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Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices
2. Orthogonal diagonalization
3. Spectral decomposition

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.
2. Construct a spectral decomposition of a matrix.

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a fast

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 – 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 – 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 – 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 – 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/3 – 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 – 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 – 3/21	Break	Break	Break	Break	Break
12	3/24 – 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 – 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 – 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 – 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 – 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 – 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

Symmetric Matrices

So. A is $n \times n$

Definition

Matrix A is **symmetric** if $A^T = A$.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

✓ $A = [*]$ ✓ $B = \begin{bmatrix} * & \star \\ \star & * \end{bmatrix}$ $C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$

✗ $D = \begin{bmatrix} * & \star \\ 0 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix}$ F_{24} F_{42}

$B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

eg: $\begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$. Symmetric.

$D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ not symmetric

$E^T = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ wrong size.

$D^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = D.$

A^TA is Symmetric

e.g. $A^TA\hat{x} = A^T\vec{b}$ normal eqns — A^TA always symmetric

A very common example: For **any** matrix A with columns $a_1, \dots, a_n$,

$$A^TA = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \cdots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \cdots & a_2^Ta_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \cdots & a_n^Ta_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

3×2.

$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

2×2

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 13 \end{bmatrix} = A^TA$ always symmetric.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 3$

$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$ same.

$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 13$

Q orthonormal cols.

$Q^TQ = I_n$

followed b/c $\|a_i\| = 1$, $a_i \cdot a_j = 0$ $(i \neq j)$

$Q^TQ = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 \\ a_2 \cdot a_1 & a_2 \cdot a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

$\lambda = -1$. look ing at null of

$A + I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

orthogonal diag'n. of A

$\boxed{A = PDP^T}$

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

???

$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$A = PDP^{-1}$

want. $P = [v_1 \ v_2 \ v_3]$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

want columns to all be orthogonal to each other. & unit length

$\lambda_1 = 1$. $A\vec{v} = \vec{v}$ i.e. $(A - I)\vec{v} = \vec{0}$

want $\text{Nul}(A - I)$

$A - I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ r s

$x = r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Symmetric Matrices and their Eigenspaces

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A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

Example 1

Diagonalize A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

$V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\lambda_1 = 1$

$V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\lambda_2 = 1 = \lambda_1$

$V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\lambda_3 = -1$.

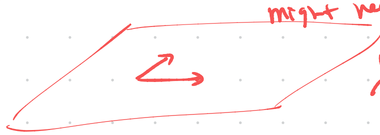
Free from theorem.

e.g. $V_1 \cdot V_3 = 0$

$V_2 \cdot V_3 = 0$

$V_1 \cdot V_2 = 0$ by blind luck.

might need to do G-S.



$\lambda = 1$

Parametric vector form usually only has lin ind vectors.

Spectral Theorem

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real
2. The dimenison of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity. geo=alg.
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is Orthogonal

Proof (if time permits):

(P has orthonormal cols)

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = [\vec{u}_1 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank ________.

Example: Find the spectral decomposition of A.

A symmetric

$\{v_1, v_2, v_3\}$ orthogonal set in $\mathbb{R}^3$

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

to get this

$A - 4I \sim \cdots$

$A - I \sim \cdots$

$A - 0I \sim \cdots$

get param vector form.

$A = PDP^T$ orthogonal diag'n.

$P = [v_1 \ v_2 \ v_3]$

$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T$

$$A = 4 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \left(1/\sqrt{2} \ \ 0 \ \ 1/\sqrt{2}\right) + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left(0 \ \ 1 \ \ 0\right) + 0 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \left(-1/\sqrt{2} \ \ 0 \ \ 1/\sqrt{2}\right)$$

$$= \underbrace{4 \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}}_{\text{best rank 1 approx of } A.} + 1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} = A$$

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \begin{pmatrix} \underbrace{1/\sqrt{2}}_{v_1} & \underbrace{-1/\sqrt{2}}_{v_2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \overset{\lambda_1}{4} & 0 \\ 0 & \underset{\lambda_2}{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

I want

$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$

FACT

$UU^T \vec{x} = \text{proj}_W(\vec{x})$

$U = [\vec{u}_1 \ \vec{u}_2]$ orthonormal cols. where $\text{Col}\,U = W$.

$$A = 4\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

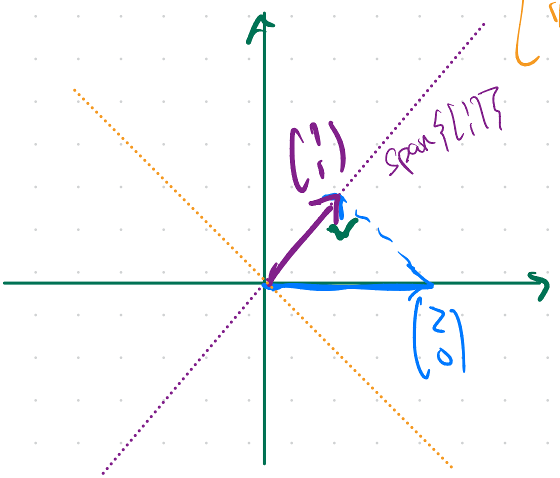
$$= 4\underbrace{\boxed{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}}_{\text{proj } v_1} + 2\underbrace{\boxed{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}}}_{\text{proj } v_2}$$

the spectral decomp is a sum of rank 1 matrices that add up to A.

