Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

"A problem isn't finished just because you've found the right answer."
- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

Topics and Objectives

Topics
We will cover these topics in this section.

- The relationships between row reductions, the invertibility of a matrix, and determinants.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.

2. Use determinants to determine whether a square matrix is invertible.

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We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

✓ cofactor expansion is
expensive $O(n!)$ BAD

Idea: have a <u>better</u> way to compute

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$\det(A)$ using the fact that

for upper/lower triangular matrix B

computing $\det(B)$ is just the of diagonal entries.

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Section 3.2	5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
	6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
	7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
	8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

Row Operations

- We saw how determinants are difficult or impossible to compute with a cofactor expansion for large N.
- Row operations give us a more efficient way to compute determinants.

Theorem: Row Operations and the Determinant
Let A be a square matrix.
1. If a multiple of a row of A is added to another row to produce B, then $\det B = \det A$.
2. If two rows are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by a scalar k to produce B, then $\det B = k \det A$.

Example 1 Compute $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

$$= 2R_1 + R_2 \rightarrow \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$
$R_1 + R_3 \rightarrow$

$$= - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = \boxed{15}$$

$$\boxed{E_3 E_2 E_1 A = B}$$
$\det(E_3)\det(E_2)\det(E_1)\det(A) = \det(B)$
easy.

elementary matrix	Row operation	$\det E_i$
$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$2R_1 + R_2 \rightarrow R_2$	$\det(E_1) = 1*1*1 = 1.$
	$\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \sim \begin{bmatrix} 1&0&0 \\ 2&1&0 \\ 0&0&1 \end{bmatrix} = E_1$ ✓	
$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\boxed{R_1 \leftrightarrow R_2}$ ✓	$\det(E_2) = 0 - 0 + 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$
$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ✓	$\boxed{-2R_3 \rightarrow R_3}$	$\det(E_3) = -2.$ So. $\det(E_3 A) = \det(E_3)\det A = -2 \det A.$
$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$??	$\boxed{-2R_2 + R_1 \rightarrow R_2}$?? row 2 got scaled!! ↶ ↗ ?	$\det(E_4) = -2$

Important practical implication: If A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times \text{(product of pivots)}, & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular} \end{cases}$$

Example 2 Compute the determinant

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{pmatrix}$$

$$A \sim \begin{bmatrix} \boxed{2} & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{bmatrix} \quad * -1$$

$$\sim \quad R_1 + R_4 \rightarrow \begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 5 \end{bmatrix} \quad * 1$$

$$\sim \quad -3R_2 + R_3 \begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 5 \end{bmatrix} \quad * 1$$

$$\sim \begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad * -1$$

$$\sim \quad \frac{1}{3}R_3 \rightarrow \quad \frac{1}{5}R_4 \rightarrow \begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

$$* \frac{1}{3} * \frac{1}{5}$$

So.

$$\frac{1}{15} \det A = \det B$$

??

$$\det A = -2 * 15$$
$$= -30$$

$$\det(E_h \cdots E_2 E_1) \det A = \det B$$
$$\det A = \frac{\det B}{\det(E_1) \cdots \det(E_h)}$$

$$(-1)^2 * \frac{1}{3} * \frac{1}{5} = \frac{1}{15}$$

$$\det B = -2$$

THEOREM 3 — Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.   *row replacement.*

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.   *row swap.*

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.   *row scaling.*

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

THEOREM 4

A square matrix A is invertible if and only if $\det A \neq 0$.   IMT DLC

FIGURE 1
Typical echelon forms of square matrices.

THEOREM 6 — Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Properties of the Determinant

For any square matrices A and B, we can show the following.

1. $\det A = \det A^T$.
2. A is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$.

$\det A^{-1} = \dfrac{1}{\det A}$

$\det(AA^{-1}) = \det I = 1.$

Additional Example (if time permits)

Use a determinant to find all values of λ such that matrix C is not invertible.

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda I_3$$

hint: use cofactor expansion.

T/F Suppose A is 3×3 and $\det A = 1$.
Then $\det 2A = 2$.

$2 \ast \det A$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \det(I_3) = 1. \qquad 2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\det(2I_3) = 2^3.$

FACT $\boxed{\det(cA) = c^n \det A}$

Additional Example (if time permits)

Determine the value of

$$\det A = \det\left(\begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^8 \right).$$

3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 3 & -5 & 2 \end{vmatrix}$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}$
6. $\begin{vmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$
8. $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

11. $\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}$
12. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$
14. $\begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix}$

Find the determinants in Exercises 15–20, where
$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$
16. $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$

17. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$
18. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

21. $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$
22. $\begin{bmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 27 and 28, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.

b. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

c. If the columns of A are linearly dependent, then $\det A = 0$.

d. $\det(A + B) = \det A + \det B$.

28. a. If three row interchanges are made in succession, then the new determinant equals the old determinant.

b. The determinant of A is the product of the diagonal entries in A.

c. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

d. $\det A^{-1} = (-1) \det A$.

29. Compute $\det B^4$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

32. Suppose that A is a square matrix such that $\det A^3 = 0$. Explain why A cannot be invertible.

33. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.

34. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

35. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

36. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}$

39. Let A and B be 3×3 matrices, with $\det A = -3$ and $\det B = 4$. Use properties of determinants (in the text and in the exercises above) to compute:

