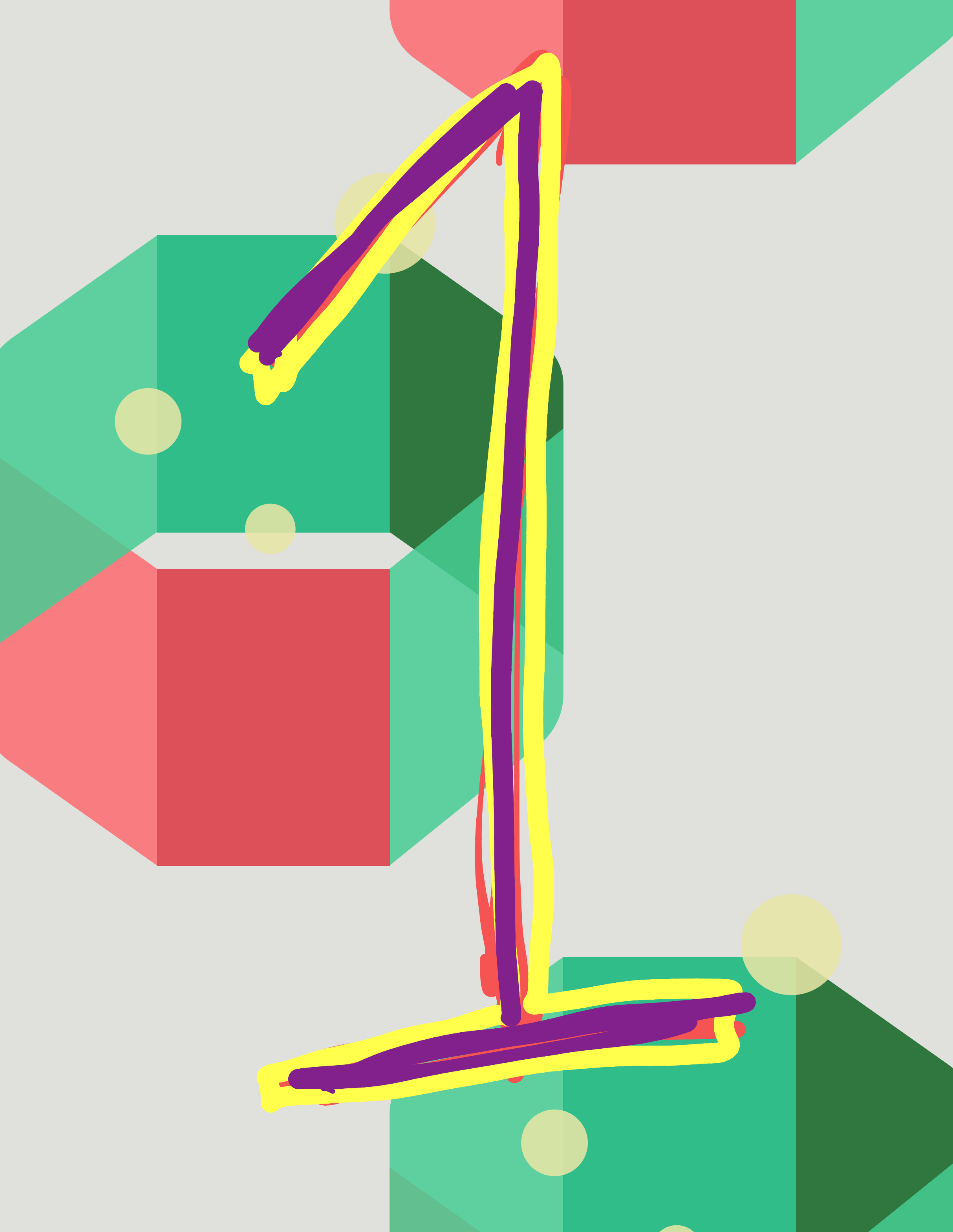

1

Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.1 Systems of Linear Equations

Topics
We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.1 Slide 1

Section 1.1 Systems of Linear Equations

Topics
We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

Section 1.1 Slide 2

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

First HW due

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 - 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 - 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 - 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 - 1/31	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2
5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	Exam 2, Review	Cancelled	5.3
9	3/3 - 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 - 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 - 3/21	Break	Break	Break	Break	Break
12	3/24 - 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 - 4/4	6.6	WS6.5,6.6	Exam 3, Review	Cancelled	PageRank
14	4/7 - 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 - 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 - 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 - 5/2	Final Exams: MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm				

// Quiz 0 Due Sunday.

