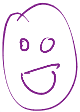
interesting!

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



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Topics and Objectives

Topics

We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question

Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

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Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

A is mxn matrix

① Col A subspace of $\mathbb{R}^m$ (m = # rows of A)

Col A = "span of cols of A"

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

Col A = Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

② Nul A = $\{ \vec{x} \mid A\vec{x} = \vec{0} \}$

Solns to $A\vec{x} = \vec{0}$

$\vec{x} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Nul A = span$\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Subsets of $\mathbb{R}^n$

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

Subspaces in $\mathbb{R}^n$

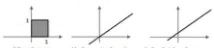
Definition

A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

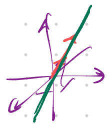
1. $c\vec{u} \in H$ closed under scalar multiplication
2. $\vec{u} + \vec{v} \in H$ closed under vector addition.

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square b) a line passing through the origin c) a line that doesn't pass through the origin



Ex.

✓ ① $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a subset of $\mathbb{R}^3$ — trivial subspace.

X ② $H = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is a subset of $\mathbb{R}^3$ $c = 5$ $v = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $cv = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} \notin H$

NP ③ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ is NOT a subset of $\mathbb{R}^2$

✓ ④ $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ (non-proper)

✓ ⑤ span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ a subset of $\mathbb{R}^3$?

Subsets of $\mathbb{R}^n$

Definition
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Definition
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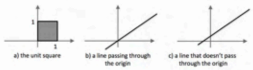

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pictures of subspaces?

e.g.,

*three vectors

has 3 vectors in it.

$H_3 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

is H a subspace of $\mathbb{R}^3$?

$c = 2 \in \mathbb{R}$ ✓, $v = \begin{bmatrix}1\\0\\0\end{bmatrix} \in H$ ✓ $\quad cv = 2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix} \in H_3$? ?

So H_3 not closed under scalar mult.
Also H_3 not closed under vector add. e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ in H_3 ... not in H_3

*the span of three vectors

This set H_4 has so-many vectors!

$H_4 = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

IS A SUBSPACE of $\mathbb{R}^3$ ✓

is a subspace!

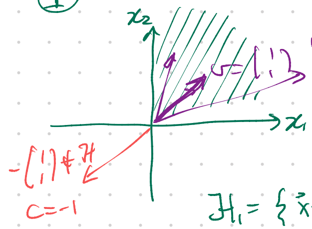
① Suppose $\vec{v} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ arb. element of H_4.
Then is $k\vec{v} \in H_4$?
$kv = ka\begin{bmatrix}1\\0\\0\end{bmatrix} + kb\begin{bmatrix}0\\1\\0\end{bmatrix} + kc\begin{bmatrix}0\\0\\1\end{bmatrix}$ } H_4 closed under scalar mult.

② Suppose $v = a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + b_1\begin{bmatrix}0\\1\\0\end{bmatrix} + c_1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}$
$u = a_2\begin{bmatrix}1\\0\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}$
is $u+v \in H_4$? $u = (a_1+a_2)\begin{bmatrix}1\\0\\0\end{bmatrix} + (b_1+b_2)\begin{bmatrix}0\\1\\0\end{bmatrix} + (c_1+c_2)\begin{bmatrix}0\\0\\1\end{bmatrix} \in H_4$. } closed under vector add.

*the zero

$H_0 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$ ✓ $\quad H_0$ is a subspace of $\mathbb{R}^3$.

① in $\mathbb{R}^2$



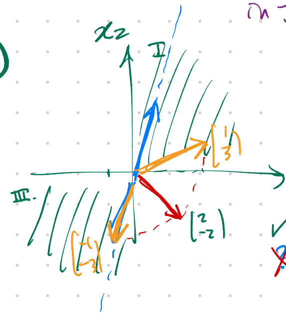
$H_1 = \{ \vec{x} \in \mathbb{R}^2 \mid x_1, x_2 \geq 0 \}$

NO 1. closed under scalar mult?
YES 2. closed under vector add?

