# LINEAR ALGEBRA

## Week 9

# Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

**Motivation**: it can be useful to take large powers of matrices, for example $A^k$, for large $k$.

**But**: multiplying two $n \times n$ matrices requires roughly $n^3$ computations. Is there a more efficient way to compute $A^k$?



## Topics and Objectives

**Topics**

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

**Learning Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

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## Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster p

| | | Mon | Tue | Wed | Thu | Fri |
|---|---|---|---|---|---|---|
| Week | Dates | Lecture | Studio | Lecture | Studio | Lecture |
| 1 | 1/8 - 1/12 | 1.1 | WS1.1 | 1.2 | WS1.2 | 1.3 |
| 2 | 1/15 - 1/19 | Break | WS1.3 | 1.4 | WS1.4 | 1.5 |
| 3 | 1/22 - 1/26 | 1.7 | WS1.5,1.7 | 1.8 | WS1.8 | 1.9 |
| 4 | 1/29 - 2/2 | 1.9,2.1 | WS1.9,2.1 | **Exam 1**, Review | Cancelled | 2.2 |
| 5 | 2/5 - 2/9 | 2.3,2.4 | WS2.2-2.4 | 2.5 | WS2.5 | 2.8 |
| 6 | 2/12 - 2/16 | 2.9 | WS2.8 | 2.9,3.1 | WS2.9,3.1 | 3.2 |
| 7 | 2/19 - 2/23 | 3.3 | WS3.2 | 4.9 | WS3.3,4.9 | 5.1 |
| 8 | 2/26 - 3/1 | 5.2 | WS5.1,5.2 | **Exam 2**, Review | Cancelled | 5.3 |
| 9 | 3/4 - 3/8 | 5.3 | WS5.3 | 5.5 | WS5.5 | 6.1 |
| 10 | 3/11 - 3/15 | 6.1,6.2 | WS6.1 | 6.2 | WS6.2 | 6.3 |
| 11 | 3/18 - 3/22 | Break | Break | Break | Break | Break |
| 12 | 3/25 - 3/29 | 6.4 | WS6.3 | 6.4,6.5 | WS6.4 | 6.5 |
| 13 | 4/1 - 4/5 | 6.6 | WS6.5,6.6 | **Exam 3**, Review | Cancelled | PageRank |
| 14 | 4/8 - 4/12 | 7.1 | WSPageRank | 7.2 | WS7.1,7.2 | 7.3 |
| 15 | 4/15 - 4/19 | 7.3,7.4 | WS7.3 | 7.4 | WS7.4 | 7.4 |
| 16 | 4/22 - 4/24 | Last lecture | Last Studio | Reading Period | | |
| 17 | 4/25 - 5/2 | Final Exams: **MATH 1554 Common Final Exam Tuesday, April 30th at 6:00pm** | | | | |

## Diagonal Matrices

A matrix is **diagonal** if the only non-zero elements, if any, are on the main diagonal.

The following are all diagonal matrices.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad [2], \quad I_n, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll only be working with diagonal square matrices in this course.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} \pi & 0 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$



## Powers of Diagonal Matrices

If $A$ is diagonal, then $A^k$ is easy to compute. For example,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3^k & 0 \\ 0 & (1/2)^k \end{bmatrix}$$

easy to compute $A^k$

But what if $A$ is not diagonal? ??

$$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^k \overset{?}{=} \text{harder?}} \quad ???$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

diagonal is $\boxed{a_{ij} = 0 \text{ if } i \neq j}$

entry on row $i$ & col $j$

Ex. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$

$p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$

$\lambda_1 = 1 \qquad A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda = 1$

$\lambda_2 = 3 \qquad A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 3$

magic??

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$$

Check

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = PDP^{-1}$

$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^3}^{A^3} = \overbrace{\left( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \right)^3}^{(PDP^{-1})^3}$$

$A = PDP^{-1}$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$P * D * \cancel{P^{-1}} * \cancel{P} * D * \cancel{P^{-1}} * \cancel{P} * D * P^{-1}$$

$$= PDDDP^{-1} = PD^3P^{-1}$$

in general $\quad A^k = (PDP^{-1})^k = \underbrace{PD\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}}_{k \text{ times.}}$

hard.

$$= \boxed{PD^kP^{-1}} \text{ easy}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^3 & 0 \\ 0 & 1^3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 27 & 27 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 28 & 26 \\ 26 & 28 \end{bmatrix}$$

easy!

$$= \boxed{\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}} = A^3$$

## Diagonalization

Suppose $A \in \mathbb{R}^{n\times n}$. We say that $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$. That is, we can write

$$A = PDP^{-1}$$

previous slide.



## Diagonalization

$A = PDP^{-1}$ $A$ is diagonalizable

**Theorem**

If $A$ is diagonalizable $\Leftrightarrow$ $A$ has $n$ linearly independent eigenvectors.

you need $n$ linearly independent eigenvector to go in the $P$ matrix.

Note: the symbol $\Leftrightarrow$ means " if and only if ".

Also note that $A = PDP^{-1}$ if and only if

$$A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]^{-1}$$

where $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors, and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (**in order**).



$$\begin{array}{cccc} & P & D & \text{same } P^{-1} \\ A = & \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} & \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} & \begin{bmatrix} | & | & | \end{bmatrix}^{-1} \end{array}$$

columns are eigenvectors.

eigenvalues on diagonal. 0's elsewhere

partial shortcut.

## Distinct Eigenvalues

**Theorem**

If $A$ is $n \times n$ and has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Why does this theorem hold?

Is it necessary for an $n \times n$ matrix to have $n$ distinct eigenvalues for it to be diagonalizable?



FACT:

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ all different then $v_1, \ldots, v_n$ corresponding eigenvectors are lin. ind.

## Non-Distinct Eigenvalues

alg > geo ? $\Rightarrow$ matrix NOT diag'ble.

Theorem. Suppose

- $A$ is $n \times n$
- $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \le n$
- $a_i$ = algebraic multiplicity of $\lambda_i$
- $d_i$ = dimension of $\lambda_i$ eigenspace ("geometric multiplicity")

Then

1. $d_i \le a_i$ for all $i$
2. $A$ is diagonalizable $\Leftrightarrow$ $\boxed{\Sigma d_i = n}$ $\Leftrightarrow$ $d_i = a_i$ for all $i$
3. $A$ is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.



Sum of geometric multiplicities is $n$

$\Rightarrow$ $A$ diag'ble.