a. $\det AB$ b. $\det 5A$ c. $\det B^T$

d. $\det A^{-1}$ e. $\det A^3$

40. Let A and B be 4×4 matrices, with $\det A = -3$ and $\det B = -1$. Compute:

a. $\det AB$ b. $\det B^5$ c. $\det 2A$

d. $\det A^T BA$ e. $\det B^{-1}AB$

41. Verify that $\det A = \det B + \det C$, where

$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$

42. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

NOTE: Cramers rule and Adjoint of a matrix are NOT covered in Math 1554

Topics and Objectives

Topics
We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

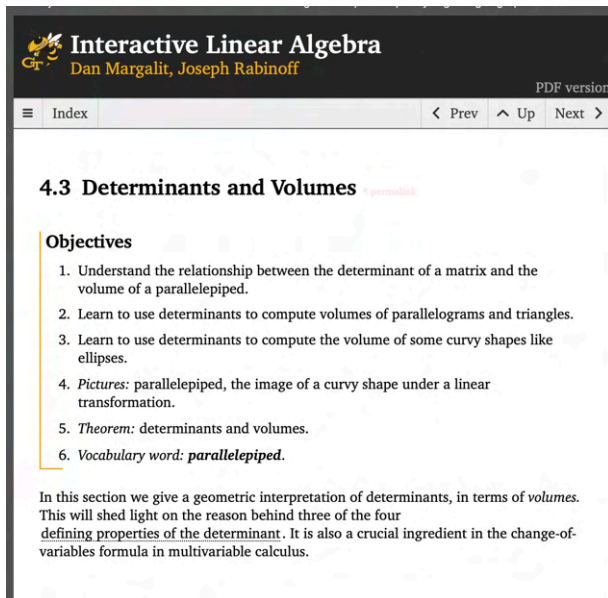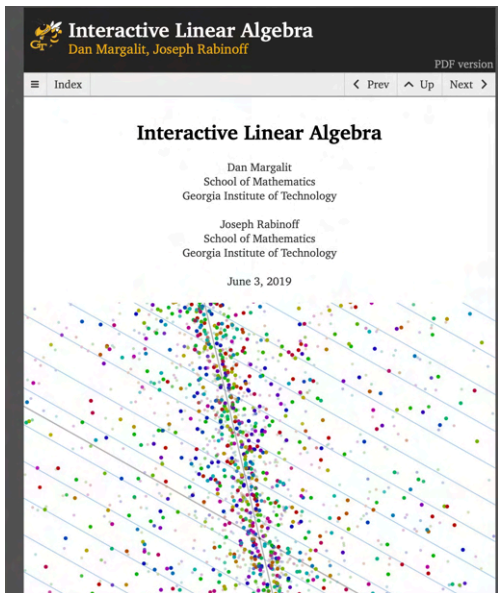Math 1554 Linear Algebra

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
		Mon	Tue	Wed	Thu	Fri
1	1/6 – 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 – 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 – 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 – 1/31	2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2
5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	Exam 2, Review	Cancelled	5.3
9	3/3 – 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 – 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 – 3/21	Break	Break	Break	Break	Break
12	3/24 – 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 – 4/4	6.6	WS6.5,6.6	Exam 3, Review	Cancelled	PageRank
14	4/7 – 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 – 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 – 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 – 5/2	Final Exams: MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm				

Topics and Objectives

Topics
We will cover these topics in this section.
1. Relationships between area, volume, determinants, and linear transformations.

Objectives
For the topics covered in this section, students are expected to be able to do the following.
1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

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Supplementary FREE textbook

https://textbooks.math.gatech.edu/ila/

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Interactive Linear Algebra
Dan Margalit, Joseph Rabinoff

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Interactive Linear Algebra

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June 3, 2019

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4.3 Determinants and Volumes

Objectives

1. Understand the relationship between the determinant of a matrix and the volume of a parallelepiped.
2. Learn to use determinants to compute volumes of parallelograms and triangles.
3. Learn to use determinants to compute the volume of some curvy shapes like ellipses.
4. *Pictures:* parallelepiped, the image of a curvy shape under a linear transformation.
5. *Theorem:* determinants and volumes.
6. *Vocabulary word:* **parallelepiped**.

In this section we give a geometric interpretation of determinants, in terms of *volumes*. This will shed light on the reason behind three of the four defining properties of the determinant. It is also a crucial ingredient in the change-of-variables formula in multivariable calculus.


Parallelograms and Paralellepipeds

The determinant computes the volume of the following kind of geometric object.

Definition. The *paralellepiped* determined by n vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$ is the subset

$$P = \{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \mid 0 \le a_1, a_2, \ldots, a_n \le 1\}.$$

In other words, a parallelepiped is the set of all linear combinations of n vectors with coefficients in $[0, 1]$. We can draw parallelepipeds using the parallelogram law for vector addition.

Example (The unit cube). The parallelepiped determined by the standard coordinate vectors $e_1, e_2, \ldots, e_n$ is the unit n-dimensional cube.
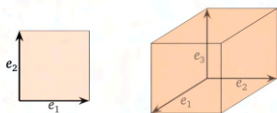
Topics

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Objectives

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Students are not expected to be familiar with Cramer's rule.

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In $\mathbb{R}^2$, determinants give us the area of a parallelogram.
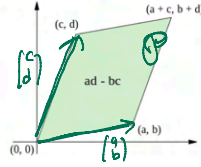
$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

FACT. $\left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right| = \mathrm{Vol}(P)$

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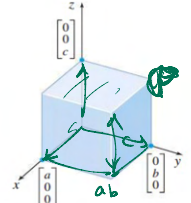
FIGURE 3

Volume $= |abc|$.

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$
$$= \mathrm{Vol}(P)$$

Example (Parallelograms). When $n = 2$, a parallelepiped is just a parallelogram in $\mathbb{R}^2$. Note that the edges come in parallel pairs.
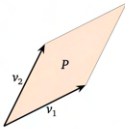
Example. When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.
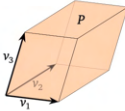
When does a parallelepiped have zero volume? This can happen only if the parallelepiped is flat, i.e., it is squashed into a lower dimension.
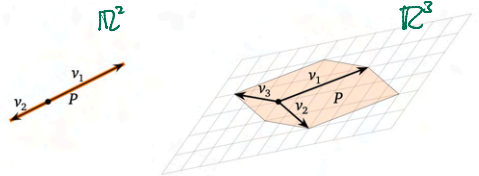

This means exactly that $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent,* which by this corollary in Section 4.1 means that the matrix with rows $v_1, v_2, \ldots, v_n$ has determinant zero. To summarize:

Key Observation. The parallelepiped defined by $v_1, v_2, \ldots, v_n$ has zero volume if and only if the matrix with rows $v_1, v_2, \ldots, v_n$ has zero determinant.
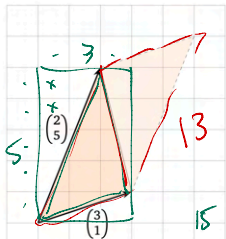
Example (Area of a triangle). ⌃

Find the area of the triangle with vertices $(-1, -2)$, $(2, -1)$, $(1, 3)$.
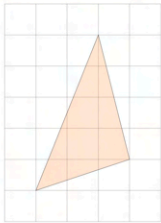
Solution

Doubling a triangle makes a parallelogram. We choose two of its sides to be the rows of a matrix.

det $E = -1$.