$\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix} = \begin{pmatrix}a+c\\b+d\end{pmatrix}$ $\quad a+c \geq 0$ ✓ $\quad b+d \geq 0$

in $H_1 \Rightarrow a, b, c, d \geq 0$

②



$H_2 = \left\{ \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 x_2 \geq 0 \right\}$

✓ 1. closed under scalar mult?
✗ 2. closed under vector add?

$\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}-1\\-3\end{bmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$

in H_2 $\quad$ in H_2 $\quad$ not in H_2

*all vectors in R^2 that are either on the x-axis or on the y-axis

$H_5 = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 = 0 \text{ or } x_2 = 0 \right\}$
$= \{ \vec{x} \in \mathbb{R}^2 \mid x_1 x_2 = 0 \}$

$v = \begin{pmatrix}0\\5\end{pmatrix}$ $\quad kv = \begin{bmatrix}0\\5k\end{bmatrix}$ ✓

$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{pmatrix}0\\5\end{pmatrix}, \cancel{\begin{pmatrix}4\\5\end{pmatrix}}$

Q: is H_5 a subspace of $\mathbb{R}^2$? $\boxed{\text{NO}}$

$u = \begin{bmatrix}3\\0\end{bmatrix}$ $\quad k\vec{u} = \begin{bmatrix}3k\\0\end{bmatrix}$ ✓

✓ Q1: Is H_5 closed under scalar mult? ✓ yes
✗ Q2: Is H_5 closed under vector add?

$u + v = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}0\\5\end{bmatrix} = \begin{pmatrix}3\\5\end{pmatrix} \notin H_5$.

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, that $\boxed{\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}}$ is:

The subset of $\mathbb{R}^n$ consisting of all linear combinations of $v_1, \dots, v_p$.

This is a **subspace**, spanned by $\vec{v}_1, \dots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$

1. The **column space of** A, Col A, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \dots, \vec{a}_n$.
2. The **null space of** A, Null A, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

Example

Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

Q: $\boxed{\text{Is } \vec{b} \in \text{Col}A\,?}$ ✓ yes.

Q₂: rewrite this question Q₁ as a better one?

$$\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} \right\} \quad (\text{rewrite of } \vec{b} \in \text{Col}A)$$

$$\Leftrightarrow \boxed{\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}$$

$A\vec{x} = \vec{b}$. is a consistent vector eqn.

$\Leftrightarrow$ row reduce $\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] = [A \mid \vec{b}]$ & no pivot in $\vec{b}$ column

Work

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & 2 & 6 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no pivot in aug col. So $Ax=b$ consistent

Q: easy way to check?

Find the weights by getting
↳ RREF & check that they work.

Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R} \qquad \lambda = 1$$

$\vec{v} \in \text{Nul}A$?

$$v = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \overset{?}{\in} \text{Nul}A = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \}$$

YES b/c $A\vec{v} = \vec{0}$.

$$A\vec{v} = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix} \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \overset{?}{=} \begin{bmatrix} -5+9-4 \\ 20-18-2 \\ 15-21+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

New Q: $\boxed{\text{Is } 2\vec{v} \in \text{Nul}A\,?}$ $\quad 2\vec{v} = \begin{bmatrix} -10 \\ -6 \\ 2 \end{bmatrix} \overset{?}{\in} \text{Nul}A\,?$

YES: idea $A(2\vec{v}) = 2A\vec{v} = 2\vec{0} = \vec{0}$ ✓

↑ b/c $A\vec{v} = \vec{0}$

New Q: $\boxed{\text{Suppose } \vec{u}, \vec{v} \in \text{Nul}A \quad \vec{u} + \vec{v} \overset{?}{\in} \text{Nul}A\,?}$

yes

$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$ ✓

$$\boxed{\text{Nul}A = \text{span}\left\{ \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \right\}}$$

Q: Is there another way to think about "Nul A is a subspace"

$A\vec{v} \dots \sim$ RREF

$$\vec{x} = s \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}$$

$\mathcal{V} = \boxed{\{ f : [0,1] \to \mathbb{R} \mid \text{cont.} \}}$ $\cancel{\text{functions}}$ $f : [0,1] \to \mathbb{R}$

$3 \sin(x) + 4 \cos(x) = g(x)$

Basis

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

The set $H = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0\}$ is a subspace.

a) H is a null space for what matrix A?

b) Construct a basis for H.