The Book!

A Single Linear Equation

$y = 3x + 1$?

A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

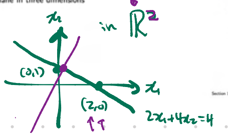
$-3x + y = 1$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the **dimension**, or number of variables.

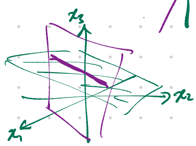
For example,

- $2x_1 + 4x_2 = 4$ is a line in two dimensions in $\mathbb{R}^2$
- $3x_1 + 2x_2 + x_3 = 6$ is a plane in three dimensions

(2,0)
(0,1)



Section 1.1 Slide 3



Systems of Linear Equations

When we have more than one linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\begin{cases} x_1 + 1.5x_2 + \pi x_3 = 4 \\ 5x_1 \qquad\quad + 7x_3 = 5 \end{cases}$$

> **Definition: Solution to a Linear System**
> The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution** to the system.

A system can have a unique solution, no solution, or an infinite number of solutions.

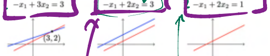
Section 1.1 Slide 4

Two Variables

consistent systems

Consider the following systems. How are they different from each other?

$$\boxed{\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}} \qquad \boxed{\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}} \qquad \boxed{\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}}$$



non-parallel lines — parallel lines — identical lines

unique soln to the system — no soln. inconsistent system — ∞-many solns.

Section 1.1 Slide 5

Three-Dimensional Case

"Soln" Solutions

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	sketch	number of solutions
line		∞-many solns
point		unique soln
empty		no soln

underlying systems are consistent

inconsistent.

Section 1.1 Slide 6

Defn: A system of linear equations is consistent iff it has at least one solution.

Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one row to another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

Proof: IF $\heartsuit = \bigstar$ and $\star = \xi$ Then $\heartsuit + \star = \bigstar + \xi$ ∎

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$$

add 1st row to second row

$$\sim \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - 7x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$$

add $-5 *$ row 2 to row 3

$$\sim \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - 7x_3 = 8 \\ 30x_3 = -30 \end{cases}$$

$\frac{1}{30} *$ row 3

$$\sim \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - 7x_3 = 8 \\ x_3 = -1 \end{cases}$$

add 7 row 3 to row 2

$$\sim \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 = 1 \\ x_3 = -1 \end{cases}$$

add $-$row 3 to row 1

$$\sim \begin{cases} x_1 - 2x_2 = 1 \\ x_1 = 1 \\ x_3 = -1 \end{cases}$$

add $-$row 2 to row 1

$$\sim \begin{cases} -2x_2 = 0 \\ x_1 = 1 \\ x_3 = -1 \end{cases}$$

swap row 1 & row 2

$$\sim \begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}}$$ is the unique soln.

Example 1

Identify the solution to the linear system.

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

$x_1 = 1$, $x_2 = 0$, $x_3 = -1$ } does it work?

$1 - 0 + (-1) = 0$ ✓

$0 + 8 = 8$ ✓

$5 - (-5) = 10$ ✓

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \xrightarrow{R_1 + R_2} \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & -7 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

coefficient matrix

augmented column.

$$\sim \xrightarrow{-5R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & -7 & 8 \\ 0 & 0 & 30 & -30 \end{array}\right]$$

$$\sim \xrightarrow{\frac{1}{30}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & -7 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \xrightarrow{7R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \xrightarrow{-R_3 + R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \xrightarrow{-R_2 + R_1} \left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{rrrl} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3.

Consistent Systems and Row Equivalence

Definition (Consistent)
A linear system is **consistent** if it has at least one Solution

Definition (Row Equivalence)
Two matrices are **row equivalent** if a sequence of row operations transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

row equivalent.

