \begin{bmatrix}1 & 0\\2 & 1\\3 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\-1 & -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\3 & 3 & 2\\4 & 5 & 3\end{bmatrix}$$

Properties of Matrix Multiplication

ABC

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B+C) = AB + AC$
3. (Right Distributive) $\cdots$
4. (Identity for matrix multiplication) $I_m A = A I_n$

FACTS

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

NON-FACTS.

The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

B goes first

$AB\vec{x}$

A gets applied to $B\vec{x}$.

g goes first

$f \circ g(x) = f(g(x))$

Schematically:

The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Not equal.

equals

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Suppose

$$AB = AC$$

Then $B \neq C$ possible!

Proof of the Associative Law

Let A be $m \times n$, $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix}$ a $n \times p$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ a $p \times 1$ matrix. Then,

$$BC = \underbrace{c_1\vec{b}_1 + \cdots + c_p\vec{b}_p}_{\text{lin combin of cols of } B}$$

So

$$\begin{aligned} A(BC) &= A(c_1\vec{b}_1 + \cdots + c_p\vec{b}_p) \\ &= c_1 A\vec{b}_1 + \cdots + c_p A\vec{b}_p && \text{(multiply by } A \text{ is linear)} \\ &= \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} && \text{(lin combin of cols of } AB\text{)} \\ &= (AB)C. \end{aligned}$$

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix B that is non-commutative with A.

$$B = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

$$AB \neq BA$$

Itempool

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\neq$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 0 & 0 \end{bmatrix}$$

$\neq$

$$\begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix}$$

The Transpose of a Matrix

If A is 2×5
Then A^T is 5×2

A^T is the matrix whose columns are the rows of A.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \\ 5 & 0 \end{bmatrix}$$

Properties of the Matrix Transpose

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r \cdot A^T$
4. $(AB)^T = B^T A^T$ ← size $n\times m$

A is $m\times p$, B is $p\times n$; B^T is $n\times p$, A^T is $p\times m$

$\boxed{AB \text{ is } m\times n}$

$(AB)^T$ size $n\times m$.

NOT DEFINED!

$$\boxed{\underset{p\times m}{A^T}\ \underset{p\times n}{B^T}}$$

Matrix Powers

For any $n\times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = AA\ldots A$$

Example: Compute C^8.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example

Define

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Which of these operations are defined, and what is the result?

1. AB
2. $3C$
3. $A + 3C$
4. $B^T A$
5. C^3
6. CB^T

Additional Example (if time permits)

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$.
2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$.

$(A+B)(A+B) \overset{?}{=} A^2 + 2AB + B^2$

$A^2 + \underbrace{BA + AB}_{\text{do not combine!!}} + B^2$

2.1 Exercises

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC, CD

2. $A + 2B$, $3C - E$, CB, EB

In the rest of this exercise set and in those to follow, you should assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.

4. Compute $A - 5I_3$ and $(5I_3)A$, when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product AB in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing AB.

5. $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$

7. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

8. How many rows does B have if BC is a 3×4 matrix?

9. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

11. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

12. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

Exercises 15–24 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False **(T/F)**. Justify each answer.

15. **(T/F)** If A and B are 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2]$.

16. **(T/F)** If A and B are 3×3 and $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.

17. **(T/F)** Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

18. **(T/F)** The second row of AB is the second row of A multiplied on the right by B.

19. **(T/F)** $AB + AC = A(B + C)$

20. **(T/F)** $A^T + B^T = (A + B)^T$

21. **(T/F)** $(AB)C = (AC)B$

22. **(T/F)** $(AB)^T = A^T B^T$

23. **(T/F)** The transpose of a product of matrices equals the product of their transposes in the same order.

24. **(T/F)** The transpose of a sum of matrices equals the sum of their transposes.

25. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

26. Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can you say about the columns of AB (if AB is defined)? Why?

27. Suppose the third column of B is the sum of the first two columns. What can you say about the third column of AB? Why?

