$$\begin{cases} x_1 + 3x_2 - 4x_3 = 2 \\ -x_1 - x_2 + 5x_2 = 3 \\ \quad\;\; x_2 + 2x_2 = 1 \end{cases}$$

$AB = I_n$

$A^{-1}\vec{b} = \vec{x}$

Section 5.1 : Eigenvectors and Eigenvalues

$x_1v_1 + x_2v_2 + x_3v_3 = b$

$Ax = b$

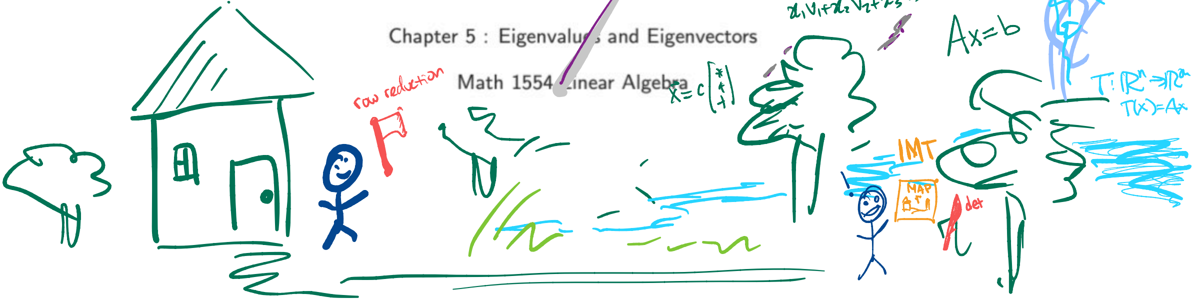
Chapter 5 : Eigenvalues and Eigenvectors

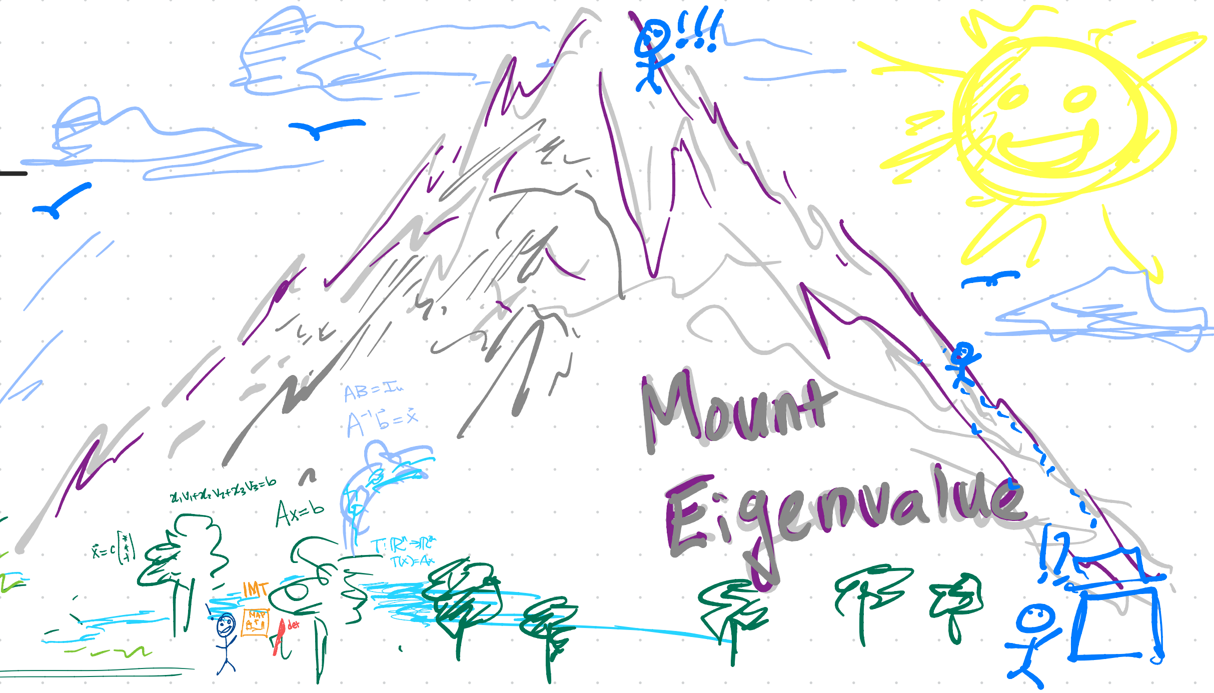
Math 1554 Linear Algebra

$\vec{x} = c\begin{bmatrix} * \\ * \\ * \end{bmatrix}$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(x) = Ax$





Warning!