$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

projection onto line $\text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$



try

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Why?

$$\underbrace{v_1 v_1^T}\vec{x} = (v_1 \cdot x) * \vec{v}_1$$

$$\text{if } \|v_1\| = 1 \rightarrow \ = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1$$

7.1 Exercises

Determine which of the matrices in Exercises 1–6 are symmetric.

1. $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

7. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

10. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

11. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$

12. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix P and a diagonal matrix D. To save you time, the eigenvalues in Exercises 17–22 are the following: (17) $-4, 4, 7$; (18) $-3, -6, 9$; (19) $-2, 7$; (20) $-3, 15$; (21) $1, 5, 9$; (22) $3, 5$.

13. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize A.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Then orthogonally diagonalize A.

In Exercises 25–32, mark each statement True or False (T/F). Justify each answer.

25. (T/F) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

26. (T/F) There are symmetric matrices that are not orthogonally diagonalizable.

27. (T/F) An orthogonal matrix is orthogonally diagonalizable.

28. (T/F) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.

29. (T/F) For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

30. (T/F) If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

31. (T/F) An $n \times n$ symmetric matrix has n distinct real eigenvalues.

32. (T/F) The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue.

33. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

34. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB, and BB^T are symmetric matrices.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

39. Construct a spectral decomposition of A from Example 2.

40. Construct a spectral decomposition of A from Example 3.

41. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.
    a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.
    b. Show that B is a symmetric matrix and $B^2 = B$.
    c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

42. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.
    a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.
    b. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^\perp$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of B?

Orthogonally diagonalize the matrices in Exercises 43–46. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue λ, find an orthonormal basis for $\mathrm{Nul}(A - \lambda I)$, as in Examples 2 and 3.

T 43. $\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$

T 44. $\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$

T 45. $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

T 46. $\begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$

40%

Q: If $\boxed{\vec{x} \notin \text{Nul}\,A,}$ then $\vec{x}$ not orthogonal to 1st row of A. ~~true~~ False.

Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

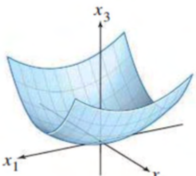
Math 1554 Linear Algebra

$$\underset{A\ \checkmark}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\vec{x}\ \checkmark}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

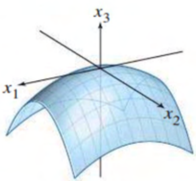
$\vec{x} \notin \text{Nul}\,A$

but $\vec{x}$ orthogonal

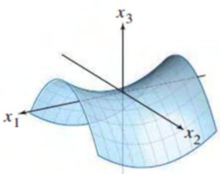
to $\begin{pmatrix} 1 & 1 \end{pmatrix}$ 1st row.



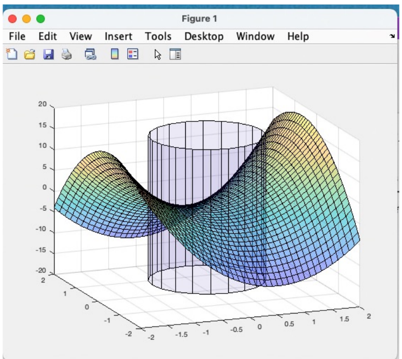
Positive definite

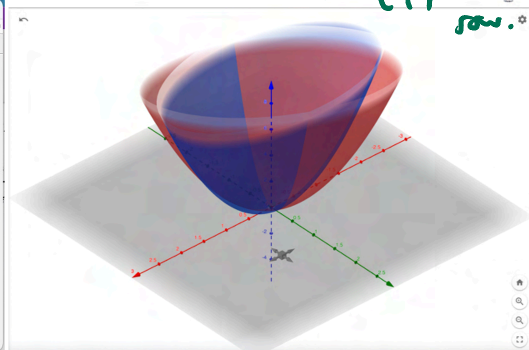


Negative definite



Indefinite





13	3/31 – 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 – 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 – 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 – 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 – 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Why $A^T = A$ is so useful?

Why this $A = PDP^T$ is so good.

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A \vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question Does this inequality hold for all x, y?

$$x^2 - 6xy + 9y^2 \geq 0$$

all terms are degree 2

Quadratic Forms

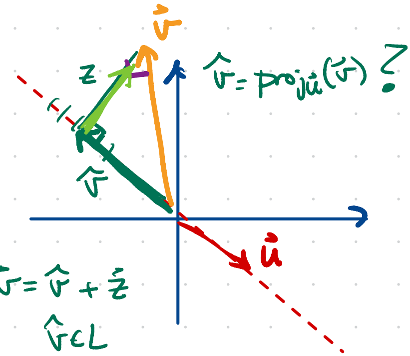
Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.



$\vec{v} = \hat{v} + \vec{z}$

$\hat{v} \in L$

$z \in L^{\perp}$

$$\boxed{Q(x,y) = x^2 - 6xy + 9y^2}$$

$Q(-\vec{x}) = Q(\vec{x})$ for any $\vec{x}$.

Q are EVEN.

Q_1: What is $Q(0,0) = 0$

$Q(1,0) = 1^2 - 6(1)(0) + 9(0)^2 = 1$

$Q(1,-1) = 1^2 - 6(1)(-1) + 9(-1)^2 = 16$

had to be equal.