$\boxed{A = PBP^{-1} \quad A \text{ similar to } B.}$

## Example 1

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} \overset{?}{=} PDP^{-1}$$

$\lambda_1 = 2 \quad A - 2I = \begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1 \quad A - (-1)I = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$$\boxed{A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}}$$

↑ a diagonalization of A.

$$\boxed{A \text{ is similar to } \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D}$$

## Example 2

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 3 \quad$ alg 2

$$\underset{A}{A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}} \overset{??}{=} \underset{P}{\begin{bmatrix} : & : \end{bmatrix}} \underset{D}{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} : & : \end{bmatrix}^{-1}}$$

$A - 3I = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

geo mult of $\lambda = 3$ is 1.

Since geo mult. is 1 you can't have columns of invertible 2x2 matrix be both eigenvectors of A.

So A is NOT diag'ble.

## Example 3

The eigenvalues of $A$ are $\boxed{\lambda = 3, 1}$. If possible, construct $P$ and $D$ such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

This or That

$\lambda_1 = 3$ alg 2 ~~alg 1~~

$\lambda_2 = 1$ alg 1 ~~alg 2~~

$p(\lambda) = -(\lambda-3)(\lambda-1)(\lambda-c)$

either 1 or 3.



## Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the $n^{th}$ number in this sequence.



Count free vars:

$$A - 3I = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \sim \cdots \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

geo 2

$$A - I = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -8 & -8 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{x} = s\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

geo 1

$$\boxed{A = \underbrace{\begin{pmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}_{P^{-1}}}$$

## Next: example of each.

| | diag'ble | not diag'ble |
|---|---|---|
| invertible | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, ... | (a) $\begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ |
| not invertible | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, ... | (b) $\begin{bmatrix} 2 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ |

$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

alg 2
geo 1

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ c & b & a \\ b & a & c \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (a+b+c)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

## Additional Example (if time permits)

## §5.3: Diagonalization.

Note that

$$\vec{x}_k = \begin{bmatrix}0&1\\1&1\end{bmatrix}\vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}, \quad k = 1,2,3,\dots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the $n^{th}$ number in this sequence.

$A \quad \vec{x}_0$

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \vec{x}_1$$



$A\vec{x}_1 = \vec{x}_2$

$A\vec{x}_2 = \vec{x}_3$

$\vdots$

$A^k\vec{x}_0 = \vec{x}_k$

not a Markov chain.

Long term: $\vec{x}_k = A^k\vec{x}_0$

$= (PDP^{-1})^k\vec{x}_0$

$= PD^kP^{-1}\vec{x}_0$

$$A = \begin{bmatrix}0&1\\1&1\end{bmatrix} \text{ diagonalize.}$$

Step 1. Find $\lambda$'s

$$p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix}-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix}$$

$$= (-\lambda)(1-\lambda) - 1$$

$$= \lambda^2 - \lambda - 1 = 0 \quad \text{solve.}$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-(-4)}}{2} = \frac{1}{2} \pm \frac{\sqrt5}{2}$$

$\lambda_1 = \frac{1+\sqrt5}{2} \quad \lambda_2 = \frac{1-\sqrt5}{2}$

$= \frac{1}{2} + \frac{\sqrt5}{2}$

$\lambda_1 = \frac{1+\sqrt5}{2}$

$$A - \lambda_1 I = \begin{bmatrix}\frac{-1-\sqrt5}{2} & 1\\ 1 & \frac{1-\sqrt5}{2}\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & \frac{1-\sqrt5}{2}\\ 0 & 0\end{bmatrix} \quad -\left(\frac{-1-\sqrt5}{2}\right)R_1 + R_2$$

$$-\left(\frac{1-\sqrt5}{2}\right) + \frac{1-\sqrt5}{2}$$

$$\vec{x} = s\begin{bmatrix}\frac{-1+\sqrt5}{2}\\ 1\end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix}-1+\sqrt5\\ 2\end{bmatrix}}$$

$$A - \lambda_2 I = \begin{bmatrix}\frac{-1+\sqrt5}{2} & 1\\ 1 & \frac{1+\sqrt5}{2}\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & \frac{1+\sqrt5}{2}\\ 0 & 0\end{bmatrix}$$

$$\vec{x} = s\begin{bmatrix}\frac{-1-\sqrt5}{2}\\ 1\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}-1+\sqrt5 & -1-\sqrt5\\ 2 & 2\end{bmatrix}\begin{bmatrix}\frac{1+\sqrt5}{2} & 0\\ 0 & \frac{1-\sqrt5}{2}\end{bmatrix}\begin{bmatrix}-1+\sqrt5 & -1-\sqrt5\\ 2 & 2\end{bmatrix}^{-1}} \quad \boxed{\vec{v}_2 = \begin{bmatrix}-1-\sqrt5\\ 2\end{bmatrix}}$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \vec{x}_0$$

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} = \vec{x}_1$$

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} = \vec{x}_2$$

$$\vec{x}_3 = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

$$\vec{x}_4 = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}5\\8\end{bmatrix}$$

1, 1, 2, 3, 5, 8, 13, 21, ...

1,2314?7?

**THEOREM 5** **The Diagonalization Theorem**

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$. In this case, the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.

**EXAMPLE 4** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Sorry.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Step 1: Find $\lambda$'s

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix} = +(2-\lambda)\begin{vmatrix} -6-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 4\begin{vmatrix} -4 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -4 & -6-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= -(\lambda-2)\left[(\lambda+6)(\lambda-1)+9\right] - 4\left[4(\lambda-1)+9\right] + 3\left[-12+3(\lambda+6)\right]$$

don't distribute!!

$$= -(\lambda-2)\left[\lambda^2+5\lambda-6+9\right] - 4(4\lambda+5) + 3\left[3\lambda+6\right]$$

$$= -(\lambda-2)\left[\lambda^2+5\lambda+3\right] - 16\lambda - 20 + 9\lambda + 18$$

$$= -(\lambda-2)\left[\lambda^2+5\lambda+3\right] - 7\lambda - 2$$

??

$$= -\lambda^3 + \dots + 4$$

**THEOREM 6** An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

**THEOREM 7** Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvalue $\lambda_k$.

b. The matrix $A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals $n$, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

c. If $A$ is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$ for each $k$, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

## Basis of Eigenvectors

Express the vector $\vec{x}_0 = \begin{bmatrix}4\\5\end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and find the coordinates of $\vec{x}_0$ in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_0\}$.