> We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

Example: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because → to get $\vec{v}_1, \vec{v}_2$ eigenvectors you row reduce $A - \lambda_i I$ & find parametric vector form of solns

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \lambda = 2$$

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \lambda = 0$$

- But the reduced echelon form of A is:
- The reduced echelon form is triangular, and its eigenvalues are:

Additional Resource

3Blue1Brown
A beautiful, animated, and visual explanation of eigenvalues and eigenvectors.

http://bit.ly/2lXyJPg

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda = 1, 0$

new eigenvector?

new eigenvalues for RREF A ≠

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \lambda_1 = 1$

$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = 0$

Some new
some old eigenvectors.

$(A - \lambda I)\vec{x} = \vec{0}$

↑ the eigenvectors of A w/ λ (if $\vec{x} \neq \vec{0}$)

$\Updownarrow$

$A\vec{x} = \lambda_i \vec{x}$ $\quad$ x is an eigenvector of A w/ λ.

$\Updownarrow$

Nul$(A - \lambda I)$ eigenspace.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

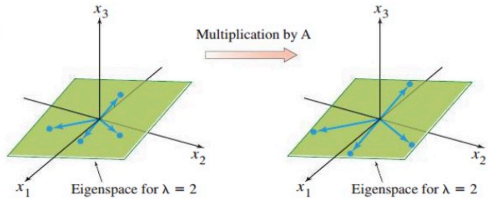


FIGURE 3 A acts as a dilation on the eigenspace.

THEOREM 2

> If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3

$A-2I \sim \cdots \sim \boxed{B}$

RREF of $A-2I$

Solns $(A-2I)\vec{x}=0$ same as solns $Bx=0$

The Characteristic Polynomial

Recall:

λ is an eigenvalue of A $\Leftrightarrow$ $(A-\lambda I)$ is not invertible

Therefore, to calculate the eigenvalues of A, we can solve

$\det(A-\lambda I) = 0$

The quantity $\det(A-\lambda I)$ is the **characteristic polynomial** of A.

The quantity $\det(A-\lambda I)=0$ is the **characteristic equation** of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The degree of $p(\lambda)$ is n if A is $n\times n$

Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is: *

$$p(\lambda) = \det(A-\lambda I_2) = \det\begin{bmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(1-\lambda) - 4$$

$$= (-1)^2(\lambda-5)(\lambda-1) - 4$$

$$= \lambda^2 - 6\lambda + 5 - 4$$

The char poly of A

$$= \boxed{\lambda^2 - 6\lambda + 1}$$

So the eigenvalues of A are:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6\pm\sqrt{32}}{2} = 3 \pm \frac{4\sqrt{2}}{2}$$

$$= \boxed{3 \pm 2\sqrt{2}}$$

$$\approx 0.2, 5.8$$

Fun only eigenvalue of $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\det(I_3 - \lambda I_3) = \det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3$$

$$= (1-\lambda)(1-\lambda)(1-\lambda)$$

$$= \boxed{-(\lambda-1)(\lambda-1)(\lambda-1)}$$

$$= -(\lambda-1)(\lambda^2-2\lambda+1) = -\lambda^3+2\lambda^2-\lambda+\lambda^2-2\lambda+1$$

$$= \boxed{-\lambda^3+3\lambda^2-3\lambda+1}$$

Characteristic Polynomial of 2 × 2 Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when M is singular?

$\det(M - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$= (a-\lambda)(d-\lambda) - bc$

$= \lambda^2 - a\lambda - d\lambda + ad - bc$

$= \lambda^2 - \underbrace{(a+d)}_{\text{trace}(M)}\lambda + \underbrace{(ad-bc)}_{\det M}$

$= \lambda^2 - \text{trace}(M) * \lambda + \det M$

Algebraic Multiplicity

$p(\lambda) = (\lambda - c_1)^2(\lambda - c_2)^4$

$\lambda_1 = c_1$ alg mult is 2
$\lambda_2 = c_2$ alg mult. is 4.

