MATH 2550 G/J Midterm 2
VERSION A
Fall 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: Key ______________________ **GT ID:**______________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 50 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	4
3	6
4	6
5	6
6	10
7	8
8	8
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $f(x,y)$ is continuous on $R = [2,4] \times [3,5]$ then the value of $\int_3^5 \int_2^4 f(x,y)\,dy\,dx$ equals the value of $\iint_R f(x,y)\,dA$.

○ **TRUE** ● **FALSE**

2. (4 points) Compute f_{xx} and f_{yx} for the function $f(x,y) = x\tan(2y) + \sqrt{3x - 4y}$. [AJN]

$f_{xx} =$ $\boxed{-\frac{9}{4}(3x-4y)^{-3/2}}$

$f_{yx} =$ $\boxed{2\sec^2(2y) + 3(3x-4y)^{-3/2}}$

$f_x = \tan(2y) + \frac{3}{2\sqrt{3x-4y}}$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{3}{2}(3x-4y)^{-1/2}\right) = \frac{3}{2} * -\frac{1}{2}(3x-4y)^{-3/2} * 3$

$= -\frac{9}{4}(3x-4y)^{-3/2}$

$f_{xy} = 2\sec^2(2y) + \frac{3}{2} * -\frac{1}{2}(3x-4y)^{-3/2} * (-4)$

$= 2\sec^2(2y) + 3(3x-4y)^{-3/2}$

3. (6 points) Compute Df given $f(r,s,t) = \begin{bmatrix} st^2 + re^s \\ 2r - 3s + 4t \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$ [AJN]

$x_r = e^s \quad x_s = t^2 + re^s \quad x_t = 2st$

$y_r = 2 \quad y_s = -3 \quad y_t = 4$

$$Df = \begin{bmatrix} e^s & t^2 + re^s & 2st \\ 2 & -3 & 4 \end{bmatrix}$$

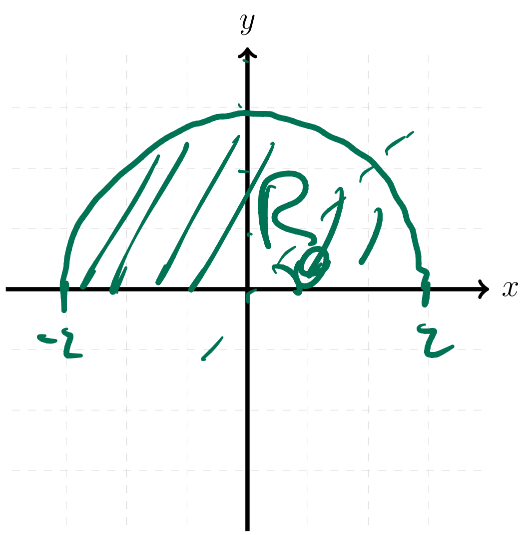
4. (6 points) Sketch R the region of integration and convert the given integral to polar coordinates. *Do not evaluate!* [AN]

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} y \, dy \, dx$$

$y = r\sin\theta$

$\theta \in [0, \pi]$

$r \in (0, 2)$



$$\text{Volume} = \int_0^{\pi} \int_0^{2} r^2 \sin\theta \, dr \, d\theta$$

5. (6 points) Use the function $z = f(x, y)$ and the point P to answer the questions below. [AJN]

$$f(x, y) = \ln(x^2 + y^2) \text{ and } P(1, -1)$$

(a) Find the gradient ∇f of f.

(b) Find the directional derivative $D_{\mathbf{u}}f$ of f in the direction of $\mathbf{u} = \langle 4, 3\rangle$ at $P(1, -1)$.

(c) Find a unit vector $\mathbf{v}$ which points in the direction which *minimizes* the value of $D_{\mathbf{u}}f$ at $P(1, -1)$.

(d) Sketch $P(1, -1)$ and the gradient vector $\nabla f(P)$ on the axes provided.