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.
b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

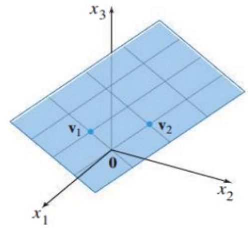


FIGURE 1
Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

Theorem
The pivotal columns a matrix A form a basis for the Column space of A.

Use the pivotal columns of A, not the pivotal columns of the Echelon form.

Theorem
Suppose that the matrix A has reduced echelon form $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, in block matrix form. Then a basis of the Null space of A is given by the columns of $\begin{bmatrix} F \\ -I \end{bmatrix}$.

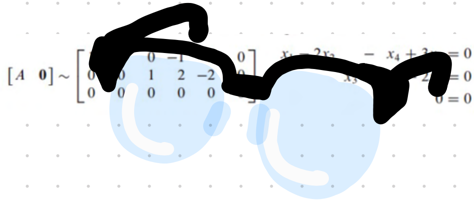
The assumption says that the first few columns are pivotal, and the last few are all free. This can be assumed, after the exchange of columns.

Additional Example (if time permits)

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$. Is V a subspace?

EXAMPLE 6 Find a basis for the null space of the matrix

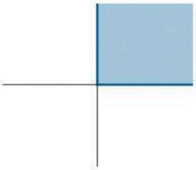
$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$



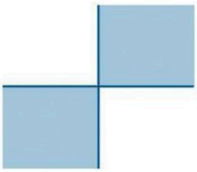
2.8 EXERCISES

Exercises 1–4 display sets in $\mathbb{R}^2$. Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of $\mathbb{R}^2$. (For instance, find two vectors in H whose sum is *not* in H, or find a vector in H with a scalar multiple that is not in H. Draw a picture.)

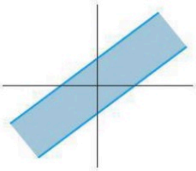
1.



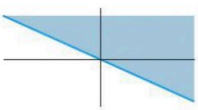
2.



3.



4.



5. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$. Determine if $\mathbf{u}$ is in the subspace of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$, and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Col A?

c. Is $\mathbf{p}$ in Col A? Why or why not?

8. Let $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$. Determine if $\mathbf{p}$ is in Col A, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

10. With $\mathbf{u} = (-2, 3, 1)$ and A as in Exercise 8, determine if $\mathbf{u}$ is in Nul A.

In Exercises 11 and 12, give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$.

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$

13. For A as in Exercise 11, find a nonzero vector in Nul A and a nonzero vector in Col A.

14. For A as in Exercise 12, find a nonzero vector in Nul A and a nonzero vector in Col A.

Determine which sets in Exercises 15–20 are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

15. $\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix}$

16. $\begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

19. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A subspace of $\mathbb{R}^n$ is any set H such that (i) the zero vector is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

b. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the same as the column space of the matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$.

c. The set of all solutions of a system of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^m$.

d. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

e. Row operations do not affect linear dependence relations among the columns of a matrix.

22. a. A subset H of $\mathbb{R}^n$ is a subspace if the zero vector is in H.

b. Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

c. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

d. The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

Exercises 23–26 display a matrix A and an echelon form of A. Find a basis for Col A and a basis for Nul A.

23. $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Col A, but $\mathbf{b}$ is not the same as any one of the columns of A.

28. Construct a nonzero 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is *not* in Col A.

29. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Nul A.

30. Suppose the columns of a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is a basis for Col A.

In Exercises 31–36, respond as comprehensively as possible, and justify your answer.

31. Suppose F is a 5×5 matrix whose column space is not equal to $\mathbb{R}^5$. What can you say about Nul F?

32. If R is a 6×6 matrix and Nul R is *not* the zero subspace, what can you say about Col R?

33. If Q is a 4×4 matrix and Col $Q = \mathbb{R}^4$, what can you say about solutions of equations of the form $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^4$?