$Q(-1,1) = (-1)^2 - 6(-1)(1) + 9(1)^2 = 16$

$$\boxed{\overset{\vec{x}^T}{\begin{bmatrix} x & y \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}}} = \begin{bmatrix} x & y \end{bmatrix} \begin{pmatrix} x - 3y \\ -3x + 9y \end{pmatrix}$$

First

$= x(x-3y) + y(-3x+9y)$

$= x^2 - 3xy - 3xy + 9y^2$

$= \boxed{x^2 - 6xy + 9y^2}$

$p(\lambda)$ char poly.

$\boxed{p(\lambda) = 0}$ char eqn.

Example 1

Compute the quadratic form $\vec{x}^T A \vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad \boxed{B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}}$$

$Q_A(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \boxed{4x^2 + 3y^2 = z}$

$Q_B(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{bmatrix} x & y \end{bmatrix} \begin{pmatrix} 4x + y \\ x - 3y \end{pmatrix}$

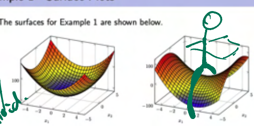
$= x(4x+y) + y(x-3y)$

$= 4x^2 + xy + xy - 3y^2$

$= \boxed{4x^2 + 2xy - 3y^2 = z}$

Example 1 - Surface Plots

The surfaces for Example 1 are shown below.