Definition

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\begin{cases} \lambda_1 = 1 & \text{alg } 1 \\ \lambda_2 = 0 & \text{alg } 2 \\ \lambda_3 = -1 & \text{alg } 1. \end{cases}$

I_3 $\lambda = 1$ only eigenvalue
alg mult is $\boxed{3}$

For $n \times n$

?? unknown terms.

$p(\lambda) = \det(A - \lambda I)$

$= (-1)^n \lambda^n \pm \text{trace}(A)\lambda^{n-1} + \dots + \det(A)$

Geometric Multiplicity

Definition

The **geometric multiplicity** of an eigenvalue λ is the dimension of $\text{Null}(A - \lambda I)$.

eigenspace of A for λ.

$\dim(\text{Nul}(A - \lambda I))$
= ~~three~~ var in $A - \lambda I$.

1. Geometric multiplicity is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

$\boxed{\lambda = 0 \quad \text{alg } 2 \quad \text{geo } 1}$

$A - 0I = A \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ The $\lambda = 0$ eigenspace of A.
is a line in $\mathbb{R}^2$.

The dim is 1 so

Example

Give an example of a 4 × 4 matrix with $\lambda = 3$ the only eigenvalue, but the geometric multiplicity of $\lambda = 3$ is one.

alg 4 ?

Check $p(\lambda) = (\lambda - 3)^4$ ✓

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Want one free var in $A - 3I$

$$A - 3I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$p(\lambda) = \det(A - \lambda I)$

$= \det\begin{bmatrix} 3-\lambda & * & * & * \\ 0 & 3-\lambda & * & * \\ 0 & 0 & 3-\lambda & * \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$

$= (3-\lambda)^4$

$= (-1)^4(\lambda - 3)^4$

geo is 1.

FACT.

$\boxed{\text{alg} \geq \text{geo} \geq 1}$

Recall: Long-Term Behavior of Markov Chains

Recall:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots$$

  as $k \to \infty$.
- If P is regular, then there is a ____________________

Now lets ask:

- If we don't know whether P is regular, what else might we do to describe the long-term behavior of the system?
- What can eigenvalues tell us about the behavior of these systems?

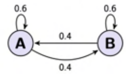
Example: Eigenvalues and Markov Chains

Note: the textbook has a similar example that you can review.

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \dots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:



Goal: use eigenvalues to describe the long-term behavior of our system.

What are the eigenvalues of P?

What are the corresponding eigenvectors of P?

Use the eigenvalues and eigenvectors of P to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

Similar Matrices

Definition

Two $n \times n$ matrices A and B are **similar** if there is a matrix P so that $A = PBP^{-1}$.

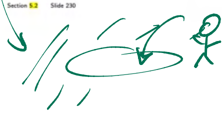
Theorem

If A and B similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section 5.2, but doesn't have exercises on this concept until 5.3.
- Two matrices, A and B, do not need to be similar to have the same eigenvalues. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = B$$



$$P_A(\lambda) = (\lambda - 0)^2 = \lambda^2$$

$$P_B(\lambda) = \lambda^2$$

Suppose $A \overset{?}{=} PBP^{-1}$

$$\boxed{A = B}$$

Additional Examples (if time permits)

1. True or false.
   a) If A is similar to the identity matrix, then A is equal to the identity matrix.
   b) A row replacement operation on a matrix does not change its eigenvalues.
2. For what values of k does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$

5.2 Exercises

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix using expansion across a row or down a column. [*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 6 & 7 & -4 \end{bmatrix}$

For the matrices in Exercises 15–17, list the eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$

In Exercises 21–30, A and B are $n \times n$ matrices. Mark each statement True or False (T/F). Justify each answer.

21. **(T/F)** If 0 is an eigenvalue of A, then A is invertible.

22. **(T/F)** The zero vector is in the eigenspace of A associated with an eigenvalue λ.

23. **(T/F)** The matrix A and its transpose, A^T, have different sets of eigenvalues.

24. **(T/F)** The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

25. **(T/F)** If 2 is an eigenvalue of A, then $A - 2I$ is not invertible.