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$\det \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = 15 - 2 = 13.$$

$$\det \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = 2 - 15 = -13$$

So area of triangle is

$\boxed{-13}$ $\boxed{13}$ $\boxed{13/2}$

Determinants as Area, or Volume

Theorem
The volume of the parallelpiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Key Geometric Fact (which works in any dimension). The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar c.
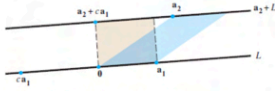
FIGURE 2 Two parallelograms of equal area.

Any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det(A)|$.
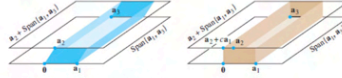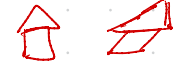
FIGURE 4 Two parallelepipeds of equal volume.

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{shear}$$

$$E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{reflect}$$

$$E_3 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \quad k\text{-dilation horizontal.}$$

Example 1

Q1: Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1), (6,4)$
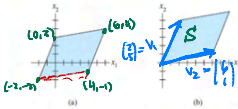
FIGURE 5 Translating a parallelogram does not change its area.

$$\begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = v_1$$

$$\begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} = v_2$$

$$\text{Vol}(S) = \left| \det \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix} \right| = |2 - 30| = \boxed{28}$$

Q2:
$T : \mathbb{R}^2 \to \mathbb{R}^2$
rotate by $\pi/2$ CCW
Then scale x_1 by 3.
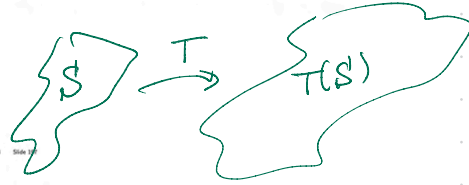$\text{Vol}(T(S)) = ?$

Linear Transformations

Theorem
If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, and S is some parallelogram in $\mathbb{R}^n$, then
$$\text{volume}(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$$

An example that applies this theorem is given in this week's worksheets.

A standard matrix of T.
$$T(x) = A\vec{x}$$

$$\text{Vol}(T(S)) = |\det A| * T(S)$$

Idea $T(\vec{x}) = Ax$

Step 1
$$A = \begin{pmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$$

$$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{scale}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
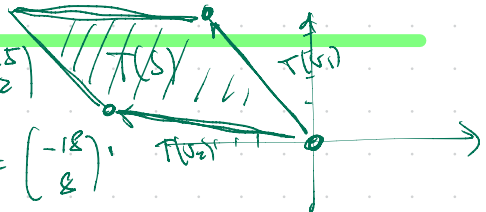
$$\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$\det A = 0 - (-3) = 3.$

So $\text{Vol}(T(S)) = 3 \cdot 28 = \boxed{84}$

$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 6 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

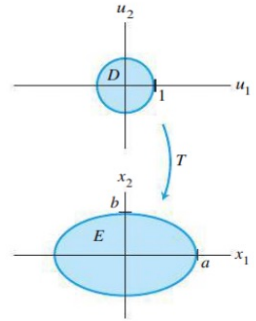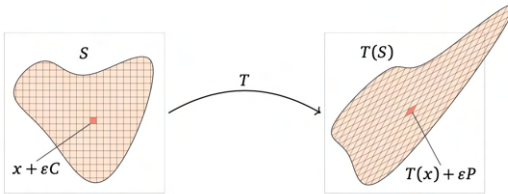$$T\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix} \right) = \begin{pmatrix} -15 \\ 2 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 8 \\ 6 \end{pmatrix} \right) = \begin{pmatrix} -18 \\ 8 \end{pmatrix}$$

THEOREM 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$

Example (Area of an ellipse). ⌃

Find the area of the interior E of the ellipse defined by the equation

$$\left(\frac{2x - y}{2}\right)^2 + \left(\frac{y + 3x}{3}\right)^2 = 1.$$

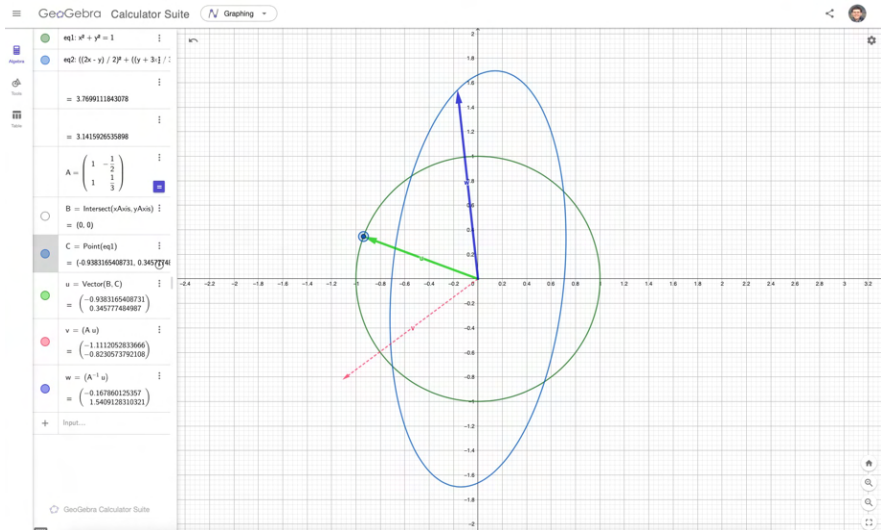https://www.geogebra.org/calculator/mkxeqfjy

Example (Area of an ellipse). ⌃

Find the area of the interior E of the ellipse defined by the equation

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In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0)$, $(5, 2)$, $(6, 4)$, $(11, 6)$

20. $(0, 0)$, $(-2, 4)$, $(4, -5)$, $(2, -1)$

21. $(-2, 0)$, $(0, 3)$, $(1, 3)$, $(-1, 0)$

22. $(0, -2)$, $(5, -2)$, $(-3, 1)$, $(2, 1)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$, and $(-1, 3, -1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [*Hint:* Think about the columns of A.]

26. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^n$.

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

30. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

[*Hint:* Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are

Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

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Exam 2 in one week from today Feb 26 @ 6:30 pm.

5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	Exam 2, Review	Cancelled	5.3

Example 1

- A small town has two libraries, A and B.
- After 1 month, among the books checked out of A,
  - 80% returned to A
  - 20% returned to B
- After 1 month, among the books checked out of B,
  - 30% returned to A
  - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is http://setosa.io/markov/index.html

$$
\begin{array}{c}
A \xrightarrow{0.8} \quad 0.3 \quad B \xrightarrow{0.7} \\
0.2
\end{array}
$$

Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

After k months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

e.g. $\begin{bmatrix} .5 \\ .5 \end{bmatrix} \begin{bmatrix} .55 \\ .45 \end{bmatrix}, \begin{bmatrix} .575 \\ .425 \end{bmatrix}, \cdots$

Markov chain.

$P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ cols are prob. vectors.

Q1: What is a Markov chain?

$\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ sequence of probability vectors.

s.t. $P\vec{x}_n = \vec{x}_{n+1}$ where P is a Stochastic matrix

Q2: What questions might one ask about them?

how to find P? (usually given)

how to find n s.t. have certain $\vec{x}_n$ below?

⊛ What is the limit as $n \to \infty$?

$\hookrightarrow \boxed{\lim_{n \to \infty} \vec{x}_n = \lim_{n \to \infty} P^n \vec{x}_0}$

compute this value or say DNE.

Q3: How do you specify a particular Markov chain?

```
clc
format shortG
% library example
A=[.8 .3 ; .2 .7]
x0=[.5;.5]

% car rental example
%A=[.8 .1 .2 ; .2 .6 .3 ; 0 .3 .5]
%x0=[.2; .2; .6]

% election example
%A=[.7 .1 .3 ; .2 .8 .3 ; .1 .1 .4]
%x0=[.55;.4;.05]

% set value of n (largest index to
compute)
%n=10;
for i=1:n
    % convert current index to string
and create xi string
    index=string(i);
    s=strcat('x',index,'=');
    % compute xi value
    xi=A^i*x0;
    % display each xi=A^i*x0
    disp(s)
    disp(xi)
end
% display final xn=A^n*x0
format longG
n=4;
index=string(n);
s=strcat('x',index,'=');
xi=A^n*x0;
disp(s)
disp(xi)
```

Command Window

```
A =

        0.8         0.3
        0.2         0.7

x0 =

        0.5
        0.5

x1=

        0.55
        0.45

x2=

        0.575
        0.425

x3=

        0.5875
        0.4125

x4=

        0.59375
        0.40625

x4=

        0.59375
        0.40625
```

any $\vec{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$

.60
.40

f_x >>

Example 1

- A small town has two libraries, A and B.
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  - 20% returned to B
- After 1 month, among the books checked out of B,
  - 30% returned to A
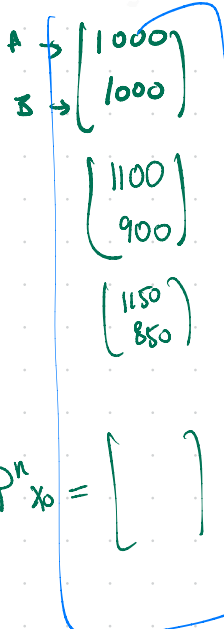 - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is http://setosa.io/markov/index.html

$$0.8 \quad A \xrightarrow{0.3} B \quad 0.7$$
$$0.2$$

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Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

After k months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

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$$A \to \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} \quad \text{initial state}$$
$$B \to$$

from A:
coming to A

from B:
coming to B

$$1000(.8) + 1000(.3) = 1100$$
$$1000(.2) + 1000(.7) = 900$$

$$\cancel{X} \to (1000)\begin{bmatrix} .8 \\ .2 \end{bmatrix} + 1000\begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} 1100 \\ 900 \end{bmatrix}$$

$$\begin{matrix} & A & \hat{x} \\ \end{matrix}$$
$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1100 \\ 900 \end{bmatrix}$$

$$\begin{bmatrix} 1100 \\ 900 \end{bmatrix} \quad \text{after 1 month to A} \\ \text{to B}$$

from A   from B
$$1100(.8) + 900(.3) = 1150$$
$$1100(.2) + 900(.7) = 850$$

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1100 \\ 900 \end{bmatrix} = \begin{bmatrix} 1150 \\ 850 \end{bmatrix}$$

$$\begin{bmatrix} 1150 \\ 850 \end{bmatrix} \quad \text{after 2 mo.}$$

$$\vdots$$

$$P^n X_0 = \begin{bmatrix} & \\ & \end{bmatrix} \quad \text{after } n \text{ months}$$

Idea: $P X_0 = X_1$ where P transition matrix

X_0 initial state

X_1 state after one iteration.

$$P X_1 = X_2$$
$$P X_2 = X_3$$
$$\vdots$$
$$P X_n = X_{n+1}$$

e.g.
$$X_3 = P(X_2) = P(P X_1) = P P P X_0 = \boxed{P^3 X_0}$$

prob vector

$$\vec{X}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}, \quad \vec{X}_1 = \begin{bmatrix} .55 \\ .45 \end{bmatrix}, \quad \vec{X}_2 = \begin{bmatrix} .575 \\ .425 \end{bmatrix}, \dots$$

$$.55 = \frac{1100}{2000} = \frac{\# @ A}{\# \text{ total}}$$

A few definitions:
- A **probability vector** is a vector, $\vec{x}$, with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, P, whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $\vec{x}_k$, and a stochastic matrix P, such that:

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

✱ A **steady-state vector** for P is a vector $\vec{q}$ such that $P\vec{q} = \vec{q}$.

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Q: is there a dist. $\vec{X}_0$
s.t. $\vec{X}_1 = \vec{X}_0$.