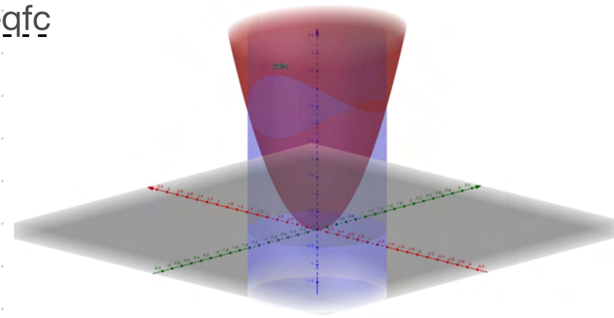


bowl paraboloid

Saddle

Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

https://www.geogebra.org/m/pbzpeqfc



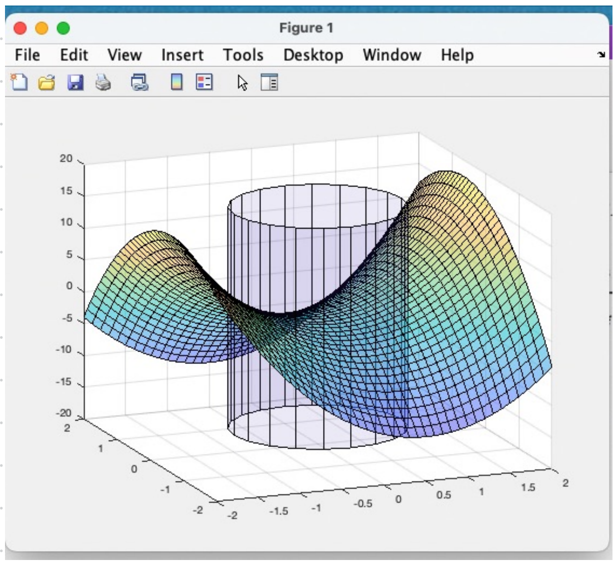
```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



Example 2

Write Q in the form $\vec{x}^T A\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$Q(x) = \boxed{5x_1^2} \boxed{- x_2^2} \boxed{+ 3x_3^2} \boxed{+ 6x_1x_3} \boxed{- 12x_2x_3}$$

$+0x_1x_2$

find this A.

$$Q(x_1,x_2,x_3) = [x_1\ x_2\ x_3]\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

row 1 col 3 $\boxed{x_1x_3}$ term. ✓

x_3x_1 term ✓

Try.

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}$$

row 3 col 1

Section 7.2 Slide 365

Section 7.2 Slide 366

$*$ do first $*$

$$Q_A(\vec{x}) = \vec{x}^T A\vec{x} = [x_1\ x_2\ x_3]\begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1\ x_2\ x_3]\begin{bmatrix} 5x_1 + 3x_3 \\ -x_2 - 6x_3 \\ 3x_1 - 6x_2 + 3x_3 \end{bmatrix}$$

$$= x_1(5x_1+3x_3) + x_2(-x_2-6x_3) + x_3(3x_1 - 6x_2 + 3x_3)$$

$$= 5x_1^2 + 3x_1x_3 - x_2^2 - 6x_2x_3 + 3x_1x_3 - 6x_2x_3 + 3x_3^2$$

$$= \boxed{5x_1^2 + 6x_1x_3 - x_2^2 - 12x_2x_3 + 3x_3^2}$$

Change of Variable

old vars $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

new vars $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\boxed{\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}}$$

With this change of variable, the quadratic form $\vec{x}^T A\vec{x}$ becomes:

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$Q_A(x_1,x_2) = (x_1\ x_2)\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2^2$$

$$Q_D(y_1,y_2) = (y_1\ y_2)\begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 + 7y_2^2$$

Try. $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ plug in $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $Q_D(\vec{y})$.

$$Q_D(1,0) = 2(1)^2 + 7(0)^2 = 2.$$

Next what $\vec{x}$ is corresponding vector if you change $\vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into $P\vec{y} = \vec{x}$?

$$P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$\vec{x} = P\vec{y}$ (what mean?)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 2y_1 + y_2 \\ -y_1 + 2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{5}\, y_1 + 1/\sqrt{5}\, y_2 \\ -1/\sqrt{5}\, y_1 + 2/\sqrt{5}\, y_2 \end{bmatrix}$$

linear change of vars.

$$\boxed{\begin{cases} x_1 = 2/\sqrt{5}\, y_1 + 1/\sqrt{5}\, y_2 \\ x_2 = -1/\sqrt{5}\, y_1 + 2/\sqrt{5}\, y_2 \end{cases}}$$

$\leftarrow y_1 = 1$, $y_2 = 0$?

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A \vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$