26. **(T/F)** If two matrices have the same set of eigenvalues, then they are similar.

27. **(T/F)** If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

28. **(T/F)** The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

29. **(T/F)** The eigenvalue of the $n \times n$ identity matrix is 1 with algebraic multiplicity n.

30. **(T/F)** The matrix A can have more than n eigenvalues.

Midterm 2 Lecture Review Activity, Math 1554

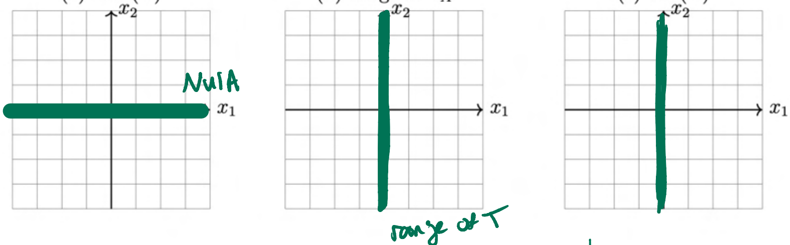
Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving throug

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 - 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 - 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 - 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 - 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/3 - 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 - 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 - 3/21	Break	Break	Break	Break	Break
12	3/24 - 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 - 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 - 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 - 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 - 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

(a) Null(A) (b) range of T_A (c) Col(A)



$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) Col A = span$\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\boxed{b = x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

(a) Nul A need to row reduce A to RREF $A \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✱ eqn.

So Nul A = span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

(b) range of T is the set of vectors that are output vector

$\boxed{Ax = b}$ collect all possible b's.

FACT:
If $T(\vec{x}) = Ax$
Then range of T
is Col A

2. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = a, x_2 = 4a, x_3 = x_1 x_2\}$ is a subspace for any ~~$a \in \mathbb{R}$~~. $a = 1$.	○	●
b) If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$ then $\det(A) \neq 0$.	○	●

for e_1, e_2 false

(a) list some vectors in S

$x_1 = 1$
$x_2 = 4$
$x_3 = 1 * 4$

$\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \in S$ and $S = \left\{\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}\right\}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{y} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Check $A\vec{x} = A\vec{e}_1 \stackrel{?}{=} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Big\}$ same
$A\vec{y} = A\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

(a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix}2\\3\end{pmatrix}$ and $\dim(\text{Null}(A)) = 1$. $A = \begin{pmatrix}2&2\\3&3\end{pmatrix}$, $\begin{pmatrix}2&0\\3&0\end{pmatrix}$, $\begin{pmatrix}1&-2\\3/2&-3\end{pmatrix}$

(b) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix}2\\3\end{pmatrix}$ & $\begin{bmatrix}-2\\-3\end{bmatrix}$ and $\dim(\text{Null}(A)) = 0$. $A = \begin{pmatrix}NP\end{pmatrix}$

$1+0 \neq 2 \leftarrow$ #cols

↑ rank A ↖ dim Nul A

(c) A is in RREF ?? and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. ✓ The vectors u and v are a basis for the range of T. ✓ Col A.

$u = \begin{pmatrix}1\\0\\0\end{pmatrix}, v = \begin{pmatrix}1\\1\\0\end{pmatrix}, A = \begin{pmatrix}1&0&*\\0&1&*\\0&0&0\end{pmatrix}$ ✓

??

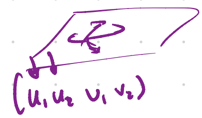
Q: $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

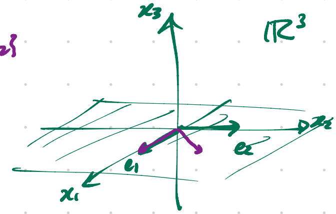
$\vec{e}_1$ $\vec{e}_2$ $\vec{e}_1$ $\vec{e}_1+\vec{e}_2$

$\mathbb{R}^3$

next? $\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$ not RREF

$\text{span}\{u_1,u_2\} = \text{span}\{v_1,v_2\}$





next? $A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ RREF ✓ 3×3 ✓

$\text{Col}\,A \overset{?}{=} \text{span}\{\vec{u}, \vec{v}\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