$[\vec{x}_0]_{\mathcal{B}} =$

Let $P = [\vec{v}_1\ \vec{v}_2]$ and $D = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, and find $[A^k\vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$[A^k\vec{x}_0]_{\mathcal{B}} =$



```
clc
P=[1 1 ; 1 -1]
% first example
%D=[1 0 ; 0 -1]
% part 2
%D=[1 0 ; 0 -1/2]
% part 3
D=[2 0 ; 0 3/2]
A=P*D*inv(P)
x0=[4;5];
s=10
format bank
for k=0:s
   % convert current index to string and
create xk and coordk strings
   index=string(k);
   s=strcat('x',index,'=');
   c=strcat('[x',index,']_B=');
   % compute xk value
   xk=A^k*x0;
   coordk=inv(P)*xk;
   % display each xk=A^k*x0
   disp(s)
   disp(xk)
   disp(c)
   disp(coordk)
end
```

$$\begin{bmatrix}4\\5\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & 4\\1 & -1 & 5\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 4\\0 & -2 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 4\\0 & 1 & -1/2\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}1 & 0 & 9/2\\0 & 1 & -1/2\end{array}\right]$$

Check $\vec{x}_0 = \begin{bmatrix}4\\5\end{bmatrix} \stackrel{?}{=} \frac{9}{2}\overset{v_1}{\begin{bmatrix}1\\1\end{bmatrix}} + \left(-\frac{1}{2}\right)\overset{v_2}{\begin{bmatrix}1\\-1\end{bmatrix}}$ ✓

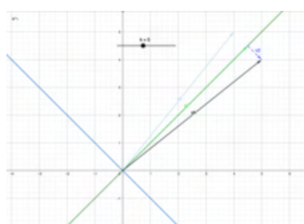
$$A = \overset{P}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}\overset{D}{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}P^{-1}$$

https://www.geogebra.org/calculator/bcah

$$\vec{x}_1 = A\vec{x}_0 = \frac{9}{2}A\vec{v}_1 + \left(-\frac{1}{2}\right)A\vec{v}_2$$

$$= \frac{9}{2}(1)\vec{v}_1 + \left(-\frac{1}{2}\right)(-1)\vec{v}_2$$

$$= \frac{9}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 = \begin{bmatrix}5\\4\end{bmatrix}$$



$$\vec{x}_2 = \frac{9}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$$

## Basis of Eigenvectors - part 2

Let $\vec{x}_0 = \begin{bmatrix}4\\5\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ as before.

Again define $P = [\vec{v}_1\ \vec{v}_2]$ but this time let $D = \boxed{\begin{bmatrix}1 & 0\\0 & -1/2\end{bmatrix}}$, and now find $[A^k\vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

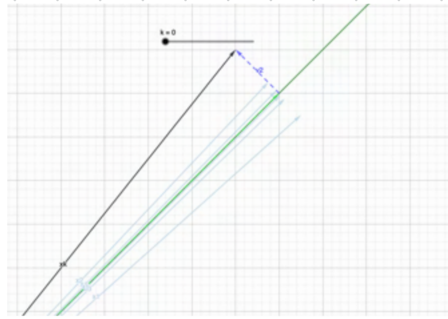
$[A^k\vec{x}_0]_{\mathcal{B}} =$

$\lambda_1 = 1$

$\lambda_2 = -\frac{1}{2}$



https://www.geogebra.org/calculator/czdnmrgc




$$\begin{bmatrix}4\\5\end{bmatrix} = \frac{9}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}1 & 0\\0 & -1/2\end{bmatrix}}_{D}\underbrace{P^{-1}}_{P^{-1}}$$

$$A\begin{bmatrix}4\\5\end{bmatrix} = \frac{9}{2}A\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}A\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{9}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\left(-\frac{1}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A^k\begin{bmatrix}4\\5\end{bmatrix} = \frac{9}{2}(1)^k - \frac{1}{2}\left(-\frac{1}{2}\right)^k\begin{bmatrix}1\\-1\end{bmatrix}$$

$\rightarrow$ going to zero.

## Basis of Eigenvectors - part 3

Let $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as before.

Again define $P = [\vec{v}_1 \ \vec{v}_2]$ but this time let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$, and now find $[A^k \vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

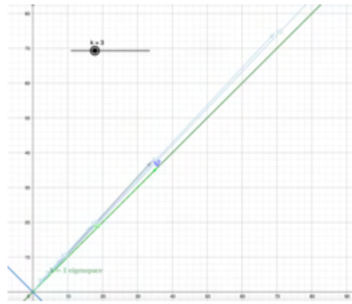
$[A^k \vec{x}_0]_{\mathcal{B}} =$

$\lambda_1 = 2$

$\lambda_2 = 3/2$



https://www.geogebra.org/calculator/ddcanyxh



$$A^k \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{9}{2}(2)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\left(\frac{3}{2}\right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \underbrace{[v_1 \ v_2]}_{P} \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} P^{-1}$$

$$\vec{x}_k = \begin{bmatrix} * \\ * \end{bmatrix}$$

## 5.3 EXERCISES

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute $A^4$.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute $A^k$, where $k$ represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

In Exercises 5 and 6, the matrix $A$ is factored in the form $PDP^{-1}$. Use the Diagonalization Theorem to find the eigenvalues of $A$ and a basis for each eigenspace.

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} =$
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

6. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} =$
$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Diagonalize the matrices in Exercises 7–20, if possible. The eigenvalues for Exercises 11–16 are as follows: (11) $\lambda = 1, 2, 3$; (12) $\lambda = 2, 8$; (13) $\lambda = 5, 1$; (14) $\lambda = 5, 4$; (15) $\lambda = 3, 1$; (16) $\lambda = 2, 1$. For Exercise 18, one eigenvalue is $\lambda = 5$ and one eigenvector is $(-2, 1, 2)$.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

18. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 21 and 22, $A$, $B$, $P$, and $D$ are $n \times n$ matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a. $A$ is diagonalizable if $A = PDP^{-1}$ for some matrix $D$ and some invertible matrix $P$.
    b. If $\mathbb{R}^n$ has a basis of eigenvectors of $A$, then $A$ is diagonalizable.
    c. $A$ is diagonalizable if and only if $A$ has $n$ eigenvalues, counting multiplicities.
    d. If $A$ is diagonalizable, then $A$ is invertible.

22. a. $A$ is diagonalizable if $A$ has $n$ eigenvectors.
    b. If $A$ is diagonalizable, then $A$ has $n$ distinct eigenvalues.
    c. If $AP = PD$, with $D$ diagonal, then the nonzero columns of $P$ must be eigenvectors of $A$.
    d. If $A$ is invertible, then $A$ is diagonalizable.