$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$
$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$\vec{x} = P\vec{y}$ (what mean?)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2y_1 + y_2 \\ -y_1 + 2y_2 \end{pmatrix}$$
$$= \begin{bmatrix} 2/\sqrt{5}\, y_1 + 1/\sqrt{5}\, y_2 \\ -1/\sqrt{5}\, y_1 + 2/\sqrt{5}\, y_2 \end{bmatrix}$$

linear change of vars.

$$\boxed{\begin{cases} x_1 = 2/\sqrt{5}\, y_1 + 1/\sqrt{5}\, y_2 \\ x_2 = -1/\sqrt{5}\, y_1 + 2/\sqrt{5}\, y_2 \end{cases}}$$

$\leftarrow y_1 = 1$, $y_2 = 0$?

$$Q_A(x_1, x_2) = (x_1\ x_2) \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2^2$$
$$Q_D(y_1, y_2) = (y_1\ y_2) \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 + 7y_2^2$$

Try: $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ plug in $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $Q_D(\vec{y})$.

$$Q_D(1,0) = 2(1)^2 + 7(0)^2 = \boxed{2.}$$

Next what $\vec{x}$ is corresponding vector if you change $\vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into $P\vec{y} = \vec{x}$?

$$\vec{x} = P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

plug in $\vec{x} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ into $Q_A(\vec{x}) = 3x_1^2 + 4x_1x_2 + 6x_3^2$

$$Q_A\left(2/\sqrt{5}, -1/\sqrt{5}\right) = 3\left(\frac{2}{\sqrt{5}}\right)^2 + 4\left(\frac{2}{\sqrt{5}}\right)\left(\frac{-1}{\sqrt{5}}\right) + 6\left(\frac{-1}{\sqrt{5}}\right)^2$$
$$= 3 \cdot \frac{4}{5} + \frac{-8}{5} + 6 \cdot \frac{1}{5} = \frac{12 - 8 + 6}{5} = \frac{10}{5}$$
$$= \boxed{2}$$

Principle Axes Theorem

Spectral thm

$A = PDP^T$
$\leftarrow$ diagonal.
$\leftarrow$ orthogonal

Theorem

If A is a Symmetric. matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{y}^T D\vec{y}$ with no cross-product terms.

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

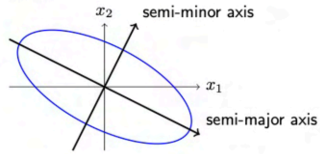
$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

Example 5

Compute the quadratic form $Q = \vec{x}^T A\vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.



$$\boxed{Q_A(\vec{x})} = \vec{x}^T A \vec{x} = \vec{x}^T (PDP^T)\vec{x}$$

$$= (x^T P) D P^T x$$

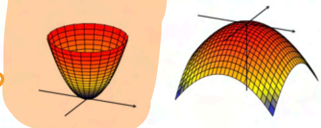
$$= (P^T x)^T D \underbrace{P^T x}_{\vec{y}}$$

Same quad form.

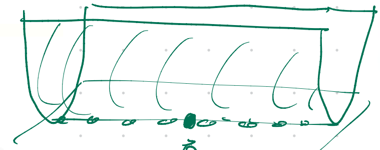
$$= \vec{y}^T D \vec{y} = \boxed{Q_D(\vec{y})}$$

Classifying Quadratic Forms

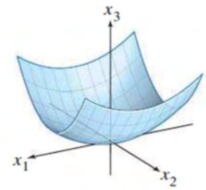
$Q = x_1^2 + x_2^2$ $\qquad$ $Q = -x_1^2 - x_2^2$



is PSD & PD



$\vec{0}$



Positive definite

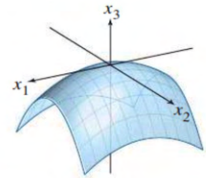
Definition