Start
$$Px = x$$
$$\Leftrightarrow Px - x = \vec{0}$$
$$\Leftrightarrow Px - Ix = \vec{0}$$
$$\Leftrightarrow (P - I)\vec{x} = \vec{0}$$

$\begin{cases} x_1 - \frac{3}{2}r = 0 \\ x_2 = r \text{ free} \end{cases} \to \begin{cases} x = \frac{3}{2}r \\ x_2 = r \end{cases} \to \sim \begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix}$ $\vec{X} = r\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ want $\vec{q} = \begin{pmatrix} * \\ * \end{pmatrix}$

Example 2

Determine a steady-state vector for the stochastic matrix

$$P = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

Notice want x_1, x_2 s.t. $\longleftrightarrow \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{cases} .8x_1 + .3x_2 = x_1 \\ .2x_1 + .7x_2 = x_2 \end{cases}$$

$$\begin{cases} .8x_1 - x_1 + .3x_2 = 0 \\ .2x_1 + .7x_2 - x_2 = 0 \end{cases} \longleftrightarrow \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\begin{cases} (.8-1)x_1 + .3x_2 = 0 \\ .2x_1 + (.7-1)x_2 = 0 \end{cases} \longleftrightarrow \left(\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\vec{X} = \vec{0}$$

Matrix!

Solve $\to \boxed{(P - I)\vec{x} = \vec{0}}$

When looking for steady-state $\vec{x}$.

$$P - I = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix}$$

Prob. vector

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$
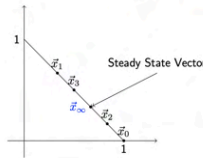
as $k \to \infty$.

Definition: a stochastic matrix P is **regular** if there is some k such that P^k only contains strictly positive entries.

Theorem

If P is a regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.   probability.   for any x_0.

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$$P = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \text{ regular? } \checkmark$$

e.g.

$$P = \begin{pmatrix} .5 & 1 \\ .5 & 0 \end{pmatrix} \text{ regular? } \boxed{\text{yes}}$$

$$P^2 = \begin{pmatrix} .5 & 1 \\ .5 & 0 \end{pmatrix}\begin{pmatrix} .5 & 1 \\ .5 & 0 \end{pmatrix} = \begin{pmatrix} .75 & .5 \\ .25 & .5 \end{pmatrix}$$

P^2 has no zero so P is regular

Stochastic Vectors in the Plane

The stochastic vectors in the plane are the line segment below, and a stochastic matrix maps stochastic vectors to themselves. Iterates $P^k\vec{x}_0$ converge to the steady state.

$P^k \to [\vec{x}_\infty \quad \vec{x}_\infty]$

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$$\frac{1}{\frac{3}{2}+1}\begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = \frac{1}{5/2}\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$
$$= \frac{2}{5}\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$
$$\boxed{\vec{q} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}}$$

pick these r-values
$r = 2$ $\vec{X} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$q = \frac{1}{3+2}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$$

$2000 * 60\% = 1200 \leftarrow$ hooks @ A
$2000 * 40\% = 800 \leftarrow$ books @ B   in the steady state.

Example 3

A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

	rented from		
	A	B	C
returned to A	.8	.1	.2
returned to B	.2	.6	.3
returned to C	.0	.3	.5

There are 10 cars at each location today.

a) Construct a stochastic matrix, P, for this problem.

b) What happens to the distribution of cars after a long time? You may assume that P is regular.
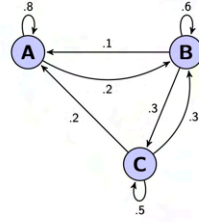
$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

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Q: Find $\hat{q}$ the unique prob steady-state vector of P.

Step 1: $P - I = \begin{pmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ 0 & .3 & .5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{bmatrix} -.2 & .1 & .2 \\ .2 & -.4 & .3 \\ 0 & .3 & -.5 \end{bmatrix} \sim \begin{pmatrix} -2 & 1 & 2 \\ 2 & -4 & 3 \\ 0 & 3 & -5 \end{pmatrix}$

$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 5 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{pmatrix} -2 & 1 & 2 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{bmatrix} -2 & 0 & 11/3 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11/6 \\ 0 & 1 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 - \frac{11}{6}r = 0 \\ x_2 - \frac{5}{3}r = 0 \\ x_3 = r \ (\text{free}) \end{cases}$

$\vec{x} = r \begin{pmatrix} 11/6 \\ 5/3 \\ 1 \end{pmatrix}$?

$\begin{cases} x_1 = 11/6 \, r \\ x_2 = 5/3 \, r \\ x_3 = r \end{cases} \longrightarrow$

Set $r = 6$ $\vec{v} = \begin{bmatrix} 11 \\ 10 \\ 6 \end{bmatrix}$ So $\frac{1}{11+10+6} \vec{v} = \begin{bmatrix} 11/27 \\ 10/27 \\ 6/27 \end{bmatrix}$

Q: P stochastic & NOT regular

A:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = P^2 * P = P$$

$$\vdots$$

$$P^n = \begin{cases} P & n \text{ odd} \\ I & n \text{ even} \end{cases}$$

$$X_0 = \begin{bmatrix} .3 \\ .7 \end{bmatrix}, \begin{bmatrix} .7 \\ .3 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} \dots$$
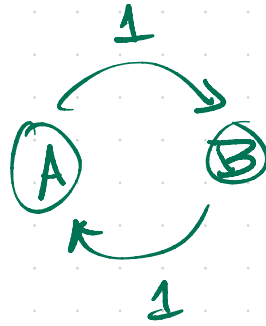
$$\begin{bmatrix} .5 \\ .5 \end{bmatrix}$$

$$P = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^2 = I$$

$$P^n = I$$

EXAMPLE 1 Section 1.10 examined a model for population movement between a city and its suburbs. See Figure 1. The annual migration between these two parts of the metropolitan region was governed by the *migration matrix M*:

$$M = \begin{array}{c} \text{From:} \\ \begin{array}{cc} \text{City} & \text{Suburbs} \end{array} \\ \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \end{array} \begin{array}{c} \text{To:} \\ \text{City} \\ \text{Suburbs} \end{array}$$

That is, each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. The columns of M are probability vectors, so M is a stochastic matrix. Suppose the 2014 population of the region is 600,000 in the city and 400,000 in the suburbs. Then the initial distribution of the population in the region is given by $\mathbf{x}_0$ in (1) above. What is the distribution of the population in 2015? In 2016?
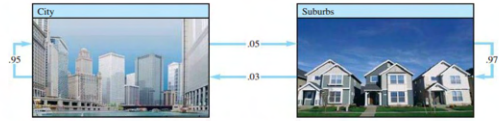
FIGURE 1 Annual percentage migration between city and suburbs.