34. If P is a 5×5 matrix and Nul P is the zero subspace, what can you say about solutions of equations of the form $P\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^5$?

35. What can you say about Nul B when B is a 5×4 matrix with linearly independent columns?

36. What can you say about the shape of an $m \times n$ matrix A when the columns of A form a basis for $\mathbb{R}^m$?

[M] In Exercises 37 and 38, construct bases for the column space and the null space of the given matrix A. Justify your work.

37. $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{bmatrix}$

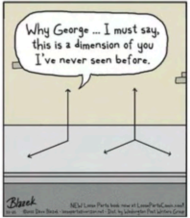
WEB Column Space and Null Space

WEB A Basis for Col A

Section 2.9 : Dimension and Rank

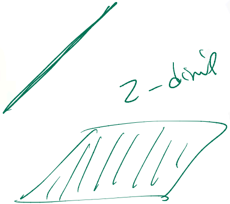
Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



1-dim'l

2-dim'l



Topics and Objectives

Topics

We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives

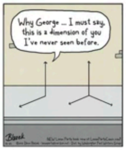
For the topics covered in this section, students are expected to be able to do the following.

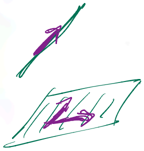
1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra





Topics and Objectives

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course mate

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 - 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 - 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 - 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 - 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/3 - 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 - 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 - 3/21	Break	Break	Break	Break	Break
12	3/24 - 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 - 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 - 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 - 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 - 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

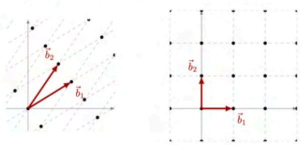
Defn: If $H \subseteq \mathbb{R}^n$ is a subspace then

$\dim H = $ # vectors in a basis of H.

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.



Coordinates

Definition
Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \dots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

And

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinate vector of $\vec{x}$**

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ basis for $\mathbb{R}^3$

standard coord.

$$\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Coordinates in other basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Example 1

Let $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{x} = \begin{bmatrix}5\\3\\5\end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

Dimension

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples:

1. $\dim \mathbb{R}^n =$
2. $H = \{(x_1, \ldots, x_n) \,:\, x_1 + \cdots + x_n = 0\}$ has dimension
3. $\dim(\mathrm{Null}\, A)$ is the number of
4. $\dim(\mathrm{Col}\, A)$ is the number of

Rank

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{bmatrix}2 & 5 & -3 & -4 & 8\\4 & 7 & -4 & -3 & 9\\6 & 9 & -5 & 2 & 4\\0 & -9 & 6 & 5 & -6\end{bmatrix} \sim \cdots \sim \begin{bmatrix}2 & 5 & -3 & -4 & 8\\0 & -3 & 2 & 5 & -7\\0 & 0 & 0 & 4 & -6\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

If a matrix A has n columns, then $\mathrm{Rank}\, A + \dim(\mathrm{Nul}\, A) = n$.

Theorem (Basis Theorem)

Any two bases for a subspace have the same cardinality.

Theorem (Invertibility Theorem)

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. *A is invertible.*
2. *The columns of A are a basis for $\mathbb{R}^n$.*
3. $\mathrm{Col}\, A = \mathbb{R}^n$.
4. $\mathrm{rank}\, A = \dim(\mathrm{Col}\, A) = n$.
5. $\mathrm{Null}\, A = \{0\}$.

Example

If possible, give an example of a 2×3 matrix A, in reduced echelon form, with the given properties.

a) $\text{rank}(A) = 3$

b) $\text{rank}(A) = 2$

c) $\dim(\text{Null}(A)) = 2$

d) $\text{Null}A = \{0\}$

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

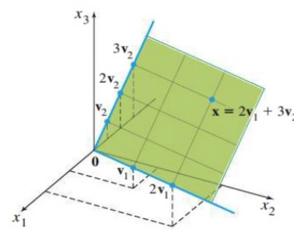


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

statements in the original theorem in Section 2.3.