To get a basis of ColA
Step 1 row reduce A
find pivots
Step 2 extract cols of A

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

$\lambda_1 \neq \lambda_2$ w/ $\lambda_1 = \lambda_2$ assume

	possible	impossible	T/F
4.i) Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u}+\vec{v}$ is also an eigenvector of A.	●	○	○ ● ●
4.ii) $T_A = A\vec{x}$ is one-to-one, $\boxed{\dim(\text{Col}(A)) = 4}$, and $T_A : \mathbb{R}^{3} \to \mathbb{R}^{4}$. (#cols, #rows)	○	●	

example: $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $u = \begin{bmatrix}1\\0\end{bmatrix}$ $v = \begin{bmatrix}3\\0\end{bmatrix}$ $\lambda = 1$

$A(u+v) = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{bmatrix}4\\0\end{bmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$ $\lambda=1$

$\text{Col}A \begin{bmatrix}2&-1\\3&4\end{bmatrix} = \text{span}\{\vec{e}_1, \vec{e}_2\}$

$A = \begin{bmatrix}*&*&*\\ *&*&*\\ *&*&*\\ *&*&*\end{bmatrix}$ size is 4x3 tall

pivot in every col b/c 1-1

rank A = 4 NP

5. (2 points) Fill in the blanks.

(a) If A is a 6×4 matrix in RREF and $\text{rank}(A) = 4$, what is the rank of A^T? $\boxed{}$

(b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? $\boxed{}$

6. (3 points) A virus is spreading in a lake. Every week,

- 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
- 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

Initially there are exactly 1000 fish in the lake.

a) What is the stochastic matrix, P, for this situation? Is P regular?

b) Write down any steady-state vector for the corresponding Markov-chain.

6. (3 points) A virus is spreading in a lake. Every week,

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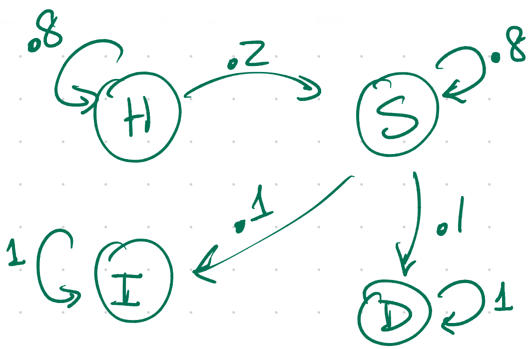
Initially there are exactly 1000 fish in the lake.

a) What is the stochastic matrix, P, for this situation? $\boxed{\text{Is } P \text{ regular?}}$

b) Write down any steady-state vector for the corresponding Markov-chain.

$$P = \begin{array}{c} \\ H \\ S \\ I \\ D \end{array} \begin{array}{c} \begin{array}{cccc} H & S & I & D \end{array} \\ \begin{bmatrix} .8 & 0 & 0 & 0 \\ .2 & .8 & 0 & 0 \\ 0 & .1 & 1 & 0 \\ 0 & .1 & 0 & 1 \end{bmatrix} \end{array}$$

(c) is P regular?



$$\boxed{\begin{array}{l} Pe_3 = e_3 \\ Pe_4 = e_4 \end{array}}$$

(b) $\begin{bmatrix} 0 \\ 0 \\ .5 \\ .5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ s \\ 1-s \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (1-s) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$(0 \le s \le 1.)$

(c) No

$\boxed{\text{There are more than one prob. steady-state vectors. So } P \text{ is NOT regular.}}$

THM IF P regular then there is unique prob steady state $\vec{q}$

Non-thm IF P has unique $\vec{q}$ $\not\Rightarrow$ P regular

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Stochastic? yes

regular ? No.

$P^2 = I$

$P^3 = P$

$P^4 = I$

$P^5 = P$

$\vdots$

Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Motivation: it can be useful to take large powers of matrices, for example A^k, for large k.

But: multiplying two $n \times n$ matrices requires roughly n^3 computations. Is there a more efficient way to compute A^k?

Topics and Objectives

Topics

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

Learning Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Motivation: it can be useful to take large powers of matrices, for example A^k, for large k.

But: multiplying two $n \times n$ matrices requires roughly n^3 computations. Is there a more efficient way to compute A^k?