23. $A$ is a $5 \times 5$ matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is $A$ diagonalizable? Why?

**24.** $A$ is a $3 \times 3$ matrix with two eigenvalues. Each eigenspace is one-dimensional. Is $A$ diagonalizable? Why?

**25.** $A$ is a $4 \times 4$ matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that $A$ is *not* diagonalizable? Justify your answer.

**26.** $A$ is a $7 \times 7$ matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that $A$ is *not* diagonalizable? Justify your answer.

**27.** Show that if $A$ is both diagonalizable and invertible, then so is $A^{-1}$.

**28.** Show that if $A$ has $n$ linearly independent eigenvectors, then so does $A^T$. [*Hint:* Use the Diagonalization Theorem.]

**29.** A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A$ in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix $P_1$ such that $A = P_1 D_1 P_1^{-1}$.

**30.** With $A$ and $D$ as in Example 2, find an invertible $P_2$ unequal to the $P$ in Example 2, such that $A = P_2 D P_2^{-1}$.

**31.** Construct a nonzero $2 \times 2$ matrix that is invertible but not diagonalizable.

**32.** Construct a nondiagonal $2 \times 2$ matrix that is diagonalizable but not invertible.

**[M]** Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

**33.** $\begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$

**34.** $\begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$

**35.** $\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$

## Chapter 5 : Eigenvalues and Eigenvectors

## 5.5 : Complex Eigenvalues

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | 3/4 - 3/8 | 5.3 | WS5.3 | 5.5 | WS5.5 | 6.1 |
| 10 | 3/11 - 3/15 | 6.1,6.2 | WS6.1 | 6.2 | WS6.2 | 6.3 |
| 11 | 3/18 - 3/22 | Break | Break | Break | Break | Break |
| 12 | 3/25 - 3/29 | 6.4 | WS6.3 | 6.4,6.5 | WS6.4 | 6.5 |
| 13 | 4/1 - 4/5 | 6.6 | WS6.5,6.6 | **Exam 3**, Review | Cancelled | PageRank |

Chapter 5 : Eigenvalues and Eigenvectors

5.5 : Complex Eigenvalues

$\mathbb{C}$ complex numbers are useful from a mathematical standpoint.

## Topics and Objectives

**Topics**

1. Complex numbers: addition, multiplication, complex conjugate
2. Diagonalizing matrices with complex eigenvalues
3. Eigenvalue theorems

**Learning Objectives**

1. Diagonalize $2 \times 2$ matrices that have complex eigenvalues.
2. Use eigenvalues to determine identify the rotation and dilation of a linear transform.
3. Apply theorems to characterize matrices with complex eigenvalues.

**Motivating Question**
What are the eigenvalues of a rotation matrix?



## Imaginary Numbers

**Recall**: When calculating roots of polynomials, we can encounter square roots of negative numbers. For example:

$$x^2 + 1 = 0$$

The roots of this equation are:

We usually write $\sqrt{-1}$ as $i$ (for "imaginary").

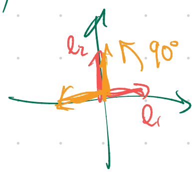


$\boxed{p(x) = x^2 + 1}$ roots? $= (x - c_1)(x - c_2) = 0$ ?

$c_1 = i \quad c_2 = -i$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $T_A$ rotate by $90°$ CCW

eig-values of A?




$p(\lambda) = \det(A - \lambda I)$

$= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - (-1)$

$= \boxed{\lambda^2 + 1}$

## Addition and Multiplication

The imaginary (or complex) numbers are denoted by C, where

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify C with $\mathbb{R}^2$: $\quad a + bi \leftrightarrow (a, b)$

$3 + 2i \in \mathbb{C}$

3 is the "real Part"

$2i$ is the imaginary part.

$i^2 = -1$. definition of $i$

We can add and multiply complex numbers as follows:

$\boxed{(2 - 3i) + (-1 + i) =}$

$(2 - 3i)(-1 + i) =$



$2 - 3i + -1 + i = 1 - 2i \in \mathbb{C}$

$(2 - 3i)(-1 + i) = -2 + 2i + 3i - 3i^2$

$= -2 + 5i - 3(-1)$

$= 1 + 5i$

## Complex Conjugate, Absolute Value, Polar Form

We can **conjugate** complex numbers: $\overline{a + bi} =$ $a - bi$

$\overline{-2 - 4i} = -2 + 4i$

If $z = 1 + 3i$ then $\bar{z} = 1 - 3i$ "z bar"

The **absolute value** of a complex number: $|a + bi| =$ $\sqrt{a^2 + b^2} = \sqrt{(a+bi)(a-bi)}$

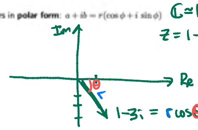
length or "norm"

$|z| = \sqrt{z \cdot \bar{z}}$

We can write complex numbers in **polar form**: $a + ib = r(\cos\phi + i \sin\phi)$

$\mathbb{C} \simeq \mathbb{R}^2$ "$\simeq$" isomorphic

$z = 1 - 3i \simeq \begin{bmatrix} 1 \\ -3 \end{bmatrix}$




$1 - 3i = r\cos\theta + i r \sin\theta$

$r = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}(b/a)$



divide??

$\boxed{\dfrac{1 + 5i}{2 - 3i}} = \dfrac{1 + 5i}{2 - 3i} \cdot \dfrac{2 + 3i}{2 + 3i}$

$= \dfrac{2 + 3i + 10i + 15i^2}{4 + 6i - 6i - 9i^2}$

$= \dfrac{2 - 15 + 13i}{4 + 9} = \dfrac{-13 + 13i}{13}$

$= \boxed{-1 + i}$

tools: $+, \times, \div, -$, conjugate.

## Complex Conjugate Properties

If $x$ and $y$ are complex numbers, $\vec{v} \in \mathbb{C}^n$, it can be shown that:

- ✓ $\overline{(x+y)} = \overline{x} + \overline{y}$
- ✓ $\overline{A\vec{v}} = A\overline{\vec{v}}$ $\quad A \in \mathbb{R}^{m \times n}$
- ✓ $\mathrm{Im}(x\overline{x}) = 0.$

**Example** True or false: if $x$ and $y$ are complex numbers, then

$$\overline{(xy)} = \overline{x}\;\overline{y}$$

$$\overline{(1+i)(1-2i)} \overset{?}{=} \overline{(1+i)} \cdot \overline{(1-2i)}$$

LHS $\overline{(1+i)(1-2i)} = \overline{(1-2i+i+2)} = \overline{3-i} = \boxed{3+i}$

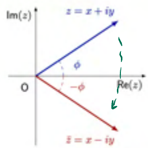
RHS $(1-i)(1+2i) = 1+2i-i+2 = 3+i$ Same!



