Final Exam Review

In-Class Final Exam Review Set A, Math 1554, Fall 2019

* LU
~~overview~~
motivation

* coord-

1. Indicate whether the statements are true or false.

true	false	
○	○	If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.
○	○	A $n \times n$ matrix A and its echelon form E will always have the same eigenvalues.
○	○	$x^2 - 2xy + 4y^2 \geq 0$ for all real values of x and y.
○	○	If matrix A has linearly dependent columns, then $\dim((\text{Row}A)^\perp) > 0$.
○	○	If λ is an eigenvalue of A, then $\dim(\text{Null}(A - \lambda I)) > 0$.
○	○	If A has QR decomposition $A = QR$, then $\text{Col}A = \text{Col}Q$.
○	○	If A has LU decomposition $A = LU$, then $\text{rank}(A) = \text{rank}(U)$.
○	○	If A has LU decomposition $A = LU$, then $\dim(\text{Null}\,A) = \dim(\text{Null}\,U))$.

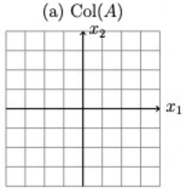
2. Give an example of the following.

i) A 4×3 lower triangular matrix, A. such that $\text{Col}(A)^\perp$ is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$. $A = \begin{pmatrix} & & \end{pmatrix}$

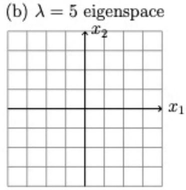
ii) A 3×4 matrix A, that is in RREF, and satisfies $\dim\left((\text{Row } A)^\perp\right) = 2$ and $\dim\left((\text{Col } A)^\perp\right) = 2$. $A = \begin{pmatrix} & & \end{pmatrix}$

3. (3 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. On the grid below, sketch a) $\text{Col}(A)$, and b) the eigenspace corresponding to eigenvalue $\lambda = 5$.

(a) $\text{Col}(A)$



(b) $\lambda = 5$ eigenspace



$\rightarrow \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix}$

4. Fill in the blanks.

   (a) If $A \in \mathbb{R}^{M \times N}$, $M \leq N$, and $A\vec{x} = 0$ does not have a non-trivial solution, how many pivot columns does A have? $\boxed{NP}$

   ← no free cols.

   (b) Consider the following linear transformation.

   output in $\mathbb{R}^3$

   $$T(x_1, x_2) = (2x_1 - x_2,\ 4x_1 - 2x_2,\ x_2 - 2x_1).$$

   ↑ input in $\mathbb{R}^2$

   The domain of T is $\boxed{\mathbb{R}^2}$. The image of $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $T(\vec{x})$ is $\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$. The co-domain of T is $\boxed{\mathbb{R}^3}$. The range of T is: Span$\left\{\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}\right\}$ the line in $\mathbb{R}^3$.

Q: give A s.t. $T(\vec{x}) = Ax$ A is size 2×3 or $\boxed{3 \times 2}$ ✓

name, domain, codomain

$$\boxed{T: \mathbb{R}^2 \to \mathbb{R}^3}$$ & A is 3x2

$$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \ T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 2 & -1 \\ 4 & -2 \\ -2 & 1 \end{bmatrix}$$

rank(A) = 1.
dim Col A = 1
Col A = range of T

5. Four points in $\mathbb{R}^2$ with coordinates (t, y) are $(0, 1)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{4}, -\frac{1}{2})$. Determine the values of c_1 and c_2 for the curve $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$ that best fits the points. Write the values you obtain for c_1 and c_2 in the boxes below.

$c_1 = \boxed{}$ $c_2 = \boxed{}$

In-Class Final Exam Review Set B, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
○	○	For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \text{proj}_W \vec{y}$ is orthogonal to W.
○	○	If A is $m \times n$ and has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.
○	○	If a matrix is invertible it is also diagonalizable.
○	○	If E is an echelon form of A, then $\text{Null}\, A = \text{Null}\, E$.
○	○	If the SVD of $n \times n$ singular matrix A is $A = U\Sigma V^T$, then $\text{Col}A = \text{Col}U$.
○	○	If the SVD of $n \times n$ matrix A is $A = U\Sigma V^T$, $r = \text{rank}A$, then the first r columns of V give a basis for $\text{Null}A$.