Defn: a matrix A is diagonalizable if $\boxed{A = PDP^{-1}}$ for some diagonal matrix D.

A must be $n \times n$ for this to be true.

Topics and Objectives

Topics

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

Learning Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

Defn: $A = PBP^{-1}$ means A is similar to B.

(aka A is similar to a diagonal matrix.)

Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course mate

		Mon	Tue	Wed	Thu	Fri
Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/6 - 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 - 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 - 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 - 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/3 - 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 - 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 - 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 - 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/3 - 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 - 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 - 3/21	Break	Break	Break	Break	Break
12	3/24 - 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 - 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 - 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 - 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 - 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 - 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

Diagonal Matrices

A matrix is **diagonal** if the only non-zero elements, if any, are on the main diagonal.

The following are all diagonal matrices.

$$2I_n = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad [2], \quad I_n, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll only be working with diagonal square matrices in this course.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{diagonal} \quad \begin{matrix} \lambda_1 = 3 & v_1 = \vec{e}_1 \\ \lambda_2 = 2 & v_2 = \vec{e}_2 \\ \lambda_1 = 1 & v_3 = \vec{e}_3 \end{matrix}$$

Check λ_1, v_1 $\quad A\vec{v}_1 = \lambda_1 \vec{v}_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$A\vec{e}_2$ $\quad$ $A\vec{e}_3$

Powers of Diagonal Matrices

If A is diagonal, then A^k is easy to compute. For example,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 0 \\ 0 & .25 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3^k & 0 \\ 0 & (.5)^k \end{bmatrix}$$

But what if A is not diagonal?

IF $A = PDP^{-1}$

Then

$$A^k = \overbrace{A * A * A * \cdots * A}^{k \text{ times}}$$

$$= PD\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$$

$$= P\underbrace{DDD\cdots D}_{k \text{ times}}P^{-1}$$

$$= PD^kP^{-1}$$

Diagonalization

Suppose $A \in \mathbb{R}^{n\times n}$. We say that A is **diagonalizable** if it is similar to a diagonal matrix, D. That is, we can write

$$A = PDP^{-1}$$

Note: D diagonal then D has n linearly ind eigenvectors $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$.

Note

$$D = IDI^{-1}$$

D is diagonalizable

Diagonalization

Theorem

If A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

Note: the symbol $\Leftrightarrow$ means " if and only if ".

Also note that $A = PDP^{-1}$ if and only if

$$A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]^{-1}$$

diagonal

$A = PDP^{-1}$

invertible

where $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent eigenvectors, and $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues (**in order**).

Goal: Diagonalize the $n\times n$ matrix A:

Step 1: Find λ's eigenvalues of A

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Step 2: Find $\{v_1, v_2, v_3\}$ lin ind eigenvectors.

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$$ columns of P are eigenvectors.

Distinct Eigenvalues

A: $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_1 = 1$ & A is 3×3 } Then A is diag'ble.

Theorem

If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.

Why does this theorem hold?

If λ's distinct then $\vec{v}_i$'s are lin ind. (apply the previous Theorem)

Is it necessary for an $n \times n$ matrix to have n distinct eigenvalues for it to be diagonalizable?

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_3, \quad O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

repeated $\lambda_1 = 0$ alg 2.

all λ's are 1

all λ's are zero

Non-Distinct Eigenvalues

If A has a repeated eigenvalue then how to get D & P.

Theorem. Suppose

- A is $n \times n$
- A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$, $k \le n$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ repeated λ's?

Then

1. $d_i \le a_i$ for all i — If alg = geo then diag'ble
2. A is diagonalizable $\Leftrightarrow$ $\Sigma d_i = n$ $\Leftrightarrow$ $d_i = a_i$ for all i
3. A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

$P = (?\ ?\ ?)$

Example 1

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Step 1 Find λ's & form D

Step 2 Find $\vec{v}$'s & form P

Step 1 $p(\lambda) = \det(A - \lambda I) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$

$-\text{trace}(A) * \lambda$; $+\det A$

$\lambda_1 = 2, \lambda_2 = -1$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Step 2. Find $\vec{v}_1, \vec{v}_2$

$\lambda_1 = 2$: $A - 2I = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{x} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$: $A + I = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{x} = r\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad ??$$

$$\boxed{\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}}$$

A

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ X

also works

$$\boxed{\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}}$$

Example 2

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Not possible

Step 1 Find $\lambda_1 = 3 \quad \lambda_2 = 3$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

($\lambda = 3$ alg mult is 2)

alg ≠ geo.