\sim

$\sim$

True $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$

False $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Fundamental Questions

Two questions that we will revisit many times throughout our course.

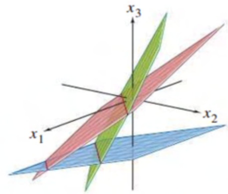
1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

ans: use matrices.

solution is unique.)

EXAMPLE 3 Determine if the following system is consistent: ANS: inconsistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \qquad (5)$$



The system is inconsistent because there is no point that lies on all three planes.

Soln. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$\begin{array}{r} \sim \\ \\ -2R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & +8 & -1 \end{array}\right]$$

$$\begin{array}{r} \sim \\ \\ 2R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right] \rightarrow \begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ 0x_1 + x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = 15 \end{cases}$$

$$0 = 15 \;?$$

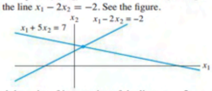
1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$

2. $\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and on the line $x_1 - 2x_2 = -2$. See the figure.



4. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $\begin{bmatrix} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{bmatrix}$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

Solve the systems in Exercises 11–14.

11. $\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$

12. $\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$

13. $\begin{aligned} x_1 \qquad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$

14. $\begin{aligned} x_1 - 3x_2 \qquad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. $\begin{aligned} x_1 \qquad + 3x_3 \qquad &= 2 \\ x_2 \qquad - 3x_4 &= 3 \\ -2x_2 + 3x_3 + 2x_4 &= 1 \\ 3x_1 \qquad\qquad + 7x_4 &= -5 \end{aligned}$

16. $\begin{aligned} x_1 \qquad\qquad - 2x_4 &= -3 \\ 2x_2 + 2x_3 \qquad &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$

17. Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

18. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$

22. $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. a. Every elementary row operation is reversible.

b. A 5×6 matrix has six rows.

c. The solution set of a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

d. Two fundamental questions about a linear system involve existence and uniqueness.

24. a. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

b. Two matrices are row equivalent if they have the same number of rows.

c. An inconsistent system has more than one solution.

d. Two linear systems are equivalent if they have the same solution set.

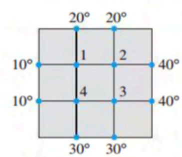
25. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

26. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2, x_2 = 1, x_3 = 0$.

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[2] For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



33. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

34. Solve the system of equations from Exercise 33. [*Hint:* To speed up the calculations, interchange rows 1 and 4 before starting "replace" operations.]

---

[2] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics
We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics
We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 – 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 – 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 – 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 – 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

Definition: Echelon Form

A rectangular matrix is in **echelon form** if

REF

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any). "staircase"
3. Below a leading entry (if any), all entries are zero.

A matrix in echelon form is in **row reduced echelon form** (RREF) if

1. The leading entry in each row is equal to 1.
2. Each leading 1 is the only nonzero entry in that column.

RREF.

Example of a Matrix in Echelon Form

■ = non-zero number, $*$ = any number

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1

Which of the following are in RREF? REF? None?

REF. a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ → d) $\begin{bmatrix} 0 & 6 & 3 & 0 \end{bmatrix}$ REF

RREF ✓ b) $\boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$? c) $\begin{bmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ RREF

e) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ NONE

REF/$\boxed{\text{RREF}}$/Neither?

→ Ex. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 4 \end{array}\right]$

Ex. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\boxed{\text{REF}}$/RREF/Neither?

Definition: Pivot Position, Pivot Column / Free column.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position. / Free column is a column that does not have a pivot 😊

Example 2: Express the matrix in row reduced echelon form and identify the pivot columns.

pivot cols

$$\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

? until REF we know pivot locations.

↑ Free column

$$\overset{-R_1}{\sim} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

$$\sim_{2R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -3 & -6 & 4 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & -3 & -6 & 4 \end{bmatrix}$$

$$\sim_{3R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

find the number of RREF.

Row Reduction Algorithm META.

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

- Step 1a Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row
- Step 1b Scale the 1st row so that its leading entry is equal to 1
- Step 1c Use row replacement so all entries above and below this 1 are 0
- Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in the 2nd row; uncover 1st row

etc.