THEOREM **The Invertible Matrix Theorem (continued)**

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

2.9 EXERCISES

In Exercises 1 and 2, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

In Exercises 3–6, the vector $\mathbf{x}$ is in a subspace H with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

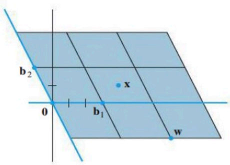
3. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

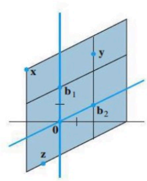
5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$

7. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{w}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}}$. Confirm your estimate of $[\mathbf{x}]_{\mathcal{B}}$ by using it and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{x}$.



8. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_{\mathcal{B}}$, $[\mathbf{y}]_{\mathcal{B}}$, and $[\mathbf{z}]_{\mathcal{B}}$. Confirm your estimates of $[\mathbf{y}]_{\mathcal{B}}$ and $[\mathbf{z}]_{\mathcal{B}}$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{y}$ and $\mathbf{z}$.



Exercises 9–12 display a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

9. $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 13 and 14, find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

13. $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}$

15. Suppose a 3×5 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

16. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here A is an $m \times n$ matrix.

17. a. If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then $c_1, \ldots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$.

b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

c. The dimension of Col A is the number of pivot columns of A.

d. The dimensions of Col A and Nul A add up to the number of columns of A.

e. If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

18. a. If $\mathcal{B}$ is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.

b. If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ makes H look and act the same as $\mathbb{R}^p$.

c. The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.

d. The dimension of the column space of A is rank A.

e. If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.

In Exercises 19–24, justify each answer or construction.

19. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

20. What is the rank of a 4×5 matrix whose null space is three-dimensional?

21. If the rank of a 7×6 matrix A is 4, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

22. Show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5\}$ in $\mathbb{R}^n$ is linearly dependent when dim Span $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5\} = 4$.

23. If possible, construct a 3×4 matrix A such that dim Nul $A = 2$ and dim Col $A = 2$.

24. Construct a 4×3 matrix with rank 1.

25. Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.

26. Suppose columns 1, 3, 5, and 6 of a matrix A are linearly independent (but are not necessarily pivot columns) and the rank of A is 4. Explain why the four columns mentioned must be a basis for the column space of A.

Section 3.1 : Introduction to Determinants

Chapter 3 : Determinants

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. The definition and computation of a determinant
2. The determinant of triangular matrices

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Compute determinants of $n \times n$ matrices using a cofactor expansion.
2. Apply theorems to compute determinants of matrices that have particular structures.

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Math 1554 Linear Algebra

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5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

A Definition of the Determinant

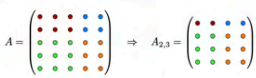
Suppose A is $n \times n$ and has elements a_{ij}.

1. If $n = 1$, $A = [a_{11}]$, and has determinant $\det A = a_{11}$.
2. Inductive case: for $n > 1$,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

where A_{ij} is the submatrix obtained by eliminating row i and column j of A.

Example



Example 1

Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Example 2

Compute $\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{vmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix}$.

Cofactors

Cofactors give us a more convenient notation for determinants.

Definition: Cofactor

The (i, j) cofactor of an $n \times n$ matrix A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

The pattern for the negative signs is

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Theorem

The determinant of a matrix A can be computed down any row or column of the matrix. For instance, down the j^{th} column, the determinant is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This gives us a way to calculate determinants more efficiently.

Example 3

Compute the determinant of $\begin{bmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$.

Triangular Matrices

Theorem

If A is a triangular matrix then

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Example 4

Compute the determinant of the matrix. Empty elements are zero.

$$\begin{bmatrix} 2 & 1 & & & & & \\ & 2 & 1 & & & & \\ & & 2 & 1 & & & \\ & & & 2 & 1 & & \\ & & & & 2 & 1 & \\ & & & & & 2 & 1 \\ & & & & & & 2 \end{bmatrix}$$

Computational Efficiency

Note that computation of a co-factor expansion for an $N \times N$ matrix requires roughly $N!$ multiplications.