2. Give an example of:

a) a vector $\vec{u} \in \mathbb{R}^3$ such that $\text{proj}_{\vec{p}} \vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$: $\quad \vec{u} = \begin{pmatrix} \quad \end{pmatrix}$

b) an upper triangular 4×4 matrix A that is in RREF, 0 is its only eigenvalue, and its corresponding eigenspace is 1-dimensional. $\quad A = \begin{pmatrix} \quad \end{pmatrix}$

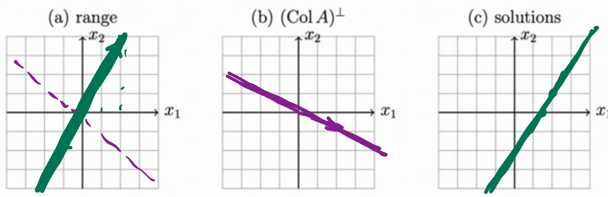
c) A 3×4 matrix, A, and $\text{Col}(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$.

d) A 2×2 matrix in RREF that is diagonalizable and not invertible.

3. Suppose $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. On the grid below, sketch a) the range of $x \to Ax$, b) $(\mathrm{Col}\,A)^{\perp}$, (c) set of solutions to $A\vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

$T(\vec{x}) = A\vec{x}$ range of T

$A\vec{x} = \vec{b}$ all b's that can come out



$\text{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$

Nul $A = \text{span}\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\}$ translated to $\begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \vec{b}$ is in $\text{span}\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}\right\}$ = Col A.

part (c) get solns to $Ax=b$ in parametric vector form

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 4 & -2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 3/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\vec{x} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}}$$

how to sketch these?

$\vec{x} = \vec{x}_p + \vec{x}_h$ ← general soln to hom.

↑ particular

$u = \frac{1}{\sigma}Av$ from svd

$\boxed{Av = \sigma u}$?? $\boxed{Ax = \lambda x}$

4. Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

1. $A(\vec{v}_1 + 4\vec{v}_2)$
2. A^{10}
3. $\lim_{k\to\infty} A^k(\vec{v}_1 + 4\vec{v}_2)$

In-Class Final Exam Review Set C, Math 1554, Fall 2019

$\lambda_i > 0.$

1. Indicate whether the statements are possible or impossible. A is symmetric & PD $\Longleftrightarrow$

possible	impossible	
○	●	$Q(\vec{x}) = \vec{x}^T A \vec{x}$ is a positive definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A. $\|Q(\vec{x})\| \geq \lambda_{small}$ $\forall$ $\|x\|=1$. $Q(c\vec{x}) \geq c^2 \lambda_{small} > 0.$
○	●	The $\boxed{\text{maximum value}}$ of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\vec{x}\| = 1$, is not unique. output. max value of $\{Q(\vec{x}) : \|\vec{x}\|=1\} = a.$
●	○	The location of the maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\vec{x}\| = 1$, is not unique. the "x". $Q\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = a$, $Q\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}\right) = a$
○	●	A is 2×2, the algebraic multiplicity of eigenvalue $\boxed{\lambda = 0}$ is 1, and $\underbrace{\dim(\text{Col}(A)^\perp) \text{ is equal to } 0}$. $\Leftrightarrow \dim \text{Col} A = 2$. IMT A not invertible
●	○	Stochastic matrix P has zero entries and is regular. entries $a_{ij} = 0.$
●	○	A is a square matrix that is not diagonalizable, but A^2 is diagonalizable. $A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ Not diag'ble. $A^2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ which is.
~~○~~	●	The map $T_A(\vec{x}) = A\vec{x}$ is ~~one-to-one~~ but not onto, A is $m \times n$, and $m < n$. not 1-1 yes onto. A is wide. e.g. $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$.

Cut

~~2. Transform $T_A = A\vec{x}$ reflects points in $\mathbb{R}^2$ through the line $y = 2 + x$. Construct a standard matrix for the transform using $\boxed{\text{homogeneous coordinates}}$. Leave your answer as a product of three matrices.~~

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\det A = \cos^2\theta + \sin^2\theta = 1$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\det A = -1.$

3. Fill in the blanks.

(a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{-1}$

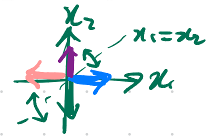
(b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? ______

(c) A is a 6×4 matrix in RREF, and $\operatorname{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? ______

(d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{1 \checkmark}$ $0 \checkmark$

(e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? ______

(f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is ______.



#3(a) (option 1)

Step 1: Find A.

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = [T(e_1)\ T(e_2)]$

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det A = -1.$

4. A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Sketch the nullspace of $Y = AC$.

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.

3. Fill in the blanks.

(a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{}$

(b) B and C are square matrices with $\det(BC) = -5$ and $\boxed{\det(C) = 2.}$ What is the value of $\det(B)\det(C^4)$? $\boxed{}$

(c) A is a 6×4 matrix in RREF, and $\text{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? $\boxed{1}$

(d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{}$

(e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? $\boxed{}$

(f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is $\boxed{}$.