EXAMPLE 4 The probability vector $\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ is a steady-state vector for the population migration matrix M in Example 1, because

$$M\mathbf{q} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .35625 + .01875 \\ .01875 + .60625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix} = \mathbf{q} \qquad \blacksquare$$

EXAMPLE 2 Suppose the voting results of a congressional election at a certain voting precinct are represented by a vector **x** in $\mathbb{R}^3$:

$$\mathbf{x} = \begin{bmatrix} \% \text{ voting Democratic (D)} \\ \% \text{ voting Republican (R)} \\ \% \text{ voting Libertarian (L)} \end{bmatrix}$$

Suppose we record the outcome of the congressional election every two years by a vector of this type and the outcome of one election depends only on the results of the preceding election. Then the sequence of vectors that describe the votes every two years may be a Markov chain. As an example of a stochastic matrix P for this chain, we take
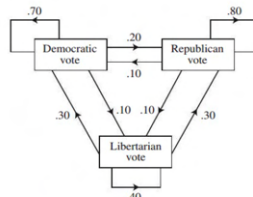

FIGURE 2 Voting changes from one election to the next.

$$
\begin{array}{cccc}
 & \text{D} & \text{R} & \text{L} & \text{To:} \\
P = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} & & & \begin{array}{c} \text{D} \\ \text{R} \\ \text{L} \end{array}
\end{array}
$$

From:

SOLUTION The outcome of the next election is described by the state vector $\mathbf{x}_1$ and that of the election after that by $\mathbf{x}_2$, where

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix}\begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} \quad \begin{array}{l} \text{44\% will vote D.} \\ \text{44.5\% will vote R.} \\ \text{11.5\% will vote L.} \end{array}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix}\begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix} \quad \begin{array}{l} \text{38.7\% will vote D.} \\ \text{47.8\% will vote R.} \\ \text{13.5\% will vote L.} \end{array}$$

To understand why $\mathbf{x}_1$ does indeed give the outcome of the next election, suppose 1000 persons voted in the "first" election, with 550 voting D, 400 voting R, and 50 voting L. (See the percentages in $\mathbf{x}_0$.) In the next election, 70% of the 550 will vote D again, 10% of the 400 will switch from R to D, and 30% of the 50 will switch from L to D. Thus the total D vote will be

$$.70(550) + .10(400) + .30(50) = 385 + 40 + 15 = 440 \tag{2}$$

Thus 44% of the vote next time will be for the D candidate. The calculation in (2) is essentially the same as that used to compute the first entry in $\mathbf{x}_1$. Analogous calculations could be made for the other entries in $\mathbf{x}_1$, for the entries in $\mathbf{x}_2$, and so on. ∎

EXAMPLE 3 Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consider a system whose state is described by the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$, for $k = 0, 1, \dots$ What happens to the system as time passes? Compute the state vectors $\mathbf{x}_1, \dots, \mathbf{x}_{15}$ to find out.

SOLUTION

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

The results of further calculations are shown below, with entries rounded to four or five significant figures.

$$\mathbf{x}_4 = \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix}$$

$$\mathbf{x}_8 = \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, \quad \mathbf{x}_{11} = \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix}$$

$$\mathbf{x}_{12} = \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, \quad \mathbf{x}_{13} = \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, \quad \mathbf{x}_{14} = \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, \quad \mathbf{x}_{15} = \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix}$$

These vectors seem to be approaching $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$. The probabilities are hardly changing from one value of k to the next. Observe that the following calculation is exact (with no rounding error):

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

When the system is in state $\mathbf{q}$, there is no change in the system from one measurement to the next. ∎

EXAMPLE 5 Let $P = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$. Find a steady-state vector for P.

SOLUTION First, solve the equation $P\mathbf{x} = \mathbf{x}$.

$$P\mathbf{x} - \mathbf{x} = \mathbf{0}$$
$$P\mathbf{x} - I\mathbf{x} = \mathbf{0} \qquad \text{Recall from Section 1.4 that } I\mathbf{x} = \mathbf{x}.$$
$$(P - I)\mathbf{x} = \mathbf{0}$$

For P as above,

$$P - I = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .3 \\ .4 & -.3 \end{bmatrix}$$

To find all solutions of $(P - I)\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $x_1 = \frac{3}{4}x_2$ and x_2 is free. The general solution is $x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

Next, choose a simple basis for the solution space. One obvious choice is $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ but a better choice with no fractions is $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (corresponding to $x_2 = 4$).

Finally, find a probability vector in the set of all solutions of $P\mathbf{x} = \mathbf{x}$. This process is easy, since every solution is a multiple of the solution $\mathbf{w}$ above. Divide $\mathbf{w}$ by the sum of its entries and obtain

$$\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

As a check, compute

$$P\mathbf{q} = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix} \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \mathbf{q} \qquad \blacksquare$$

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

4.9 EXERCISES

1. A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.

   a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.

   b. Give the initial state vector.

   c. What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?

2. A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food

on one trial, it will choose the same food on the next trial with a probability of 50%, and it will choose the other foods on the next trial with equal probabilities of 25%.

   a. What is the stochastic matrix for this situation?

   b. If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?

3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy

tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

   a. What is the stochastic matrix for this situation?

   b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?

   c. If a student is well today, what is the probability that he or she will be well two days from now?

4. The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 60% chance the weather will be good tomorrow, a 30% chance the weather will be indifferent, and a 10% chance the weather will be bad. If the weather is indifferent today, it will be good tomorrow with probability .40 and indifferent with probability .30. Finally, if the weather is bad today, it will be good tomorrow with probability .40 and indifferent with probability .50.

   a. What is the stochastic matrix for this situation?

   b. Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?

   c. Suppose the predicted weather for Monday is 40% indifferent weather and 60% bad weather. What are the chances for good weather on Wednesday?

In Exercises 5–8, find the steady-state vector.