28. Suppose the second column of B is all zeros. What can you say about the second column of AB?

29. Suppose the last column of AB is all zeros, but B itself has no column of zeros. What can you say about the columns of A?

30. Show that if the columns of B are linearly dependent, then so are the columns of AB.

31. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

32. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

33. Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint:* Think about the product CAD.]

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

Topics and Objectives

Topics
We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 - 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 - 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/5 - 2/9	2.3,2.4	WS2.2-2.4	2.5	WS2.5	2.8
6	2/12 - 2/16	2.9	WS2.8	2.9,3.1	WS2.9,3.1	3.2
7	2/19 - 2/23	3.3	WS3.2	4.9	WS3.3,4.9	5.1
8	2/26 - 3/1	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/4 - 3/8	5.3	WS5.3	5.5	WS5.5	6.1
10	3/11 - 3/15	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/18 - 3/22	Break	Break	Break	Break	Break
12	3/25 - 3/29	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	4/1 - 4/5	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/8 - 4/12	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/15 - 4/19	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/22 - 4/24	Last lecture	Last Studio	Reading Period		
17	4/25 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 30th at 6:00pm**				

The Matrix Inverse Book defn.

Definition

$A \in \mathbb{R}^{n \times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n \times n}$ so that

$$AC = CA = I_n.$$

If there is, we write $C = A^{-1}$.

$3 \cdot \frac{1}{3} = 1.$

$$\underset{n\times p}{A}\,\underset{p\times n}{C} = \underset{p\times n}{C}\,\underset{n\times p}{A} = \underset{n\times n}{I_n}$$

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

State the inverse of the matrix below.

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \frac{1}{2(-7)-(5)(-3)} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{\cancelto{1}{-14+15}} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \swarrow \text{ the inverse of } A.$$

Check

$$\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} -14+15 & -35+35 \\ 6-6 & 15-14 \end{bmatrix}$$

$$A^{-1} * A = I_2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Matrix Inverse

Why?

Theorem
$A \in \mathbb{R}^{n\times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Important: In applications, the entries of A are given in terms of units, say deflection per unit load. Then A^{-1} is given in terms of load per unit deflection. (Always keep units in mind in applications.)

Example
Solve the linear system.

$$3x_1 + 4x_2 = 7$$
$$5x_1 + 6x_2 = 7$$

$$\Leftrightarrow \left[\begin{array}{cc|c} 3 & 4 & 7 \\ 5 & 6 & 7 \end{array}\right]$$

$$\Leftrightarrow \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Step 1 compute A^{-1}

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{18-20}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} = A^{-1}$$

If $Ax = b$

Then $\cancel{A^{-1}A}\vec{x} = A^{-1}b$

I

$I\vec{x} = A^{-1}b$

$\boxed{\vec{x} = A^{-1}b}$

Step 2: compute $A^{-1}b$

$$A^{-1}\vec{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -21+14 \\ \frac{35}{2} - \frac{21}{2} \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix} = \vec{x}$$

Properties of the Matrix Inverse

What about

A and B are invertible $n \times n$ matrices.

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

$ABA^{-1}B^{-1} \overset{?}{\neq} I$

$A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1} = I.$

Example
True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

$$\left[\begin{array}{c|c} 3 & 1 \end{array}\right] \sim \left[\begin{array}{c|c} 1 & \frac{1}{3} \end{array}\right]$$

Check

$3(-7) + 4(7) \overset{?}{=} 7$ ✓

$5(-7) + 6(7) \overset{?}{=} 7$ ✓

An Algorithm for Computing A^{-1}

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \mid I_n)$
2. If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example
Compute the inverse of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [A \mid I]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-3R_3 + R_1$
$\sim$
$-2R_3 + R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= [I \mid A^{-1}]$$

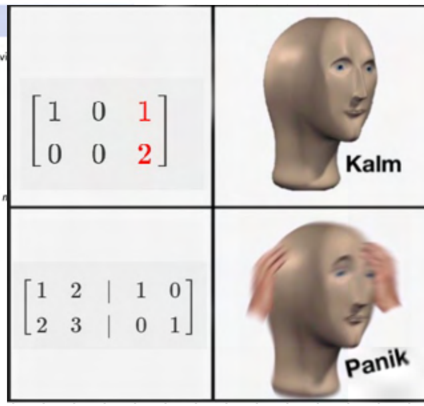
Why Does This Work?