#3(b)

true. ✓ $\det(AB) = \det A * \det B$ ✓

False in general. $\det(A+B) \neq \det A + \det B$

$$\boxed{\det(B)\det(C^4)} = \det B * \det C * \det C * \det C * \det C$$
$$= \det B * 2^4$$
$$= \frac{-5}{2} * 2^4 = \boxed{-40}$$

$\det(BC) = \det B * \det C$

$-5 = 2 \det B$

$\Rightarrow \det B = -5/2$

$$\det(BC) * \det(C^3) = -5 * 2^3 = -40 \checkmark$$

#3(c)

$$A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

3. Fill in the blanks.

(a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{}$

(b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? $\boxed{}$

(c) A is a 6×4 matrix in RREF, and $\text{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? $\boxed{}$

(d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{}$

(e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? $\boxed{3}$

(f) If A is $\boxed{30 \times 12}$ and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is $\boxed{0}$.

rows cols

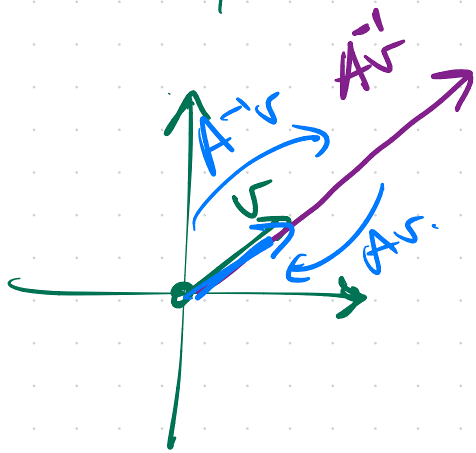
rank(A) = 12 no free vars.

#3(e) $A\vec{v} = \frac{1}{3}\vec{v}$ for some $\vec{v} \neq \vec{0}$.

$\Rightarrow A^{-1}Av = \frac{1}{3}A^{-1}v$

$\Rightarrow Iv = \frac{1}{3}A^{-1}v$

$\Rightarrow 3v = A^{-1}v$



#3(f) §6.5 If A has lin ind cols

$\Leftrightarrow Ax = b$ has unique L.S. soln

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.

Final Exam Review Worksheet, Spring 2020

1. (12 points) Indicate whether the statements are true or false.

		true	false
i)	If $A\vec{x} = \vec{b}$ has infinitely many solutions, then the RREF of A must have a row of zeros.	○	○
ii)	If A is $n \times n$ and $A\vec{x} = \vec{b}$ is inconsistent, then the columns of A are linearly dependent.	○	○
iii)	If A is a 3×3 matrix and $\det(A) = 2$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for Col(A).	○	○
iv)	A basis for a subspace must include the zero vector.	○	○
v)	If the columns of an $n \times n$ matrix span $\mathbb{R}^n$, then the matrix must be invertible.	○	○
vi)	A matrix, A, and any echelon form of A will have the same column space.	○	○
xii)	An $n \times n$ diagonalizable matrix must have n distinct eigenvalues.	○	○
xiii)	The geometric multiplicity of an eigenvalue is greater than or equal to the algebraic multiplicity of the same eigenvalue.	○	○
ix)	If S is a subspace of $\mathbb{R}^8$ and $\dim(S) = 6$, then $S^\perp$ is a two-dimensional subspace.	○	○
x)	If two vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then they are linearly independent.	○	○
xi)	If A is symmetric, and $v_1 \neq v_2$ are two eigenvectors of A, then v_1 and v_2 are orthogonal.	○	○
xii)	For a symmetric matrix A, the largest value of $\|Ax\|$ subject to the constraint that $\|x\| = 1$ is the largest singular value of A.	○	○

2. (10 points) Fill in the blanks.

(a) List all values of $k \in \mathbb{R}$ such that the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \\ -1 \end{pmatrix}$ are linearly dependent. $\boxed{}$.

(b) Suppose $\det(A^2B) = 4$, $\det(B) = \frac{1}{3}$, and A and B are $n \times n$ real matrices. List all possible values of $\det(A)$. $\boxed{}$.

(c) List all values of k such that $A\vec{x} = \vec{b}$ is inconsistent where $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2k \\ 0 & 0 & k \end{pmatrix}$. $k = \boxed{}$

(d) Consider the row operation that reduces matrix A to RREF.

$$A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1 A} = E_1 A$$

By inspection, E_1 is the elementary matrix $E_1 = \begin{pmatrix} \end{pmatrix}$.

(e) If $S = \{\vec{x} \in \mathbb{R}^4 \mid x_1 = x_2\}$ then $\dim S = \boxed{}$.

(f) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{pmatrix}$, then a non-zero vector in Null$A$ is $\begin{pmatrix} \end{pmatrix}$.