Basic And Free Variables

Consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right] \longleftrightarrow \begin{cases} x_1 + 3x_2 + 7x_4 = 4 \\ x_3 + 4x_4 = 5 \\ x_5 = 6 \end{cases}$$

pivot cols are col 1, col 3, col 5

Free cols are col 2, col 4, col 6 (aug col)

The leading one's are in first, third, and fifth columns. So:

- Its pivot variables are x_1, x_3, and x_5.
- The free variables are x_2 and x_4. **Any choice** of the free variables leads to a solution of the system.

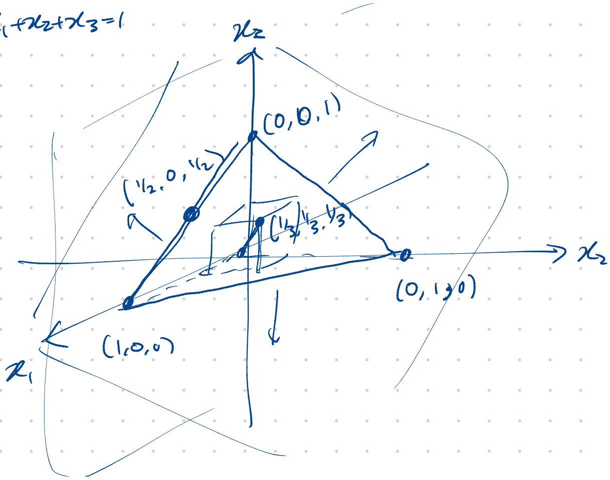
Free vars are $\boxed{x_2 \ \& \ x_4}$

pivot vars $\boxed{x_1, x_3, x_5}$

list as many solns as you want by choosing values for free var.

$$\begin{cases} x_1 = 4 - 3x_2 - 7x_4 \\ x_2 = x_2 \text{ free} \\ x_3 = 5 - 4x_4 \\ x_4 = x_4 \text{ free} \\ x_5 = 6 \end{cases}$$

$x_1+x_2+x_3=1$



Existence and Uniqueness

Theorem

A linear system is consistent if and only if (exactly when) the last column of the augmented matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$[0 \quad 0 \quad 0 \quad \cdots 0 \; 1]$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no Free vars
2. ∞ - many many solutions that are parameterized by free variables.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols $\blacksquare$, $*$, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3×2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

15. a. $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$

16. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

In Exercises 17 and 18, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

17. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$

In Exercises 19 and 20, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19. $x_1 + hx_2 = 2$
$4x_1 + 8x_2 = k$

20. $x_1 + 3x_2 = 2$
$3x_1 + hx_2 = k$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.[4]

21. a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

b. The row reduction algorithm applies only to augmented matrices for a linear system.

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

d. Finding a parametric description of the solution set of a linear system is the same as *solving* the system.

e. If one row in an echelon form of an augmented matrix is $[0\ \ 0\ \ 0\ \ 5\ \ 0]$, then the associated linear system is inconsistent.

22. a. The echelon form of a matrix is unique.

b. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

c. Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.

d. Whenever a system has free variables, the solution set contains many solutions.

e. A general solution of a system is an explicit description of all solutions of the system.

23. Suppose a 3×5 *coefficient* matrix for a system has three pivot columns. Is the system consistent? Why or why not?

24. Suppose a system of linear equations has a 3×5 *augmented* matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics
We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives
For the topics covered in this section, students are expected to be able to do the following.

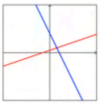
1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Week	Dates	Mon Lecture	Tue Studio	Wed Lecture	Thu Studio	Fri Lecture
1	1/6 – 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 – 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 – 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 – 1/31	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$x - 3y = -3$
$2x + y = 8$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

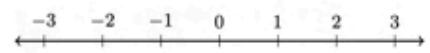
$\mathbb{R}^n$

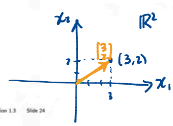
Recall that $\mathbb{R}$ denotes the collection of all real numbers.