Step 2 Find v_1, v_2

$A - 3I = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \vec{x} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

($\lambda = 3$ geo mult. is 1)

$$A \overset{?}{=} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\boxed{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^{-1}}}_{P^{-1}}$$

Example 3

Step 3? Q: $D=\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}$ or $D=\begin{bmatrix}3&0&0\\0&3&0\\0&0&1\end{bmatrix}$

The eigenvalues of A are $\boxed{\lambda = 3, 1.}$ If possible, construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

need $\boxed{\text{alg}=\text{geo}}$ for both λ's.

Step 2:

$\lambda_1=3$

$$A-3I=\begin{bmatrix}4&4&16\\2&2&8\\-2&-2&-8\end{bmatrix}\sim\begin{bmatrix}1&1&4\\0&0&0\\0&0&0\end{bmatrix}$$

$$\vec{x}=r\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

$$P=\begin{bmatrix}-1&-4&*\\1&0&*\\0&1&*\end{bmatrix}$$

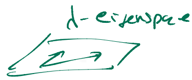
Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the n^{th} number in this sequence.



$\lambda_2=1$

$$A-I=\begin{bmatrix}6&4&16\\2&4&8\\-2&-2&-6\end{bmatrix}\sim\begin{bmatrix}6&4&16\\1&2&4\\0&2&2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&4\\0&1&1\\0&-8&-8\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\end{bmatrix}$$

$$\vec{x}=r\begin{bmatrix}-2\\-1\\1\end{bmatrix}$$

So

$$P=\begin{bmatrix}-1&-4&-2\\1&0&-1\\0&1&1\end{bmatrix} \qquad D=\begin{bmatrix}3&0&0\\0&\boxed{3}&0\\0&0&1\end{bmatrix}?$$

$\lambda_1 \quad \lambda_1 \quad \lambda_2$

Where.

$$\boxed{A=PDP^{-1}}$$

also worked ✓

$$P=\begin{bmatrix}-2&-4&-1\\-1&0&1\\1&1&0\end{bmatrix} \quad D=\begin{bmatrix}1&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

NOTICE the diag'zation is not unique

Additional Example (if time permits)

$A^k = (PDP^{-1})^k = PD^kP^{-1}$

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the n^{th} number in this sequence.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \vec{X}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $\vec{X}_n = A\vec{X}_{n-1}$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Please tell me

Fibbonacci

$\vec{X}_0, \vec{X}_1, \vec{X}_2, \ldots$ $\quad 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 13 \end{bmatrix}, \ldots$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$

nth vector

$\vec{X}_n = A\vec{X}_{n-1} = A^n \vec{X}_0$

$\lambda_1 =$

$\vec{X}_{100} = A^{100}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 =$

$= PD^{100}P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE 4 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

THEOREM 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

THEOREM 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

5.3 EXERCISES

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

In Exercises 5 and 6, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} =$
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

6. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} =$
$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Diagonalize the matrices in Exercises 7–20, if possible. The eigenvalues for Exercises 11–16 are as follows: (11) $\lambda = 1, 2, 3$; (12) $\lambda = 2, 8$; (13) $\lambda = 5, 1$; (14) $\lambda = 5, 4$; (15) $\lambda = 3, 1$; (16) $\lambda = 2, 1$. For Exercise 18, one eigenvalue is $\lambda = 5$ and one eigenvector is $(-2, 1, 2)$.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

18. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 21 and 22, A, B, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.
    b. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
    c. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
    d. If A is diagonalizable, then A is invertible.

22. a. A is diagonalizable if A has n eigenvectors.
    b. If A is diagonalizable, then A has n distinct eigenvalues.
    c. If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.
    d. If A is invertible, then A is diagonalizable.

23. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

27. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

28. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

29. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix A in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

30. With A and D as in Example 2, find an invertible P_2 unequal to the P in Example 2, such that $A = P_2 D P_2^{-1}$.

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

[M] Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

33. $\begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$ **34.** $\begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$

35. $\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$