FYI $\varphi : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ "ring homomorphism" extra.

## Polar Form and the Complex Conjugate

Conjugation reflects points across the real axis.




$\mathbb{C} \simeq \mathbb{R}^2$

Conjugation is really just "reflect about the real axis of $\mathbb{C}$"



$\begin{bmatrix} 2+i \\ 3-4i \end{bmatrix} \in \mathbb{C}^2 \simeq \mathbb{R}^4$ $\quad \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \in \mathbb{C}^{2\times 2}$

natural.

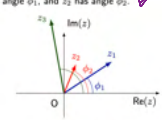
$\simeq \begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix} \in \mathbb{R}^4$

complex vector space of 2-dim'l over $\mathbb{C}$.

Come back

## Euler's Formula

Suppose $z_1$ has angle $\phi_1$, and $z_2$ has angle $\phi_2$.




The product $z_1 z_2$ has angle $\phi_1 + \phi_2$ and modulus $|z|\,|w|$. Easy to remember using Euler's formula.

$$z = |z|e^{i\phi}$$

The product $z_1 z_2$ is:

$$z_3 = z_1 z_2 = (|z_1|e^{i\phi_1})(|z_2|e^{i\phi_2}) = |z_1|\,|z_2|e^{i(\phi_1+\phi_2)}$$



why $\mathbb{C}$ has to be in 1554 ??

## Complex Numbers and Polynomials

**Theorem: Fundamental Theorem of Algebra**
Every polynomial of degree $n$ has exactly $n$ complex roots, counting multiplicity.

$\mathbb{C}$-roots come in pairs.

$p(\lambda) = \pm\lambda^n \pm a_{n-1}\lambda^{n-1} + \dots + a_0 = \pm(\lambda - c_1)\dots(\lambda - c_n)$

$a_i \in \mathbb{R}$ real coeffs

either $\mathbb{R}$ or $\mathbb{C}$ roots.

**Theorem**

1. If $\lambda \in \mathbb{C}$ is a root of a real polynomial $p(x)$, then the conjugate $\overline{\lambda}$ is also a root of $p(x)$.
2. If $\lambda$ is an eigenvalue of real matrix $A$ with eigenvector $\vec{v}$, then $\overline{\lambda}$ is an eigenvalue of $A$ with eigenvector $\overline{\vec{v}}$.

irreducible quadratic pair



$p(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$

$\lambda_1 = i$
$\lambda_2 = -i$

roots are either REAL or they come in CONJUGATE PAIRS.

## Example

Four of the eigenvalues of a $7 \times 7$ matrix are $-2, 4+i, -4-i$, and $i$. What are the other eigenvalues?



## Example

The matrix that rotates vectors by $\phi = \pi/4$ radians about the origin, and then scales (or dilates) vectors by $r = \sqrt{2}$, is

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

What are the eigenvalues of $A$? Express them in polar form.

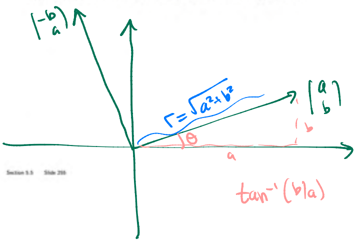


## Example rotation-dilation

The matrix in the previous example is a special case of this matrix:

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Calculate the eigenvalues of $C$ and express them in polar form.






$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$p(\lambda) = \lambda^2 - 2a\lambda + a^2 + b^2$$

$$\boxed{\lambda = a \pm bi}$$

## Diagonalization

**Theorem**

Let $A$ be a real $2 \times 2$ matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector $\vec{v}$. Then we may construct the diagonalization

$$A = PCP^{-1}$$

where

$$P = (\operatorname{Re}\vec{v} \quad \operatorname{Im}\vec{v}) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Note the following.

- $C$ is referred to as a **rotation dilation** matrix, because it is the composition of a rotation by $\phi$ and dilation by $r$.
- The proof for why the columns of $P$ are always linearly independent is a bit long, it goes beyond the scope of this course.

## Example

Find the complex eigenvalues and an associated complex eigenvector for each eigenvalue for the matrix.

Step 1: Find λ's. $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

$p(\lambda) = \det(A - \lambda I)$

$= \det \begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - (-2)$

$= 3 - \lambda - 3\lambda + \lambda^2 + 2$

$= \lambda^2 - 4\lambda + 5 = 0$

$\cancel{p(\lambda) = \lambda(\lambda - i) = \lambda^2 - i\lambda}$

$\sqrt{-4} = \sqrt{4 \cdot -1} = \sqrt{4} \cdot \sqrt{-1} = 2i$

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roots $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \frac{\sqrt{-4}}{2} = 2 \pm \frac{2i}{2} = \boxed{2 \pm i}$

$\lambda_1 = 2+i$ $\boxed{\lambda_2 = 2-i}$ start

$\boxed{\lambda_1 = 2+i \quad \lambda_2 = 2-i \quad \vec{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}}$

$\vec{V}_1 = \overline{\vec{V}_2}$

$\cancel{\frac{27}{3}} \neq \boxed{9}$

Final answer should be ℂ-numbers $\boxed{a+bi}$ only

Step 2: Find $\vec{v}_1, \vec{v}_2$ eigenvectors.

$\lambda = 2-i$: $A - \lambda I = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} - (2-i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} - \begin{bmatrix} 2-i & 0 \\ 0 & 2-i \end{bmatrix}$

$= \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i \\ -1+i & -2 \end{bmatrix} \sim_{-(-1+i)R_1 + R_2} \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$

$x_1 + (1+i)x_2 = 0$

$x_2 =$ free

$\boxed{\vec{X} = s\begin{bmatrix} -1-i \\ 1 \end{bmatrix}}$

$\boxed{\vec{V}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \quad \lambda_2 = 2-i}$

$-(-1+i)(1+i) - 2$

$= -(i-1)(i+1) - 2$

$= -(i^2 - 1) - 2$

$= -(-1-1) - 2$

$= -(-2) - 2 = 0$

## 5.5 EXERCISES

Let each matrix in Exercises 1–6 act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

1. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

In Exercises 7–12, use Example 6 to list the eigenvalues of $A$. In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle $\varphi$ of the rotation, where $-\pi < \varphi \le \pi$, and give the scale factor $r$.

7. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$

9. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

10. $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$

11. $\begin{bmatrix} .1 & .1 \\ -.1 & .1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & .3 \\ -.3 & 0 \end{bmatrix}$

In Exercises 13–20, find an invertible matrix $P$ and a matrix $C$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$. For Exercises 13–16, use information from Exercises 1–4.

13. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 \\ .4 & .6 \end{bmatrix}$

19. $\begin{bmatrix} 1.52 & -.7 \\ .56 & .4 \end{bmatrix}$

20. $\begin{bmatrix} -1.64 & -2.4 \\ 1.92 & 2.2 \end{bmatrix}$

21. In Example 2, solve the first equation in (2) for $x_2$ in terms of $x_1$, and from that produce the eigenvector $\mathbf{y} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix}$ for the matrix $A$. Show that this $\mathbf{y}$ is a (complex) multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Let $A$ be a complex (or real) $n \times n$ matrix, and let $\mathbf{x}$ in $\mathbb{C}^n$ be an eigenvector corresponding to an eigenvalue $\lambda$ in $\mathbb{C}$. Show that for each nonzero complex scalar $\mu$, the vector $\mu\mathbf{x}$ is an eigenvector of $A$.

Chapter 7 will focus on matrices $A$ with the property that $A^T = A$. Exercises 23 and 24 show that every eigenvalue of such a matrix is necessarily real.

23. Let $A$ be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \bar{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that $q$ is a real number by verifying that $\bar{q} = q$. Give a reason for each step.

$$\bar{q} = \overline{\bar{\mathbf{x}}^T A\mathbf{x}} \underset{(a)}{=} \mathbf{x}^T \overline{A\mathbf{x}} \underset{(b)}{=} \mathbf{x}^T A\bar{\mathbf{x}} \underset{(c)}{=} (\mathbf{x}^T A\bar{\mathbf{x}})^T \underset{(d)}{=} \bar{\mathbf{x}}^T A^T \mathbf{x} \underset{(e)}{=} q$$

# Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

# Section 6.1 : Inner Product, Length, and Orthogonality

Chapter 6: Orthogonality and Least Squares

Math 1554 Linear Algebra

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | 3/4 - 3/8 | 5.3 | WS5.3 | 5.5 | WS5.5 | 6.1 |
| 10 | 3/11 - 3/15 | 6.1,6.2 | WS6.1 | 6.2 | WS6.2 | 6.3 |
| 11 | 3/18 - 3/22 | Break | Break | Break | Break | Break |
| 12 | 3/25 - 3/29 | 6.4 | WS6.3 | 6.4,6.5 | WS6.4 | 6.5 |
| 13 | 4/1 - 4/5 | 6.6 | WS6.5,6.6 | Exam 3, Review | Cancelled | PageRank |

## Topics and Objectives

$$[\cdot] * [\cdot] = [\cdot]$$

$$* : S \times S \to S$$

**Topics**

1. Dot product of vectors
2. Magnitude of vectors, and distances in $\mathbb{R}^n$
3. Orthogonal vectors and complements
4. Angles between vectors

**Learning Objectives**

1. Compute (a) dot product of two vectors, (b) length (or magnitude) of a vector, (c) distance between two points in $\mathbb{R}^n$, and (d) angles between vectors.
2. Apply theorems related to orthogonal complements, and their relationships to Row and Null space, to characterize vectors and linear systems.

**Motivating Question**
For a matrix $A$, which vectors are orthogonal to all the rows of $A$? To the columns of $A$?



product "multiplication"

## The Dot Product

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

**Example 1:** For what values of $k$ is $\vec{u} \cdot \vec{v} = 0$?

$$\vec{u} = \begin{pmatrix} -1 \\ 3 \\ k \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 4 \\ 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\vec{u} \cdot \vec{v} = -4 + \cancel{6} + k - \cancel{6} = 0$$

$$\Rightarrow k - 4 = 0 \Rightarrow \boxed{k = 4}$$



## Properties of the Dot Product

The dot product is a special form of matrix multiplication, so it inherits linear properties.

**Theorem (Basic Identities of Dot Product)**

Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$.

1. (Symmetry) $\vec{u} \cdot \vec{w} =$ $\vec{v} \cdot \vec{u}$
2. (Linear in each vector) $(\vec{v} + \vec{w}) \cdot \vec{u} =$ $\vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{u}$
3. (Scalars) $(c\vec{u}) \cdot \vec{w} =$ $c(\vec{u} \cdot \vec{w})$
4. (Positivity) $\vec{u} \cdot \vec{u} \geq 0$, and the dot product equals $\|\vec{u}\|^2$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1^2 + 0^2 + (-1)^2 = 2$$



$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (2)(-1) + (3)(0) + (1)(1) = -2 + 0 + 1 = -1$$

$$\vec{v} \circ \vec{w} = \vec{v}^T * \vec{w}$$

$1 \times 3$ $\quad$ $3 \times 1$ $\quad$ $1 \times 1$

$$\begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (2)(-1) + (3)(0) + (1)(1) = -1$$


**THEOREM 1** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let $c$ be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{a^2 + b^2 + c^2 = 0}$$

Then $a, b, c$ are all zero

## The Length of a Vector

$\|\vec{u}\|$ length of $\vec{u}$.

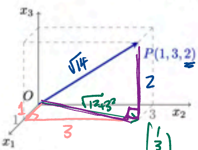
**Definition**

The **length** of a vector $\vec{u} \in \mathbb{R}^n$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2+u_2^2+\cdots+u_n^2}$$

**Example**: the length of the vector $\overrightarrow{OP}$ is

$$\sqrt{1^2+3^2+2^2} = \sqrt{14}$$




$\begin{bmatrix}1\\3\\0\end{bmatrix}$

$$\left\|\begin{bmatrix}1\\3\\2\end{bmatrix}\right\| = \sqrt{\left(\sqrt{1^2+3^2}\right)^2+2^2} = \sqrt{1^2+3^2+2^2} = \sqrt{14}$$

$$\left\|\begin{bmatrix}1\\3\\0\end{bmatrix}\right\| = \sqrt{1^2+3^2}$$



$$\begin{bmatrix}1\\3\\2\end{bmatrix}\cdot\begin{bmatrix}1\\3\\2\end{bmatrix} = 1^2+3^2+2^2 \overset{??}{=} \left\|\begin{bmatrix}1\\3\\2\end{bmatrix}\right\|^2$$

$$\boxed{\vec{v}\cdot\vec{v} = \|\vec{v}\|^2}$$

## Example

General approach??