We can think of our algorithm as simultaneously solvi

$$A\vec{x}_1 = \vec{e}_1$$
$$A\vec{x}_2 = \vec{e}_2$$
$$\vdots$$
$$A\vec{x}_n = \vec{e}_n$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

There's another explanation, which uses elementary m



Why?

First Check

$A * A^{-1} \overset{?}{=} I_3$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_3 !?

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

Example

Suppose

$$E\begin{bmatrix}1&1&1\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&3&2\\0&0&1\end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

doing a row operation is the same as multiplying on the left by an elem. matrix E_1A

$$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=E_1$$

$$E_1 * A$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\g&h&i\\d&e&f\end{bmatrix}$$

$$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim_{6R_2\to}\begin{bmatrix}1&0&0\\0&6&0\\0&0&1\end{bmatrix}=E_2$$

$$E_2 * A$$

$$\begin{bmatrix}1&0&0\\0&6&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\6d&6e&6f\\g&h&i\end{bmatrix}$$

$$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\sim_{2R_1+R_3}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=E_3$$

$$\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\2a+g&2b+h&2c+i\end{bmatrix}$$

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k\cdots E_3E_2E_1)A=I_n$$

Thus, $E_k\cdots E_3E_2E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

> **Theorem**
> Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x}=\vec{b}$$

We would calculate A^{-1} and then:

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

$$[A\,|\,I]\sim\ldots\sim[I\,|\,A^{-1}]$$

$$\sim[E_1A\,|\,E_1I]$$

$$\sim[E_2E_1A\,|\,E_2E_1I]$$

$$\sim[\underbrace{E_3E_2E_1A}_{I}\,|\,\underbrace{E_3E_2E_1I}_{E_3E_2E_1}]$$

why A^{-1}?

$$\boxed{\boxed{E_3E_2E_1}*A=I}$$

A^{-1} !!

2.2 EXERCISES

Find the inverses of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

5. Use the inverse found in Exercise 1 to solve the system

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

6. Use the inverse found in Exercise 3 to solve the system

$$\begin{aligned} 8x_1 + 5x_2 &= -9 \\ -7x_1 - 5x_2 &= 11 \end{aligned}$$

7. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

   a. Find A^{-1}, and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$

   b. The four equations in part (a) can be solved by the *same* set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $[A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4]$.

8. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.

   b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

   c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

   d. If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$.

   e. Each elementary matrix is invertible.

10. a. A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

    b. If A is invertible, then the inverse of A^{-1} is A itself.

    c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.

    d. If A can be row reduced to the identity matrix, then A must be invertible.

    e. If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

11. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

12. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

$$\text{If } [A \;\; B] \sim \cdots \sim [I \;\; X], \text{ then } X = A^{-1}B.$$

If A is larger than 2×2, then row reduction of $[A \;\; B]$ is much faster than computing both A^{-1} and $A^{-1}B$.

13. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

14. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

15. Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)\,D = I$ and $D\,(ABC) = I$.

16. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [*Hint:* Let $C = AB$, and solve this equation for A.]

17. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

18. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

19. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

Find the inverses of the matrices in Exercises 29–32, if they exist. Use the algorithm introduced in this section.

29. $\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

30. $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

33. Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

34. Repeat the strategy of Exercise 33 to guess the inverse of
$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}.$$ Prove that your guess is correct.

35. Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1}

38. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D using only 1 and 0 as entries, such that $AD = I_2$. Is it possible that $CA = I_4$ for some 4×2 matrix C? Why or why not?