- A 10×10 matrix requires roughly $10! = 3.6$ million multiplications
- A 20×20 matrix requires $20! \approx 2.4 \times 10^{18}$ multiplications

This doesn't mean that determinants are not useful.

- We will explore other methods that further the efficiency of their calculation.
- Determinants are very useful in multivariable calculus for solving certain integration problems.

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

9. $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$

10. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 5 & -6 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

$$\begin{array}{ccccc} & & - & - & - \\ \left[\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right. & \left.\right] & \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \\ & + & + & + & \end{array}$$

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to 4×4 or larger matrices.*

15. $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 3 & 1 \\ 4 & -5 & 0 \\ 3 & 4 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -3 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix}$

18. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

21. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

22. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$

23. $\begin{bmatrix} a & b & c \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ a & b & c \\ 4 & 5 & 6 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39. a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.
    b. The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

"A problem isn't finished just because you've found the right answer."
- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

Topics and Objectives

Topics
We will cover these topics in this section.

- The relationships between row reductions, the invertibility of a matrix, and determinants.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.
2. Use determinants to determine whether a square matrix is invertible.

Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

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- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

cofactor expansion is expensive $\mathcal{O}(n!)$ BAD

Idea: have a better way to compute $\det(A)$ using the fact that for upper/lower triangular matrix B computing $\det(B)$ is just the of diagonal entries.

Topics and Objectives

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We will cover these topics in this section.

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1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.
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6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

Row Operations

- We saw how determinants are difficult or impossible to compute with a cofactor expansion for large N.
- Row operations give us a more efficient way to compute determinants.

Theorem: Row Operations and the Determinant
Let A be a square matrix.
1. If a multiple of a row of A is added to another row to produce B, then $\det B = \det A$.
2. If two rows are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by a scalar k to produce B, then $\det B = k \det A$.

Example 1 Compute $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

elementary matrix

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ $\boxed{R_1 \leftrightarrow R_2}$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ✓ $\boxed{-2R_3 \to R_3}$

$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?? $\boxed{-2R_2 + R_1 \to R_2}$??

row 2 got scaled!!

?

Invertibility

Important practical implication: If A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times \text{(product of pivots)}, & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular.} \end{cases}$$

Example 2 Compute the determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{vmatrix}$$

THEOREM 3 **Row Operations**

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$
$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$\det U = 0$

FIGURE 1
Typical echelon forms of square matrices.

THEOREM 4 A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM 6 **Multiplicative Property**

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Properties of the Determinant

For any square matrices A and B, we can show the following.

1. $\det A = \det A^T$.
2. A is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$.

Additional Example (if time permits)

Use a determinant to find all values of λ such that matrix C is not invertible.

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda I_3$$

Additional Example (if time permits)

Determine the value of

$$\det A = \det\left(\begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^8\right).$$

3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 3 & -5 & 2 \end{vmatrix}$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

11. $\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

14. $\begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix}$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$

16. $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$

17. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

18. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

21. $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 27 and 28, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.

b. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

c. If the columns of A are linearly dependent, then $\det A = 0$.

d. $\det(A + B) = \det A + \det B$.

28. a. If three row interchanges are made in succession, then the new determinant equals the old determinant.

b. The determinant of A is the product of the diagonal entries in A.

c. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

d. $\det A^{-1} = (-1)\det A$.

29. Compute $\det B^4$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

32. Suppose that A is a square matrix such that $\det A^3 = 0$. Explain why A cannot be invertible.

33. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.

34. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

35. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

36. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

39. Let A and B be 3×3 matrices, with $\det A = -3$ and $\det B = 4$. Use properties of determinants (in the text and in the exercises above) to compute:

a. $\det AB$ b. $\det 5A$ c. $\det B^T$

d. $\det A^{-1}$ e. $\det A^3$

40. Let A and B be 4×4 matrices, with $\det A = -3$ and $\det B = -1$. Compute:

a. $\det AB$ b. $\det B^5$ c. $\det 2A$

d. $\det A^T BA$ e. $\det B^{-1}AB$

41. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, \ B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$

42. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

NOTE: Cramers rule and Adjoint of a matrix are NOT covered in Math 1554

Topics and Objectives

Topics
We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

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Supplementary FREE textbook

https://textbooks.math.gatech.edu/ila/

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Interactive Linear Algebra

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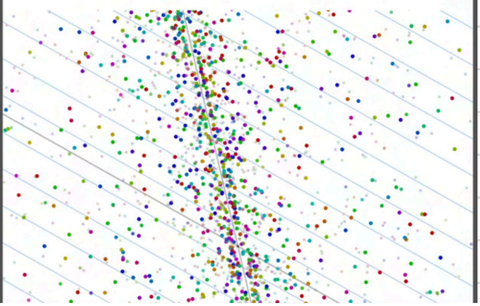
June 3, 2019



4.3 Determinants and Volumes

Objectives

1. Understand the relationship between the determinant of a matrix and the volume of a parallelepiped.
2. Learn to use determinants to compute volumes of parallelograms and triangles.
3. Learn to use determinants to compute the volume of some curvy shapes like ellipses.
4. *Pictures:* parallelepiped, the image of a curvy shape under a linear transformation.
5. *Theorem:* determinants and volumes.
6. *Vocabulary word:* ***parallelepiped***.

In this section we give a geometric interpretation of determinants, in terms of *volumes*. This will shed light on the reason behind three of the four defining properties of the determinant. It is also a crucial ingredient in the change-of-variables formula in multivariable calculus.

Parallelograms and Paralellepipeds

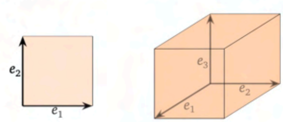
The determinant computes the volume of the following kind of geometric object.

Definition. The ***paralellepiped*** determined by n vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$ is the subset

$$P = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n \mid 0 \le a_1, a_2, \ldots, a_n \le 1\}.$$

In other words, a parallelepiped is the set of all linear combinations of n vectors with coefficients in $[0,1]$. We can draw parallelepipeds using the parallelogram law for vector addition.

Example (The unit cube). The parallelepiped determined by the standard coordinate vectors $e_1, e_2, \ldots, e_n$ is the unit n-dimensional cube.



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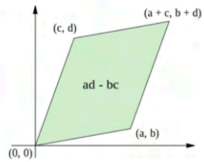
Determinants, Area and Volume

In $\mathbb{R}^2$, determinants give us the area of a parallelogram.



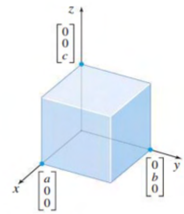
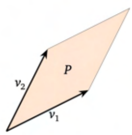
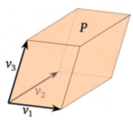


FIGURE 3
Volume $= |abc|$.

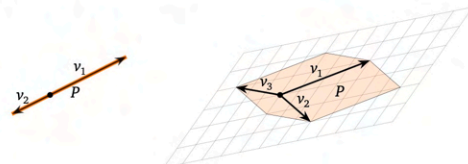
Example (Parallelograms). When $n = 2$, a paralellepiped is just a paralellogram in $\mathbf{R}^2$. Note that the edges come in parallel pairs.



Example. When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.



When does a parallelepiped have zero volume? This can happen only if the parallelepiped is flat, i.e., it is squashed into a lower dimension.

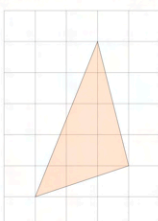


This means exactly that $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent,* which by this corollary in Section 4.1 means that the matrix with rows $v_1, v_2, \ldots, v_n$ has determinant zero. To summarize:

Key Observation. The parallelepiped defined by $v_1, v_2, \ldots, v_n$ has zero volume if and only if the matrix with rows $v_1, v_2, \ldots, v_n$ has zero determinant.