$\|\vec{u}\|^2 = 25 = u\cdot u$

idea: $\boxed{\vec{w}\cdot\vec{w} = \|\vec{w}\|^2}$

Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$ with $\|\vec{u}\| = 5$, $\|\vec{v}\| = \sqrt{3}$, and $\vec{u}\cdot\vec{v} = -1$.
Compute the value of $\|\vec{u}+\vec{v}\|$.

Want: $\|\vec{u}+\vec{v}\| = ?$ $\quad \cancel{\vec{u}^2}$ $\quad \vec{u}, \vec{v}$

Step 1: Instead find

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} + 2\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= 25 + 2(-1) + 3$$
$$= \boxed{26}$$

So $\boxed{\|\vec{u}+\vec{v}\| = \sqrt{26}}$



**DEFINITION** The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2+v_2^2+\cdots+v_n^2}, \quad \text{and} \quad \boxed{\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}}$$

more useful geometrically.

more useful for calculations & manipulating dot products.

## Length of Vectors and Unit Vectors

$\left\|\begin{bmatrix}1\\3\\2\end{bmatrix}\right\| = \sqrt{14}$

$\sqrt{14}$ 1.

**Note**: for any vector $\vec{v}$ and scalar $c$, the length of $c\vec{v}$ is

$$\|c\vec{v}\| = |c| * \|v\|$$

$\left\|\begin{bmatrix}2\\6\\4\end{bmatrix}\right\| = 2\sqrt{14}$

**Definition**

If $\vec{v} \in \mathbb{R}^n$ has length one, we say that it is a **unit vector.**

$\|cv\|^2 \stackrel{?}{=} c^2\|v\|^2$

**Example**: Let $W$ be a subspace of $\mathbb{R}^4$ spanned by

$$\vec{v} = \begin{bmatrix}-1\\-3\\-2\\1\end{bmatrix}$$

$\left\|\begin{bmatrix}1/\sqrt{14}\\3/\sqrt{14}\\2/\sqrt{14}\end{bmatrix}\right\| = 1$

a) Construct a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

b) Construct a basis for $W$ using unit vectors.



$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1+9+4+1}}\begin{bmatrix}-1\\-3\\-2\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{15}\\-3/\sqrt{15}\\-2/\sqrt{15}\\1/\sqrt{15}\end{bmatrix}$$

unit vector in span$\{\vec{v}\}$

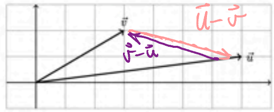
## Distance in $\mathbb{R}^n$

**Definition**

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the **distance** between $\vec{u}$ and $\vec{v}$ is given by the formula

$$\boxed{\|\vec{u} - \vec{v}\|} \text{ or } \|\vec{v} - \vec{u}\|$$

**Example:** Compute the distance from $\vec{u} = \begin{pmatrix}7\\1\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}3\\2\end{pmatrix}$.




$\|\vec{u} - \vec{v}\| = \left\|\begin{pmatrix}7\\1\end{pmatrix} - \begin{pmatrix}3\\2\end{pmatrix}\right\|$

$= \left\|\begin{pmatrix}4\\-1\end{pmatrix}\right\| = \sqrt{16+1}$

$= \boxed{\sqrt{17}}$

$\vec{v} + \vec{u} - \vec{v} = \vec{u}$



$\ell^2$ - norm Euclidean norm $\sqrt{a^2+b^2+c^2}$

$\ell^1$ - norm $|a|+|b|+|c|$

**DEFINITION**

For **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as dist(**u**, **v**), is the length of the vector **u** − **v**. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

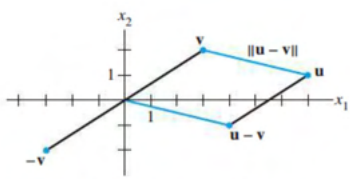



**FIGURE 4** The distance between **u** and **v** is the length of **u** − **v**.

## Orthogonality

Chapter 6: Orthogonality

**Definition (Orthogonal Vectors)**

Two vectors $\vec{u}$ and $\vec{w}$ are **orthogonal** if $\vec{u}\cdot\vec{w}=0$. This is equivalent to:

$$\|\vec{u}+\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{w}\|^2$$

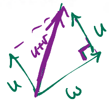
Note: The zero vector is orthogonal to every vector. But we usually only mean non-zero vectors.

If $\vec{u}\cdot\vec{w}=0$

then $\|\vec{u}+\vec{w}\|^2 = (\vec{u}+\vec{w})\cdot(\vec{u}+\vec{w})$

$= \vec{u}\cdot\vec{u} + \cancel{2\vec{u}\cdot\vec{w}}^{\,0} + \vec{w}\cdot\vec{w}$

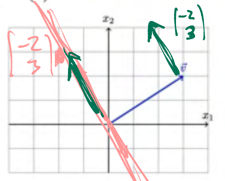
$= \|\vec{u}\|^2 + \|\vec{v}\|^2$






## Example

Sketch the subspace spanned by the set of all vectors $\vec{u}$ that are orthogonal to $\vec{v}=\begin{pmatrix}3\\2\end{pmatrix}$.




$\text{Span}\left\{\begin{bmatrix}-2/3\\1\end{bmatrix}\right\}$

$=\text{Span}\left\{\begin{bmatrix}-2\\3\end{bmatrix}\right\}$



$\vec{v}\cdot\vec{x}=0$

$\boxed{v^T \cdot \vec{x}=0}$

Nul$(v^T)$

Solve for a, b

$\boxed{\begin{bmatrix}3\\2\end{bmatrix}}\cdot\begin{bmatrix}a\\b\end{bmatrix}=0$

$3a+2b=0$

$\boxed{[3\ \ 2]} \sim [1\ \ 2/3]$

$\vec{x}=s\begin{bmatrix}-2/3\\1\end{bmatrix}$

Summarize

$\vec{u}\cdot\vec{w}=0$ says the dot product is zero.

$\|\vec{u}+\vec{w}\|^2 = \|u\|^2+\|\vec{w}\|^2$ says the vectors $\vec{u}$ & $\vec{w}$ are perpendicular (90° angle w/ each other)

## Orthogonal Compliments

**Definitions**

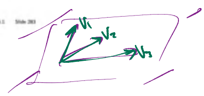
Let $W$ be a subspace of $\mathbb{R}^n$. A vector $\vec{z}\in\mathbb{R}^n$ is said to be **orthogonal** to $W$ if $\vec{z}$ is orthogonal to each vector in $W$.