A quadratic form Q is

PD 1. **positive definite** if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.
2. **negative definite** if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.

PSD 3. **positive semidefinite** if $Q(\vec{x}) \geq 0$ for all $\vec{x}$.
4. **negative semidefinite** if $Q(\vec{x}) \leq 0$ for all $\vec{x}$.
5. **indefinite** if none of the above.

NOTICE every PD quadratic form is PSD but not visa versa.



Negative definite

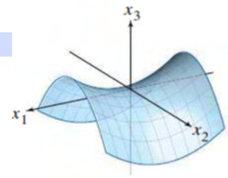
$Q_A(x_1, x_2) = (x_1\ x_2)\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2^2$

$Q_D(y_1, y_2) = (y_1\ y_2)\begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2y_1^2 + 7y_2^2$

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A\vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$
$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$
$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$



Indefinite

Quadratic Forms and Eigenvalues

Theorem

If A is a Symmetric matrix with eigenvalues λ_i, then $Q = \vec{x}^T A\vec{x}$ is

1. **positive definite** iff $\lambda_i > 0$
2. **negative definite** iff $\lambda_i < 0$
3. **indefinite** iff λ_i some pos. & some neg.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

$\underbrace{\qquad}$ control the shape of Q.

PSD Q_A iff $\lambda_i \geq 0$ eigenvalues of A

NSD Q_A iff $\lambda_i \leq 0$ eigenvalues of A.

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y?

$$\underbrace{x^2 - 6xy + 9y^2}_{} \geq 0$$

$Q_A(x,y) \geq 0$? PSD?

$\boxed{\text{yes}}$

Same

$Q_D\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 10y_1^2 + 0y_2^2$

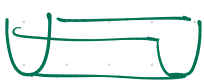
Step 1:

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

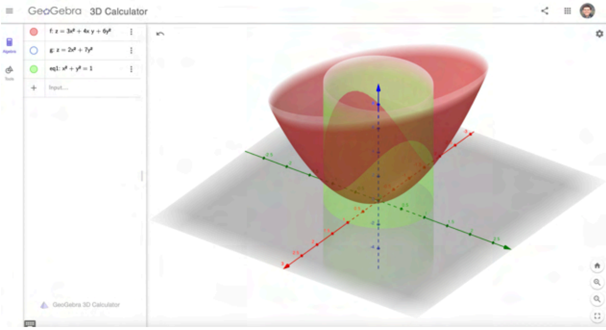
Step 2: Find λ_1, λ_2 of A.

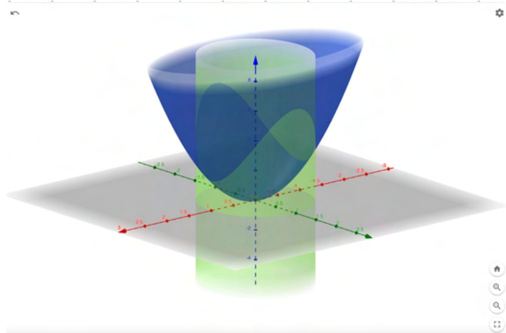
$p(\lambda) = \lambda^2 - 10\lambda + 0$

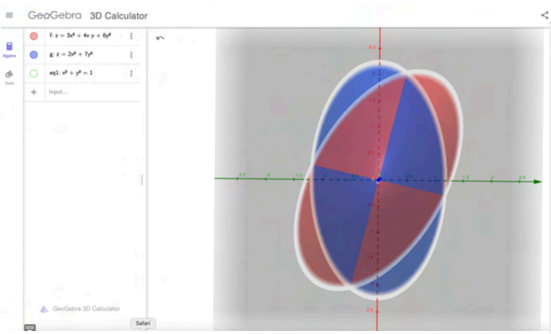
$= \lambda(\lambda - 10) = 0$

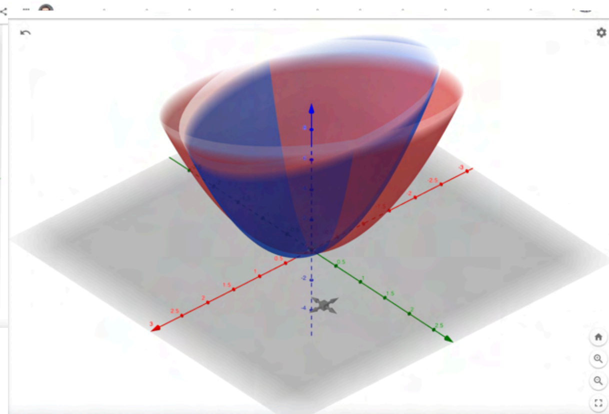


$\boxed{\lambda = 0, 10}$

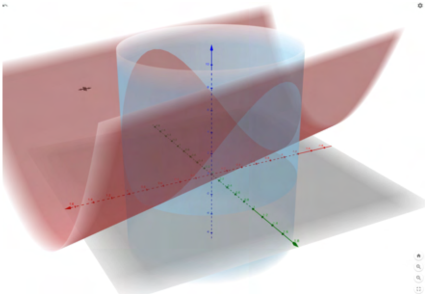


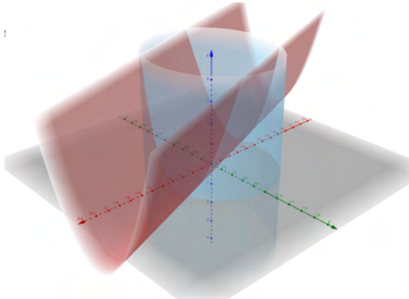


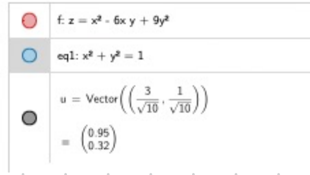


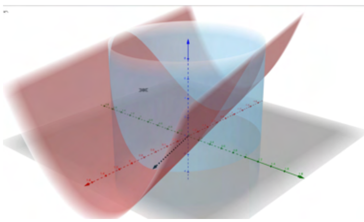


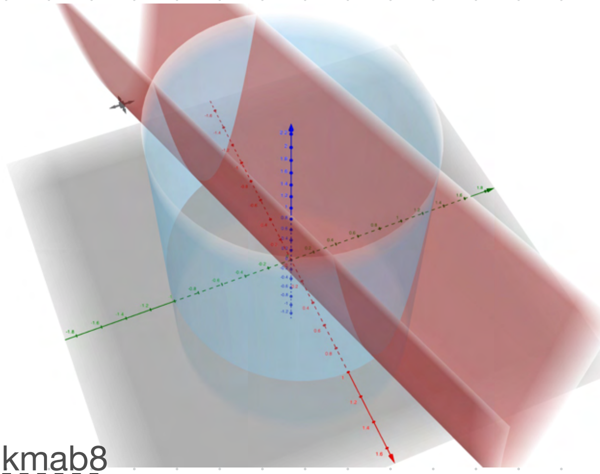
https://www.geogebra.org/m/c6yg2agh











https://www.geogebra.org/m/akbkmab8

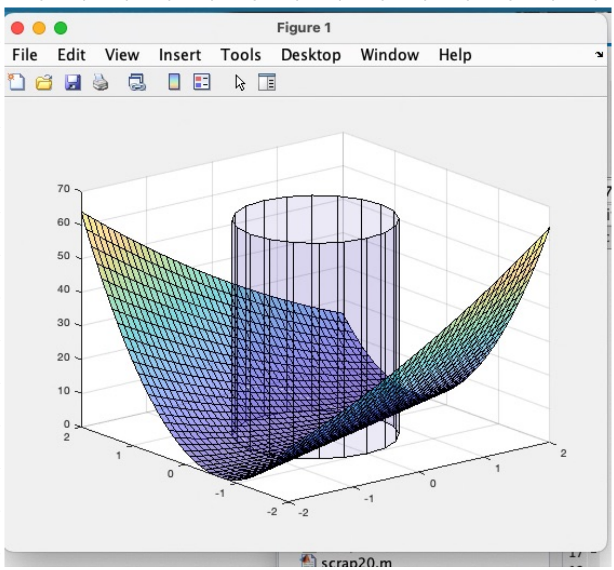
```
clc
format bank
%% example 1a
[X,Y]=meshgrid(-2:.1:2);
Z=4.*X.^2+3.*Y.^2;
[X1,Y1,Z1]=cylinder(1);
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 1b
Z=4.*X.^2+2.*X.*Y-3.*Y.^2;
%s=surf(X,Y,Z,'FaceAlpha',0.5); hold on