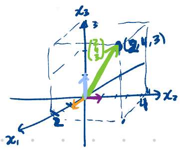
Let n be a positive whole number. We define

$\mathbb{R}^n$ = all ordered n-tuples of real numbers $(x_1, x_2, x_3, \dots, x_n)$.

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.







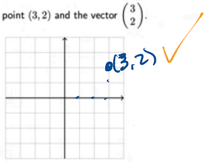
$$\begin{bmatrix}2\\4\\3\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\mathbb{R}^2$

Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

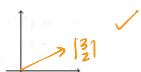
Example: Sketch the point $(3,2)$ and the vector $\begin{pmatrix}3\\2\end{pmatrix}$.



Vectors

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.



For example, the vector $\begin{pmatrix}3\\2\end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

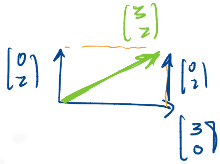
Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

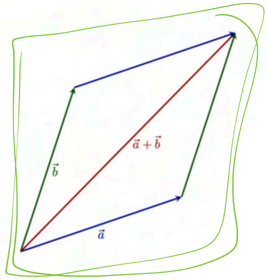
Vectors have the following properties.

1. **Scalar Multiple**:
$$c\vec{u} =$$
2. **Vector Addition**:
$$\vec{u} + \vec{v} =$$

Note that vectors in higher dimensions have the same properties.



Parallelogram Rule for Vector Addition



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \quad \leftarrow \text{vector addition}$$

$$5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \quad \leftarrow \text{scalar mult.}$$

Linear Combinations and Span

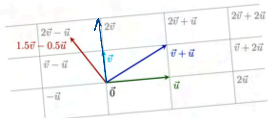
Difficult concept

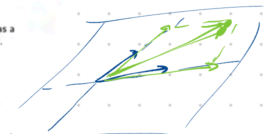
Definition

1. Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$, the vector below
$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$
is called a **linear combination of** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ **with weights** $c_1, c_2, \ldots, c_p$.
2. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Geometric Interpretation of Linear Combinations

Note that any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$. In other words, any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors that are not multiples of each other.





Ex. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

pick $c_1 = 3, c_2 = -1, c_3 = 0$

$$3\begin{bmatrix} 3 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

So $\boxed{\begin{bmatrix} 7 \\ -1 \end{bmatrix}}$ is a linear combination of $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ is in span$\left\{\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$

Soln

$$\begin{bmatrix} 3c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ 4c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

So
$$\begin{bmatrix} 3c_1 + 2c_2 + c_3 \\ c_2 + 4c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

So
$$\begin{cases} 3c_1 + 2c_2 + c_3 = 4 \\ c_2 + 4c_3 = 3 \end{cases}$$

Ex. Is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$?

Ans $\boxed{\text{Yes.}}$

Row reduce $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & 1 & 4 & 3 \end{array}\right]$ consistent & ∞-many solns.

Q: Can you find new c_1, c_2, c_3 such that $c_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Example

Start here

Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$

Is y in $\text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \right\}$

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.

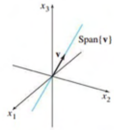


FIGURE 10 Span {v} as a line through the origin.

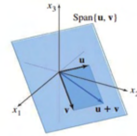


FIGURE 11 Span {u, v} as a plane through the origin.

1.3 EXERCISES

In Exercises 1 and 2, compute $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-2\mathbf{v}$.

1. $\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{v}=\begin{bmatrix}-3\\-1\end{bmatrix}$

2. $\mathbf{u}=\begin{bmatrix}3\\2\end{bmatrix}, \mathbf{v}=\begin{bmatrix}2\\-1\end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy-graph: $\mathbf{u}, \mathbf{v}, -\mathbf{v}, -2\mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}$. Notice that $\mathbf{u}-\mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$, and $-\mathbf{v}$.

3. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 1

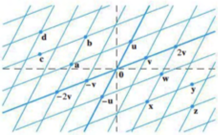
4. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1\begin{bmatrix}6\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$

6. $x_1\begin{bmatrix}-2\\3\end{bmatrix}+x_2\begin{bmatrix}8\\5\end{bmatrix}+x_3\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?



7. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$

8. Vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and $\mathbf{z}$

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9.
$$\begin{aligned} x_2+5x_3&=0\\ 4x_1+6x_2-x_3&=0\\ -x_1+3x_2-8x_3&=0\end{aligned}$$

10.
$$\begin{aligned} 4x_1+x_2+3x_3&=9\\ x_1-7x_2-2x_3&=2\\ 8x_1+6x_2-5x_3&=15\end{aligned}$$

In Exercises 11 and 12, determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$.

11. $\mathbf{a}_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}, \mathbf{a}_2=\begin{bmatrix}0\\1\\2\end{bmatrix}, \mathbf{a}_3=\begin{bmatrix}5\\-6\\8\end{bmatrix}, \mathbf{b}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$

12. $\mathbf{a}_1=\begin{bmatrix}1\\-2\\2\end{bmatrix}, \mathbf{a}_2=\begin{bmatrix}0\\5\\5\end{bmatrix}, \mathbf{a}_3=\begin{bmatrix}2\\0\\8\end{bmatrix}, \mathbf{b}=\begin{bmatrix}-5\\11\\-7\end{bmatrix}$

In Exercises 13 and 14, determine if $\mathbf{b}$ is a linear combination of the vectors formed from the columns of the matrix A.

13. $A=\begin{bmatrix}1&-4&2\\0&3&5\\-2&8&-4\end{bmatrix}, \mathbf{b}=\begin{bmatrix}3\\-7\\-3\end{bmatrix}$

14. $A=\begin{bmatrix}1&-2&-6\\0&3&7\\1&-2&5\end{bmatrix}, \mathbf{b}=\begin{bmatrix}11\\-5\\9\end{bmatrix}$

In Exercises 15 and 16, list five vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on $\mathbf{v}_1$ and $\mathbf{v}_2$ used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $\mathbf{v}_1=\begin{bmatrix}7\\1\\-6\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-5\\3\\0\end{bmatrix}$

16. $\mathbf{v}_1=\begin{bmatrix}3\\0\\2\end{bmatrix}, \mathbf{v}_2=\begin{bmatrix}-2\\0\\3\end{bmatrix}$

17. Let $\mathbf{a}_1=\begin{bmatrix}1\\4\\-2\end{bmatrix}$, $\mathbf{a}_2=\begin{bmatrix}-2\\-3\\7\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}4\\1\\h\end{bmatrix}$. For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

18. Let $\mathbf{v}_1=\begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}-3\\1\\8\end{bmatrix}$, and $\mathbf{y}=\begin{bmatrix}h\\-5\\-3\end{bmatrix}$. For what value(s) of h is $\mathbf{y}$ in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

19. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors
$$\mathbf{v}_1=\begin{bmatrix}8\\2\\-6\end{bmatrix} \text{ and } \mathbf{v}_2=\begin{bmatrix}12\\3\\-9\end{bmatrix}.$$

20. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 16.

21. Let $\mathbf{u}=\begin{bmatrix}2\\-1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2\\1\end{bmatrix}$. Show that $\begin{bmatrix}h\\k\end{bmatrix}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is *not* in the set spanned by the columns of A.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector $\begin{bmatrix}-4\\3\end{bmatrix}$ is $[-4 \quad 3]$.

b. The points in the plane corresponding to $\begin{bmatrix}-2\\5\end{bmatrix}$ and $\begin{bmatrix}-5\\2\end{bmatrix}$ lie on a line through the origin.

c. An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$.

d. The solution set of the linear system whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ is the same as the solution set of the equation $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_3\mathbf{a}_3=\mathbf{b}$.

e. The set Span $\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

24. a. Any list of five real numbers is a vector in $\mathbb{R}^5$.

b. The vector $\mathbf{u}$ results when a vector $\mathbf{u}-\mathbf{v}$ is added to the vector $\mathbf{v}$.

c. The weights $c_1, \ldots, c_p$ in a linear combination $c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p$ cannot all be zero.

d. When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line through $\mathbf{u}$ and the origin.

e. Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ has a solution amounts to asking whether $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.