(g) If the basis for the column space of an 11×15 matrix consists of exactly three vectors, how many pivot columns does the matrix have? $\boxed{}$

(h) If A is a 3×3 matrix with eigenvalues 5 and $1 - i$, then the third eigenvalue is $\boxed{}$.

(i) If $\vec{v}$ is the steady-state vector for a regular stochastic matrix, then $\vec{v}$ is an eigenvector of that matrix corresponding to the eigenvalue $\lambda = \boxed{}$.

(j) List all values of k such that $A = \begin{pmatrix} 4 & k \\ 0 & 4 \end{pmatrix}$ is diagonalizable. $\boxed{}$

3. (6 points) Fill in the blanks.

(a) The distance between the vector $\vec{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and the line spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\boxed{}$.

(b) If W is the plane spanned by the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, a basis of $W^{\perp}$ is given by $\vec{w} = \begin{pmatrix} \\ \\ \end{pmatrix}$.

(c) If A is a 3×3 matrix and $\dim(\text{Row}(A)) = 2$, then $\dim(\text{Null}(A^T)) = \boxed{}$.

(d) If $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^2$ satisfying $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$ and $\vec{u} \cdot \vec{v} = \frac{3}{2}$, then the length of the sum of the two vectors is $\|\vec{u} + \vec{v}\| = \boxed{}$.

(e) Let U be an $n \times n$ matrix with orthonormal columns. Then $U^t U =$ ______

(f) The maximum value of $Q(\vec{x}) = 10x_1^2 - 7x_2^2 - 4x_3^2$ subject to the constraints $\vec{x} \cdot \vec{x} = 1$ and $\vec{x} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ is equal to $\boxed{}$.

4. (8 points) Indicate whether the statements are possible or impossible.

		possible	impossible
i)	The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. $T = Ax$, and A has linearly independent columns.	○	○
ii)	The columns of a matrix with N rows are linearly dependent and span $\mathbb{R}^N$.	○	○
iii)	Matrix A is $n \times n$, $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, and $\dim(\text{Null}A) = 0$.	○	○
iv)	P is a stochastic matrix which has zero in the first entry of the first row, but is regular.	○	○
v)	There is a 2×2 real matrix A and a vector $\vec{u} \neq \vec{0}$, such that $\vec{u} \in \text{Null}(A)$ and $\vec{u} \in \text{Row}(A)$.	○	○
vi)	A is a non-singular matrix which is not diagonalizable.	○	○
vi)	$\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of matrix A that correspond to distinct eigenvalues, $A = A^T$, and $\vec{v}_1 \cdot \vec{v}_2 = 1$.	○	○
viii)	$\vec{y}$ is a non-zero vector in $\mathbb{R}^5$. The projection of $\vec{y}$ onto a subspace of $\mathbb{R}^5$ is the zero vector.	○	○

5. (2 points) Suppose A and B are $n \times n$ matrices and A is symmetric. Fill in the circles next to the expressions (if any) that are equal to

$$(B^T AB)^T$$

Leave the other circles empty.

- ○ $BA^T B^T$
- ○ $B^T AB$

6. (2 points) List the singular values of the matrix below. (No need to justify your answer.)

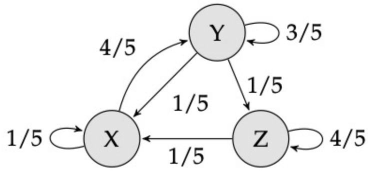
$$\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \_\_\_\_\_, \qquad \sigma_2 = \_\_\_\_\_,$$

7. (6 points) Let $A = \begin{pmatrix} -2 & -4 & 0 & 0 & 2 \\ -2 & -4 & 1 & 0 & 0 \\ -2 & -4 & 0 & 2 & 4 \\ -2 & -4 & 0 & 3 & 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 7 \\ 8 \end{pmatrix}$.

(a) Solve the system $A\vec{x} = \vec{b}$ where A and $\vec{b}$ are as above. Write your answer in parametric vector form for full credit.

(b) Write down a basis for $\mathrm{Col}(A)$.

8. (4 points) Consider the following Markov chain.



(a) What is the transition matrix, P?

$$P = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

(b) Use your transition matrix from part (a) to calculate the steady-state probability vector, $\vec{q}$. Show your work.

9. (3 points) Apply the Gram-Schmidt process to construct an orthogonal basis for Col(A). Show your work.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

10. (3 points) Construct the LU factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 0 \end{pmatrix}$. Clearly indicate matrices L and U.

11. (5 points) Compute Σ and V in the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^T$$

$$\Sigma = \begin{bmatrix} \_\_\_ & 0 \\ 0 & \_\_\_ \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

12. (5 points) Find matrices D and P to construct the orthogonal diagonalization of A. Show your work.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} = PDP^T$$

$$D = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}, \qquad P = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$