5. $\begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$

6. $\begin{bmatrix} .8 & .5 \\ .2 & .5 \end{bmatrix}$

7. $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$

8. $\begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix}$

9. Determine if $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ is a regular stochastic matrix.

10. Determine if $P = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ is a regular stochastic matrix.

11. a. Find the steady-state vector for the Markov chain in Exercise 1.

    b. At some time late in the day, what fraction of the listeners will be listening to the news?

12. Refer to Exercise 2. Which food will the animal prefer after many trials?

13. a. Find the steady-state vector for the Markov chain in Exercise 3.

    b. What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?

14. Refer to Exercise 4. In the long run, how likely is it for the weather in Columbus to be good on a given day?

15. [M] The Demographic Research Unit of the California State Department of Finance supplied data for the following migration matrix, which describes the movement of the United

Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics

We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

If $A \in \mathbb{R}^{n \times n}$, and there is a $\vec{v} \neq 0$ in $\mathbb{R}^n$, and

$$A\vec{v} = \lambda \vec{v}$$

$\vec{v} \neq \vec{0}$

then $\vec{v}$ is an **eigenvector** for A, and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

Note that

- We will only consider square matrices.
- If $\lambda \in \mathbb{R}$, then
  - when $\lambda > 0$, $A\vec{v}$ and $\vec{v}$ point in the same direction
  - when $\lambda < 0$, $A\vec{v}$ and $\vec{v}$ point in opposite directions
- Even when all entries of A and $\vec{v}$ are real, λ can be complex (a rotation of the plane has no **real** eigenvalues.)
- We explore complex eigenvalues in Section 5.5.

A has eigenvector $\vec{v}$ w/ eigenvalue λ

$$A\vec{v} = \lambda\vec{v} \quad \& \quad \vec{v} \neq \vec{0}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$A\vec{w} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

5.1 EIGENVECTORS AND EIGENVALUES

Although a transformation $\mathbf{x} \mapsto A\mathbf{x}$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Figure 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, $\mathbf{v}$. ∎
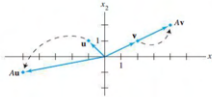

FIGURE 1 Effects of multiplication by A.

As another example, readers of Section 4.9 will recall that if A is a stochastic matrix, then the steady-state vector $\mathbf{q}$ for A satisfies the equation $A\mathbf{x} = \mathbf{x}$. That is, $A\mathbf{q} = 1 \cdot \mathbf{q}$.

5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

Q: Is $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of A?

A: check $A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \neq \lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So NO

Idea: want to check if $A\vec{v} \in$ span $\{\vec{v}\}$

Q: how about $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

A: check $A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \lambda = 2$

Yes w/ $\lambda = 2$

Example 1

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_1 = 2$ ✓

yes

b) $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $A\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \overset{\checkmark}{=} 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda_2 = 0$ ✓

yes

c) $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\lambda = ?$

No b/c every λ would be an eigenvalue if $v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ were considered to be an eigenvector.

Section 5.1   Slide 212

Check $A\vec{v} = 3\vec{v}$?

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$ ✓

Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

remember $\boxed{P\vec{q} = \vec{q}} \Leftrightarrow (P - I)\vec{q} = 0$ ie $A\vec{x} = \vec{x}$

want $\vec{v} \neq 0$ s.t. $A\vec{v} = 3\vec{v}$

$\Leftrightarrow A\vec{v} - 3\vec{v} = 0$

$\Leftrightarrow (A - 3I)\vec{v} = \vec{0}$

lots of ways IMT

looking for a $\vec{v} \in \text{Nul}(A - 3I)$ other than $\vec{v} = 0$.

$A - 3I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

$\vec{x} = r\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ $\boxed{\vec{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}}$

$\text{span}\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}$

$= \text{Nul}(A - 3I)$

$=$ "3-eigenspace of A"

Section 5.1   Slide 213

Eigenspace

Definition

Suppose $A \in \mathbb{R}^{n \times n}$. The eigenvectors for a given λ span a subspace of $\mathbb{R}^n$ called the λ-**eigenspace** of A.

Note: the λ-eigenspace for matrix A is $\text{Nul}(A - \lambda I)$.

Example 3
Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \quad \lambda = -1, 2$$

Note A is invertible, but $A - (-1)I$, $A - 2I$ are NOT invertible.

Section 5.1   Slide 215

Theorems

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.

2. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A.

3. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

$\underline{\lambda_1 = 2}$ $A - 2I = \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ basis for $\lambda = 2$ eigenspace is $\boxed{\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}}$

$\underline{\lambda_2 = -1}$ $A + I = \begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ basis is $\boxed{\cancel{\text{span}}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}}$

If need basis don't write span

If need subspace write span.

Proofs for the following theorems are stated in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

✓ 1. The diagonal elements of a triangular matrix are its eigenvalues.

✓ 2. A invertible ⇔ 0 is not an eigenvalue of A. — IMT DLC

③ 3. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

Section 5.1

does NOT say

"lin ind eigenvectors must came from distinct eigenvalues"

false.

eigenvalues.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 3$
$\lambda_2 = -1$
$\lambda_3 = 4$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
lin ind

want $\vec{v}_i \neq \vec{0}$ $i = 1, 2, 3$.
$A\vec{v}_i = \lambda_i \vec{v}_i$

DON'T DO ROW OPERATIONS

on A

when looking for e-vals!!

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = 1$

$A\vec{v} = \vec{v}$

$v_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$v_3 = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ lin ind
but λ's all same

3Blue1Brown
A beautiful, animated, and visual explanation of eigenvalues and eigenvectors.

http://bit.ly/21XyJPg

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

Example: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \lambda = 2$$

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \lambda = 0$$

- But the reduced echelon form of A is:

- The reduced echelon form is triangular, and its eigenvalues are:

$$A \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \lambda = 1, 0$$

new eigenvector?

DEFINITION   An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

EXAMPLE 2   Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?
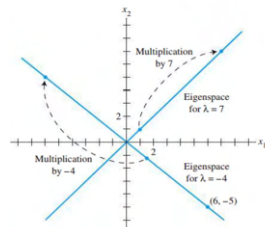

FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
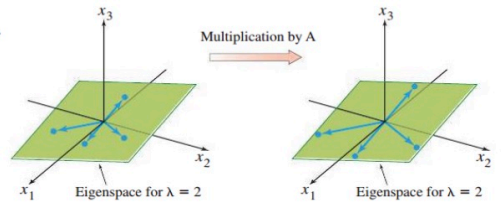

FIGURE 3 A acts as a dilation on the eigenspace.