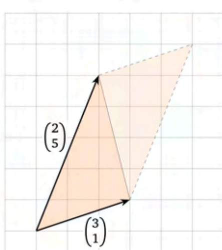
Example (Area of a triangle).

Find the area of the triangle with vertices $(-1,-2)$, $(2,-1)$, $(1,3)$.



Solution

Doubling a triangle makes a paralellogram. We choose two of its sides to be the rows of a matrix.



Determinants as Area, or Volume

Theorem

The volume of the parallelpiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar c.

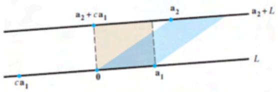


FIGURE 2 Two parallelograms of equal area.

Any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det(A)|$.

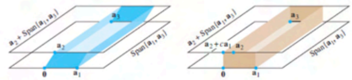


FIGURE 4 Two parallelepipeds of equal volume.

Example 1

Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1), (6,4)$

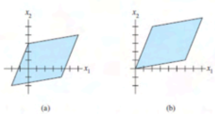


FIGURE 5 Translating a parallelogram does not change its area.

Linear Transformations

Theorem

If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then

$$\text{volume}\,(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$$

An example that applies this theorem is given in this week's worksheets.

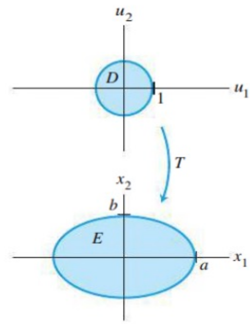
THEOREM 10

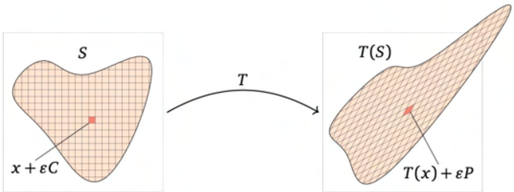
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$



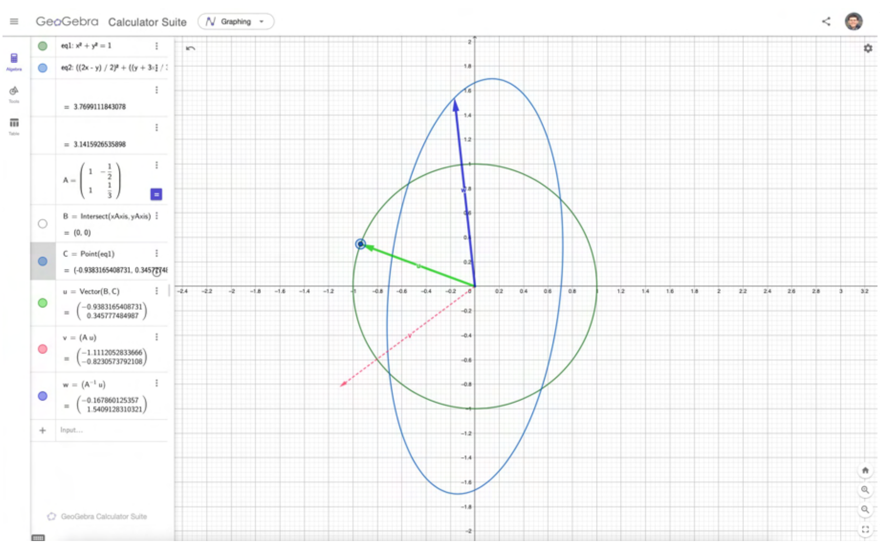


Example (Area of an ellipse).

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1.$$

https://www.geogebra.org/calculator/mkxeqfjy



Example (Area of an ellipse).

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2+\left(\frac{y+3x}{3}\right)^2=1.$$

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0), (5, 2), (6, 4), (11, 6)$

20. $(0, 0), (-2, 4), (4, -5), (2, -1)$

21. $(-2, 0), (0, 3), (1, 3), (-1, 0)$

22. $(0, -2), (5, -2), (-3, 1), (2, 1)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$, and $(-1, 3, -1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [*Hint:* Think about the columns of A.]

26. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^n$.

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

30. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

[*Hint:* Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are