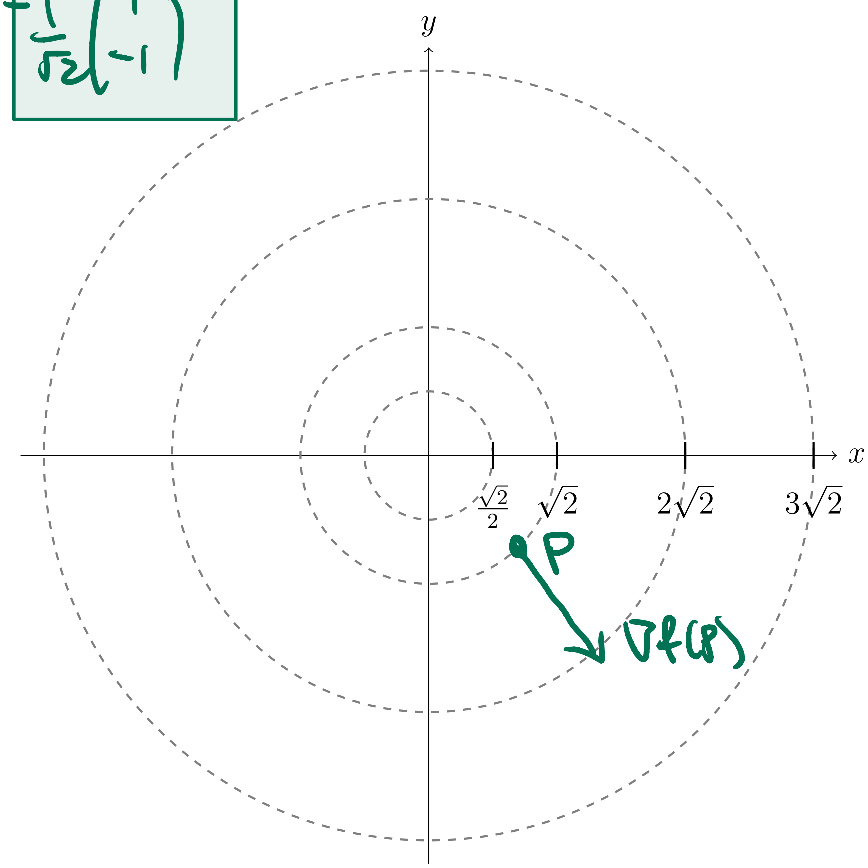
(a) $\boxed{\nabla f = \begin{bmatrix} 2x/x^2+y^2 \\ 2y/x^2+y^2 \end{bmatrix}}$ @P(1,-1) $\nabla f(P) = \begin{bmatrix} 2/2 \\ -2/2 \end{bmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $\hat{u} \sim \frac{1}{\sqrt{16+9}}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ so $D_u f(P) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} = \frac{4}{5} - \frac{3}{5}$

$= \boxed{1/5}$

(c) $\hat{v} = \frac{-1}{\|\nabla f(P)\|}\nabla f(P) = \boxed{-\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(d) see sketch



6. (10 points) Evaluate $\frac{\partial W}{\partial t}$ at the point $(s,t) = (2, \frac{\pi}{4})$. [AJN]

$$W = xy + \ln x,\ x = s + 2t,\ y = \sin t.$$

Formula

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial t}$$

$\frac{\partial W}{\partial x} = y + \frac{1}{x}$ @ $(s,t) = (2, \pi/4)$ Then $x = 2 + 2(\frac{\pi}{4}) = 2 + \frac{\pi}{2}$

$y = \sin(\pi/4) = \sqrt{2}/2$

So $\frac{\partial W}{\partial x} = \frac{\sqrt{2}}{2} + \frac{1}{2+\pi/2}$

$\frac{\partial W}{\partial y} = x$ @ $(s,t) = (2, \pi/4)$ Then $x = 2 + \pi/2$

So $\frac{\partial W}{\partial y} = 2 + \pi/2$

$\frac{\partial x}{\partial t} = 2$ and $\frac{\partial y}{\partial t} = \cos t$ @ $(s,t) = (2, \pi/4)$ Then