THEOREM 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

5.1 EXERCISES

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not?

3. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If so, find the eigenvalue.

4. Is $\begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$? If so, find the eigenvalue.

5. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.

6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

8. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

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9. $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \lambda = 1, 5$

10. $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \lambda = 4$

11. $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \lambda = 10$

12. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \lambda = 1, 5$

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}, \lambda = -2$

15. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \lambda = 3$

16. $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

18. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

19. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify your answer.

In Exercises 21 and 22, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

21. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

   b. A matrix A is not invertible if and only if 0 is an eigenvalue of A.

   c. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

   e. To find the eigenvalues of A, reduce A to echelon form.

22. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

   b. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

   c. A steady-state vector for a stochastic matrix is actually an eigenvector.

   d. The eigenvalues of a matrix are on its main diagonal.

   e. An eigenspace of A is a null space of a certain matrix.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

24. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

25. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [Hint: Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

26. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [Hint: Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

28. Use Exercise 27 to complete the proof of Theorem 1 for the case when A is lower triangular.

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [Hint: Find an eigenvector.]

30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [Hint: Use Exercises 27 and 29.]

In Exercises 31 and 32, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.

31. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

32. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

33. Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ and μ, and let c_1 and c_2 be scalars. Define
$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v} \quad (k = 0, 1, 2, \ldots)$$

   a. What is $\mathbf{x}_{k+1}$, by definition?

   b. Compute $A\mathbf{x}_k$ from the formula for $\mathbf{x}_k$, and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$. This calculation will prove that the sequence $\{\mathbf{x}_k\}$ defined above satisfies the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$.

34. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$ if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. [Hint: How might you relate $\mathbf{x}_0$ to eigenvectors of A?]

35. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure, and suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.

36. Repeat Exercise 35, assuming $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A that correspond to eigenvalues -1 and 3, respectively.

[M] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

37. $\begin{bmatrix} 8 & -10 & -5 \\ 2 & 17 & 2 \\ -9 & -18 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$

39. $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$

40. $\begin{bmatrix} -4 & -4 & 20 & -8 & -1 \\ 14 & 12 & 46 & 18 & 2 \\ 6 & 4 & -18 & 8 & 1 \\ 11 & 7 & -37 & 17 & 2 \\ 18 & 12 & -60 & 24 & 5 \end{bmatrix}$

Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

5	9/18 – 9/22	2.3,2.4	WS2.2,2.3	2.5	WS2.4,2.5	2.8
6	9/25 – 9/29	2.9	WS2.8,2.9	3.1,3.2	WS3.1,3.2	3.3
7	10/2 – 10/6	4.9	WS3.3,4.9	5.1,5.2	WS5.1,5.2	5.2
8	10/9 – 10/13	Break	Break	**Exam 2**, Review	Cancelled	5.3

The Characteristic Polynomial

Recall:

λ is an eigenvalue of A $\Leftrightarrow$ $(A - \lambda I)$ is not _____

Therefore, to calculate the eigenvalues of A, we can solve

$$\det(A - \lambda I) =$$

The quantity $\det(A - \lambda I)$ is the **characteristic polynomial** of A.

The quantity $\det(A - \lambda I) = 0$ is the **characteristic equation** of A.

The roots of the characteristic polynomial are the _____ of A.

Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

So the eigenvalues of A are:

Characteristic Polynomial of 2×2 Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when M is singular?

Algebraic Multiplicity

> **Definition**
>
> The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Geometric Multiplicity

> **Definition**
>
> The **geometric multiplicity** of an eigenvalue λ is the dimension of $\text{Null}(A - \lambda I)$.

1. Geometric multiplicity is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

Example

Give an example of a 4×4 matrix with $\lambda = 0$ the only eigenvalue, but the geometric multiplicity of $\lambda = 0$ is one.

Recall:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

  as $k \to \infty$.

- If P is regular, then there is a _____

Now lets ask:

- If we don't know whether P is regular, what else might we do to describe the long-term behavior of the system?
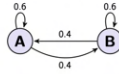- What can eigenvalues tell us about the behavior of these systems?

Example: Eigenvalues and Markov Chains

Note: the textbook has a similar example that you can review.

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:

Goal: use eigenvalues to describe the long-term behavior of our system.

What are the eigenvalues of P?

What are the corresponding eigenvectors of P?

Use the eigenvalues and eigenvectors of P to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

Similar Matrices

Definition

Two $n \times n$ matrices A and B are **similar** if there is a matrix P so that $A = PBP^{-1}$.

Theorem

If A and B similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section <mark>5.2</mark>, but doesn't have exercises on this concept until 5.3.
- Two matrices, A and B, do not need to be similar to have the same eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Additional Examples (if time permits)

1. True or false.

   a) If A is similar to the identity matrix, then A is equal to the identity matrix.

   b) A row replacement operation on a matrix does not change its eigenvalues.

2. For what values of k does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$

5.2 Exercises

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix using expansion across a row or down a column. [*Note:* Finding the characteristic polynomial of a 3 × 3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 6 & 7 & -4 \end{bmatrix}$

For the matrices in Exercises 15–17, list the eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

In Exercises 21–30, A and B are $n \times n$ matrices. Mark each statement True or False (T/F). Justify each answer.

21. (T/F) If 0 is an eigenvalue of A, then A is invertible.

22. (T/F) The zero vector is in the eigenspace of A associated with an eigenvalue λ.

23. (T/F) The matrix A and its transpose, A^T, have different sets of eigenvalues.

24. (T/F) The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

25. (T/F) If 2 is an eigenvalue of A, then $A - 2I$ is not invertible.

26. (T/F) If two matrices have the same set of eigenvalues, then they are similar.

27. (T/F) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

28. (T/F) The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

29. (T/F) The eigenvalue of the $n \times n$ identity matrix is 1 with algebraic multiplicity n.

30. (T/F) The matrix A can have more than n eigenvalues.