%% example 6
%[X,Y]=meshgrid(-2:.2:2);
Z=X.^2-6.*X.*Y+9.*Y.^2;
s=surf(X,Y,Z,'FaceAlpha',0.5); hold on
[P,D]=eig([1 -3 ; -3 9])
A=P*D*inv(P)
rref(A-10*eye(2))

%% plots cylinder
h=max(Z(:));
Z1=Z1*h;
%Z1(1,:)=-Z1(2,:);
c=surf(X1,Y1,Z1,'FaceAlpha',0.1); hold on


%% no errors check
1+1
```



7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and

   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and

   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $3x_1^2 - 4x_1x_2 + 5x_2^2$ b. $3x_1^2 + 2x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $5x_1^2 + 16x_1x_2 - 5x_2^2$ b. $2x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$
   b. $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $3x_1^2 - 2x_2^2 + 5x_3^2 + 4x_1x_2 - 6x_1x_3$
   b. $4x_3^2 - 2x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

8. Let A be the matrix of the quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

   It can be shown that the eigenvalues of A are 3, 9, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct P using the methods of Section 7.1.

9. $4x_1^2 - 4x_1x_2 + 4x_2^2$
10. $2x_1^2 + 6x_1x_2 - 6x_2^2$
11. $2x_1^2 - 4x_1x_2 - x_2^2$
12. $-x_1^2 - 2x_1x_2 - x_2^2$
13. $x_1^2 - 6x_1x_2 + 9x_2^2$
14. $3x_1^2 + 4x_1x_2$
15. [M] $-3x_1^2 - 7x_2^2 - 10x_3^2 - 10x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$
16. [M] $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 + 8x_3x_4 - 6x_1x_4 + 6x_2x_3$
17. [M] $11x_1^2 + 11x_2^2 + 11x_3^2 + 11x_4^2 + 16x_1x_2 - 12x_1x_4 + 12x_2x_3 + 16x_3x_4$
18. [M] $2x_1^2 + 2x_2^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.
    b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.
    c. The principal axes of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are eigenvectors of A.
    d. A positive definite quadratic form Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
    e. If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.
    f. A Cholesky factorization of a symmetric matrix A has the form $A = R^T R$, for an upper triangular matrix R with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is not a quadratic form.
    b. If A is symmetric and P is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no cross-product term.
    c. If A is a 2×2 symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola.
    d. An indefinite quadratic form is neither positive semidefinite nor negative semidefinite.
    e. If A is symmetric and the quadratic form $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of A are all positive.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of A.

23. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.
    a. Q is positive definite if $\det A > 0$ and $a > 0$.
    b. Q is negative definite if $\det A > 0$ and $a < 0$.
    c. Q is indefinite if $\det A < 0$.

25. Show that if B is $m \times n$, then $B^T B$ is positive semidefinite; and if B is $n \times n$ and invertible, then $B^T B$ is positive definite.

26. Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.]

B works.]

27. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let A be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

SG Mastering: Diagonalization and Quadratic Forms 7-7

Section 7.3 : Constrained Optimization

$Q_A(x_1, x_2) = 3x_1^2 + 7x_2^2$ max?

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ unit length?

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

$Q(x_1, x_2, x_3) = 2x_1^2 + x_2^2 - x_3^2$

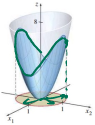


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

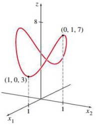
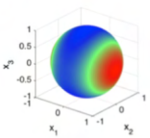


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.



13	3/31 – 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 – 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 – 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 – 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 – 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

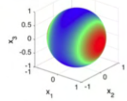
Example 1

The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = ||\vec{x}||^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$



Find the largest and smallest values of Q on the surface of the sphere.

Sanity check: plug in random unit vectors $\vec{x}$ & evaluate $Q(\vec{x})$

$Q(1,0,0) = 9(1)^2 + 4(0)^2 + 3(0)^2 = \boxed{9}$ MAX

$Q(0,1,0) = 9(0)^2 + 4(1)^2 + 3(0)^2 = 4$

$Q(0,0,1) = 9(0)^2 + 4(0)^2 + 3(1)^2 = \boxed{3}$ MIN

$Q(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}) = 9(\frac{1}{\sqrt3})^2 + 4(\frac{1}{\sqrt3})^2 + 3(\frac{1}{\sqrt3})^2 = \frac{9+4+3}{3} = \frac{16}{3} \approx 5.\overline{33}$

$Q(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}, 0) = 9(\frac{1}{\sqrt2})^2 + 4(\frac{-1}{\sqrt2})^2 + 3(0)^2 = \frac{9(1)+4(1)+3(0)}{2} = \frac{13}{2} = 6.5$

$Q(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) = 9(\frac{2}{3})^2 + 4(\frac{2}{3})^2 + 3(\frac{1}{3})^2 = \frac{9(4)+4(4)+3(1)}{9} = \frac{55}{9} \approx 6.\overline{11}$

Punchline. $Q_A(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ has max value 9 min value 3 when inputs have $||\vec{x}|| = 1$.

Ex. Find the largest output z-value with restricted input ||x||=1 where z is given by:

$Q(\vec{x}) = 3x_1^2 + 7x_2^2.$

Q_1: what is MAX/MIN of $Q(\vec{x})$ when $||\vec{x}|| = 1$?

Q_2: what inputs give MAX/MIN value?

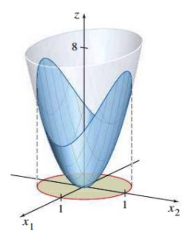


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

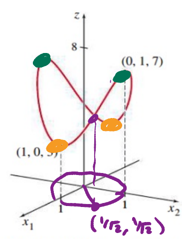


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.

MAX is 7 happens at $(0,1)$ & $(0,-1)$

MIN is 3 happen at $(1,0)$ & $(-1,0)$

Bonus.

$Q(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}) = 3(\frac{1}{\sqrt5})^2 + 7(\frac{2}{\sqrt5})^2 = 3(\frac{1}{5}) + 7(\frac{4}{5}) = \frac{3(1)+7(4)}{5} = \frac{31}{5} = 6.2$

EXAMPLE 3 Let $A = \begin{array}{ccc} x_1 & x_2 & \\ \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \end{array}$. Find the maximum value of the quadratic form $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, and find a unit vector at which this maximum value is attained.

Q_A ???

$A = PDP^T$

$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$x \cdot x = 1$ so $\|x\|^2 = 1$.

$P = [v_1 \; v_2 \; v_3]$ ↑ ↑ ↑ eigenvectors (orthonormal)

SOLUTION By Theorem 6, the desired maximum value is the greatest eigenvalue of A. The characteristic equation turns out to be

$$0 = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = \boxed{-(\lambda-6)(\lambda-3)(\lambda-1)}$$

$\lambda_1 = 6, \; \lambda_2 = 3, \; \lambda_3 = 1.$

The greatest eigenvalue is 6.

$$\to \boxed{Q_A(x_1, x_2, x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3} \quad (?)$$

So

$$\boxed{Q_D(y_1, y_2, y_3) = 6y_1^2 + 3y_2^2 + y_3^2}$$

$\boxed{\begin{matrix} y = P^T x \\ x = Py \end{matrix}}$ Change of variables.

$\boxed{Q_A(\vec{x}) = x^T A x}$ $\boxed{Q_D(\vec{y}) = y^T D y.}$