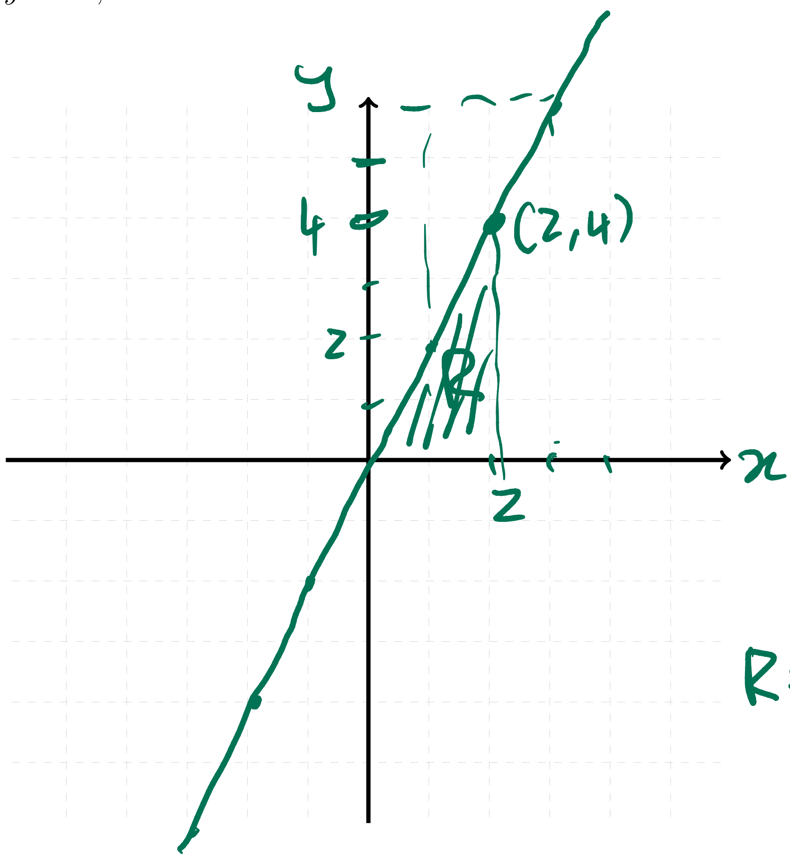
$\frac{\partial y}{\partial t} = \cos(\pi/4) = \sqrt{2}/2$.

So $$\frac{\partial W}{\partial t} = \left(\frac{\sqrt{2}}{2} + \frac{1}{2+\pi/2}\right)(2) + \left(2 + \frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} + \frac{2}{2+\frac{\pi}{2}} + \sqrt{2} + \frac{\sqrt{2}\pi}{4} = \boxed{2\sqrt{2} + \frac{4}{4+\pi} + \frac{\sqrt{2}\pi}{4}}$$

7. (8 points) Sketch the region R including all labels for each of the axes, each boundary curve, and each intersection point between curves and axes. Then, compute the average value of $f(x, y) = x^2 y$ over the region R in the first quadrant bounded by the lines $y = 0$, $y = 2x$, and $x = 2$.

[AJN]



$R: x \in [0, 2]$
$y \in [0, 2x]$

$\text{Area } R = \frac{1}{2}(2)(4) = 4$

$$\text{Vol} = \int_0^2 \int_0^{2x} x^2 y \, dy \, dx = \int_0^2 \frac{1}{2} x^2 y^2 \Big|_0^{2x} dx$$

$$= \int_0^2 \frac{1}{2} x^2 (4x^2 - 0) \, dx = \int_0^2 2x^4 \, dx$$

$$= \frac{2}{5} x^5 \Big|_0^2 = \frac{2^6}{5} = 64/5$$

So Avg value

$\frac{\text{Vol}}{\text{Area } R} = \frac{1}{4} * \frac{64}{5} = \boxed{\frac{16}{5}}$

8. (8 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x, y) = xy + y^2 + 3x + 4y + 5$$

Solve $\nabla f = 0$, $\nabla f = \begin{bmatrix} y+3 \\ x+2y+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $y=-3$ and $x-6+4=0 \Rightarrow x=2$

So only crit point is $(2,-3)$.

and. $Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

and $\det Hf = 0-1 = -1 < 0$

So $\boxed{(2,-3) \text{ is saddle}}$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $\quad dA = r\ dr\ d\theta$

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