The set of all vectors orthogonal to $W$ is a subspace, the **orthogonal compliment** of $W$, or $W^\perp$ or '$W$ perp.'

$$W^\perp = \{\vec{z}\in\mathbb{R}^n : \vec{z}\cdot \qquad \}$$

$W=\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ in $\mathbb{R}^n$

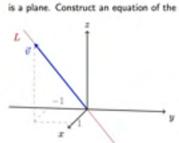





$$\begin{cases}\vec{v}_1\cdot\vec{x}=0\\ \vec{v}_2\cdot\vec{x}=0\\ \vec{v}_3\cdot\vec{x}=0\end{cases}\qquad \begin{matrix}v_1^T\vec{x}=0\\ v_2^T\vec{x}=0\\ v_3^T\vec{x}=0\end{matrix}$$

## Example

Line $L$ is a subspace of $\mathbb{R}^3$ spanned by $\vec{v}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$. Then the space $L^\perp$ is a plane. Construct an equation of the plane $L^\perp$.




Can also visualise line and plane with CalcPlot3D: web.monroecc.edu/calcNSF



$$\begin{bmatrix}v_1^T\\v_2^T\\v_3^T\end{bmatrix} = \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T$$

## Row$A$

**Definition**

Row$A$ is the space spanned by the rows of matrix $A$.

We can show that

- dim(Row($A$)) = dim(Col($A$))
- a basis for Row$A$ is the pivot rows of $A$



## Example

Describe the Null($A$) in terms of an orthogonal subspace.

A vector $\vec{x}$ is in Null $A$ if and only if

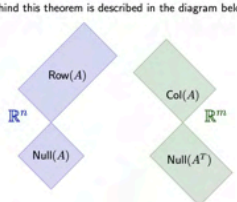
1. $A\vec{x} =$
2. This means that $\vec{x}$ is $\boxed{\phantom{xxxxxx}}$ to each row of $A$.
3. Row $A$ is $\boxed{\phantom{xxxxxxxx}}$ to Null $A$.
4. The dimension of Row $A$ plus the dimension of Null $A$ equals $\boxed{\phantom{xxxxxx}}$



**Theorem (The Four Subspaces)**

For any $A \in \mathbb{R}^{m\times n}$, the orthogonal complement of Row $A$ is Null $A$, and the orthogonal complement of Col $A$ is Null $A^T$.

The idea behind this theorem is described in the diagram below.






## Additional Example (if time permits)

$A$ has the LU factorization:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Construct a basis for $(\text{Row}A)^\perp$

b) Construct a basis for $(\text{Col}A)^\perp$

*Hint: it is not necessary to compute $A$. Recall that $A^T = U^T L^T$, matrix $L^T$ is invertible, and $U^T$ has a non-empty nullspace.*



**THEOREM 3**

Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the null space of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

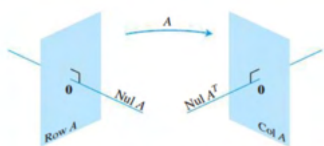



**FIGURE 8** The fundamental subspaces determined by an $m \times n$ matrix $A$.

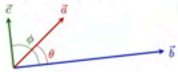
## Angles

**Theorem**

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$. Thus, if $\vec{a} \cdot \vec{b} = 0$, then:

- $\vec{a}$ and/or $\vec{b}$ are _______ vectors, or
- $\vec{a}$ and $\vec{b}$ are _______.

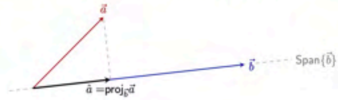
For example, consider the vectors below.






## Looking Ahead - Projections

Suppose we want to find the closed vector in Span$\{\vec{b}\}$ to $\vec{a}$.




- Later in this Chapter, we will make connections between dot products and **projections**.
- Projections are also used throughout multivariable calculus courses.



## 6.1 EXERCISES

Compute the quantities in Exercises 1–8 using the vectors

$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$

**1.** $\mathbf{u}\cdot\mathbf{u}$, $\mathbf{v}\cdot\mathbf{u}$, and $\dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$

**2.** $\mathbf{w}\cdot\mathbf{w}$, $\mathbf{x}\cdot\mathbf{w}$, and $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$

**3.** $\dfrac{1}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$

**4.** $\dfrac{1}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

**5.** $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$

**6.** $\left(\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$

**7.** $\|\mathbf{w}\|$

**8.** $\|\mathbf{x}\|$

In Exercises 9–12, find a unit vector in the direction of the given vector.

**9.** $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

**10.** $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

**11.** $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

**12.** $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

**13.** Find the distance between $\mathbf{x} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

**14.** Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Determine which pairs of vectors in Exercises 15–18 are orthogonal.

**15.** $\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

**16.** $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

**17.** $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

**18.** $\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

In Exercises 19 and 20, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

**19.** a. $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$.

b. For any scalar $c$, $\mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$.

c. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

d. For a square matrix $A$, vectors in Col $A$ are orthogonal to vectors in Nul $A$.

Lay Linear Algebra.

e. If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace $W$ and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^{\perp}$.

**20.** a. $\mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} = 0$.

b. For any scalar $c$, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

c. If $\mathbf{x}$ is orthogonal to every vector in a subspace $W$, then $\mathbf{x}$ is in $W^{\perp}$.

d. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

e. For an $m \times n$ matrix $A$, vectors in the null space of $A$ are orthogonal to vectors in the row space of $A$.

**21.** Use the transpose definition of the inner product to verify parts (b) and (c) of Theorem 1. Mention the appropriate facts from Chapter 2.

**22.** Let $\mathbf{u} = (u_1, u_2, u_3)$. Explain why $\mathbf{u}\cdot\mathbf{u} \geq 0$. When is $\mathbf{u}\cdot\mathbf{u} = 0$?

**23.** Let $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. Compute and compare $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|^2$, $\|\mathbf{v}\|^2$, and $\|\mathbf{u}+\mathbf{v}\|^2$. Do not use the Pythagorean Theorem.

**24.** Verify the *parallelogram law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

**25.** Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set $H$ of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{v}$. [*Hint:* Consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.]

**26.** Let $\mathbf{u} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let $W$ be the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{u}\cdot\mathbf{x} = 0$. What theorem in Chapter 4 can be used to show that $W$ is a subspace of $\mathbb{R}^3$? Describe $W$ in geometric language.

**27.** Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u}+\mathbf{v}$.

**28.** Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$. [*Hint:* An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to such a vector $\mathbf{w}$.]

**29.** Let $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in $W$.