ANS. $Q_A(\vec{x})$ is between $\underline{\underline{6}}$ and $\underline{\underline{1}}$ for any $\|\vec{x}\| = 1$.

ans.

Question: What is the input(s) that give output 6 for $Q_A(\vec{x})$?

$Q_D(1,0,0) = 6$

$\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{e}_1$

to get $\vec{v}_1$ unit length eigenvector for A w/ $\lambda_1 = 6$.

$P\vec{y} = \vec{x}$

$P\vec{e}_1 = \vec{v}_1$

$\underline{\lambda = 6}$

$$A - 6I = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} - 6I_3 = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Q_A(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 3(1/\sqrt{3})^2 + 3(1/\sqrt{3})^2 + 4(1/\sqrt{3})^2 + 4(1/\sqrt{3})(1/\sqrt{3}) + 2(1/\sqrt{3})(1/\sqrt{3}) + 2(1/\sqrt{3})(1/\sqrt{3})$$

$$= \frac{1}{3}(3 + 3 + 4 + 4 + 2 + 2) = 18/3 = \boxed{6}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

~~**EXAMPLE 5** Let A be the matrix in Example 3 and let $\mathbf{u}_1$ be a unit eigenvector corresponding to the greatest eigenvalue of A. Find the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the conditions~~

$$\mathbf{x}^T\mathbf{x} = 1, \quad \mathbf{x}^T\mathbf{u}_1 = 0 \tag{4}$$

$$Q_A(\vec{v}) = v^T A\vec{v} = v^T 6v = 6v^T v = 6\, v \cdot v$$

$$= 6\|v\|^2 = \boxed{6}$$

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

A Symmetric.

subject to

$$||\vec{x}|| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$
$$M = \max\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $||\vec{x}|| = 1$,
- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm\vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm\vec{u}_n$.

Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

$$\boxed{Q(\vec{x}) = x_1^2 + 2x_2x_3}$$

✓ Step 1 : Find A Symmetric s.t. $Q(\vec{x}) = x^T A x$.

✓ Step 2: $A = PDP^T$ Find P, D.

Step 3: write down $Q_D(\vec{y}) = y^T D y$ the quadratic form for D

Soln: ① $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

② $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ ($v_1/||v_1||$, $v_2/||v_2||$, $v_3/||v_3||$)

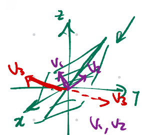
$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ & $y^T D y$

reflecting across $\boxed{y = z}$

e.g. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in W$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in W$ $\Big\}$ $\lambda_1 = 1 = \lambda_2$

$v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in W^\perp$ $\lambda_3 = -1$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. plane $y = z$ is W.



v_1, v_2 in plane reflecting through

$\boxed{\begin{aligned} Av_1 &= v_1 \\ Av_2 &= v_2 \end{aligned}}$ $\lambda_1 = 1$, $\lambda_2 = 1$

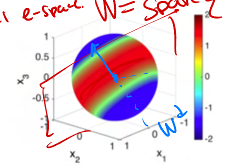
$Av_3 = -v_3$ $\lambda_3 = -1$

if $v_3 \in W^\perp$

Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.

$\lambda = 1$ e-space $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$



Sanity check

$Q(0, 1/\sqrt{2}, 1/\sqrt{2}) =$

$0^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= 2 \cdot \frac{1}{2}$

$= \boxed{1}$ ✓

③ $Q_D(y_1, y_2, y_3) = y_1^2 + y_2^2 - y_3^2$

MAX VALUE w/ any input $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ as big as you want!

$\boxed{y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} ✓ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \ldots}$

these all $\vec{y}$'s give max value of 1 for Q_D.

MAX value is 1 at restrict $||y|| = 1$.

$\vec{y} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in W. ($\lambda = 1$ eigenspace)

For $\vec{x}$'s. pick one $\vec{y}$, push it to an $\vec{x}$ by $\boxed{\vec{x} = P\vec{y}}$

e.g. $y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{x} = P\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

e.g. $y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then $\vec{x} = P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A \vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_2$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $\boxed{||\vec{x}|| = 1 \text{ and to } \vec{x} \cdot \vec{u}_1 = 0,}$ and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$Q_D(y_1, y_2, y_3) = \cancel{y_1^2} + y_2^2 - y_3^2$

plug in y s.t. $||y|| = 1$

& $\boxed{y \cdot \vec{e}_1 = 0}$

So $y = \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}$

new location

$\cancel{u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$, $\boxed{u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\cancel{\lambda_1 = 1}$ $\boxed{\lambda_2 = 1}$ $\lambda_3 = -1$

new max value.

Move along the list of eigenvalues eigenvectors to the next in the list

For A

$\cancel{u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ $\boxed{u_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$, $u_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\cancel{\lambda_1 = 1}$ $\boxed{\lambda_2 = 1}$ $\lambda_3 = -1$

new location

Example 4 (if time permits)

new max value.

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A \vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

New restriction $||\vec{y}|| = 5$ $\longrightarrow$ $Q(\vec{x}) = x_1^2 + 2x_2x_3$

recall max value w/ $||y|| = 1$ was 1 occurred at e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$Q(c\vec{x}) = (cx_1)^2 + 2(cx_2)(cx_3)$
$= c^2 x_1^2 + c^2 2x_2x_3 = c^2 Q(\vec{x})$

$Q_A\left(5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = Q(5,0,0) = (5)^2 + 2(0)(0) = \boxed{25}$

$Q_A\left(5\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right) = (0)^2 + 2\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = 2\frac{25}{2} = \boxed{25}$

punchline if $Q(\vec{x}_0) = k$

Then $\boxed{Q(c\vec{x}) = c^2 k}$

7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2+6x_2^2+7x_3^2+4x_1x_2-4x_2x_3=9y_1^2+6y_2^2+3y_3^2$

2. $3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3=7y_1^2+4y_2^2$

Hint: $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for y_3^2.

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$ (See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2+3x_2^2+5x_3^2+6x_1x_2+2x_1x_3+2x_2x_3$ (See Exercise 2.)

5. $Q(\mathbf{x}) = x_1^2 + x_2^2 - 10x_1x_2$

6. $Q(\mathbf{x}) = 3x_1^2 + 9x_2^2 + 8x_1x_2$

7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 2, −1, and −4.]

8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form Q are 9 and −3.]

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^T A\mathbf{x}$?

12. Let λ be any eigenvalue of a symmetric matrix A. Justify the statement made in this section that $m \le \lambda \le M$, where m and M are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^T A\mathbf{x}$.]

13. Let A be an $n \times n$ symmetric matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, where $\mathbf{x}^T\mathbf{x} = 1$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

[M] In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$

15. $4x_1^2-6x_1x_2-10x_1x_3-10x_1x_4-6x_2x_3-6x_2x_4-2x_3x_4$

16. $-6x_1^2-10x_2^2-13x_3^2-13x_4^2-4x_1x_2-4x_1x_3-4x_1x_4+6x_3x_4$

17. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$