MATH 2550 G/J Midterm 2
VERSION B
Fall 2025
COVERS SECTIONS 14.3-14.8, 15.1-15.4

Full name: Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I do not have a phone within reach, and I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) All of the knowledge presented in this exam is entirely my own. I am initialing to the left to attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 50 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work [J] and annotate your work using proper notation [N].
- Good luck!

Question	Points
1	2
2	4
3	6
4	6
5	6
6	10
7	8
8	8
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. For all problems on this page please be sure to neatly fill in the bubble corresponding to your answer choice. [A]

1. (2 points) If $f(x,y)$ is continuous on $R = [1,2] \times [3,4]$ then the value of $\int_1^2 \int_3^4 f(x,y)\,dy\,dx$ equals the value of $\int_3^4 \int_1^2 f(x,y)\,dy\,dx$.

   ○ **TRUE** ● **FALSE**

2. (4 points) Compute f_{xx} and f_{yx} for the function $f(x,y) = \sin(xy) + ye^{3x}$. [AJN]

$f_{xx} =$ $\boxed{-y^2 \sin(xy) + 9ye^{3x}}$

$f_{yx} =$ $\boxed{\cos xy - xy \sin xy + 3e^{3x}}$

$f_x = y\cos(xy) + 3ye^{3x}$

$f_{xx} = -y^2 \sin(xy) + 9ye^{3x}$

$f_{xy} = (1)\cos(xy) + y(-x\sin(xy)) + 3e^{3x}$

$= \cos xy - xy \sin xy + 3e^{3x} = f_{yx}$ ✓

3. (6 points) Compute Df given $f(r,s,t) = \begin{bmatrix} s^2t + r/s \\ 3r - 4s + 5t \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$ [AJN]

$x_r = 1/s$

$x_s = 2st - \frac{r}{s^2}$

$x_t = s^2$

$y_r = 3$

$y_s = -4$

$y_t = 5$

$$Df = \boxed{\begin{bmatrix} 1/s & 2st - \frac{r}{s^2} & s^2 \\ 3 & -4 & 5 \end{bmatrix}}$$

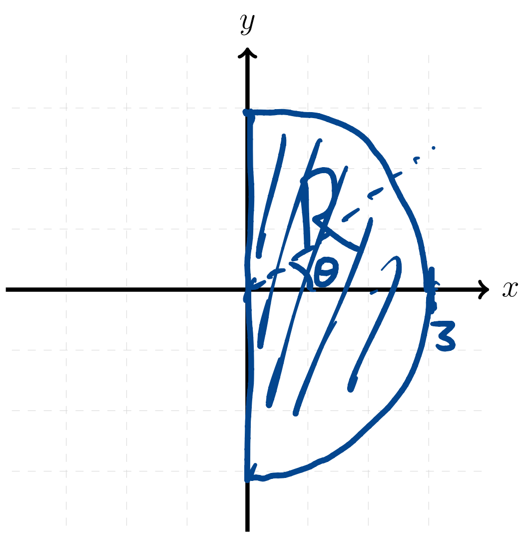
4. (6 points) Sketch R the region of integration and convert the given integral to polar coordinates. *Do not evaluate!* [AN]

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x \, dy \, dx$$

$x = r\cos\theta$

$\theta \in [-\pi/2, \pi/2]$

$r \in [0, 3]$



$$\text{Volume} = \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^2 \cos\theta \, dr \, d\theta}$$

5. (6 points) Use the function $z = f(x, y)$ and the point P to answer the questions below. [AJN]

$$f(x, y) = \ln(x^2 + y^2) \text{ and } P(-1, 1)$$

(a) Find the gradient ∇f of f.

(b) Find the directional derivative $D_{\mathbf{u}}f$ of f in the direction of $\mathbf{u} = \langle 4, 3\rangle$ at $P(-1, 1)$.

(c) Find a unit vector $\mathbf{v}$ which points in the direction which *maximizes* the value of $D_{\mathbf{u}}f$ at $P(-1, 1)$.

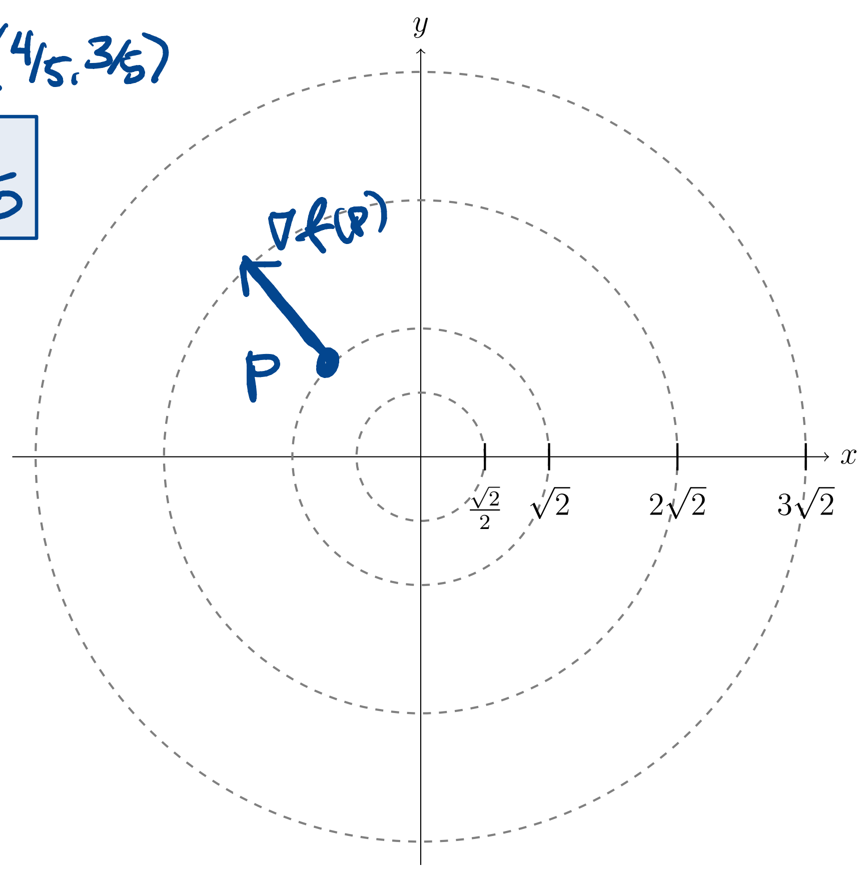
(d) Sketch $P(-1, 1)$ and the gradient vector $\nabla f(P)$ on the axes provided.

(a) $\nabla f = \boxed{\begin{bmatrix} 2x / x^2+y^2 \\ 2y / x^2+y^2 \end{bmatrix}}$

(b) $\vec{u} \sim \frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{16+9}}\langle 4, 3\rangle = \langle 4/5, 3/5\rangle$ $\qquad \nabla f(P) = \begin{bmatrix} -2/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $D_u f(P) = \nabla f(P) \cdot \langle 4/5, 3/5\rangle$

$= -\frac{4}{5} + \frac{3}{5} = \boxed{-1/5}$



(c) $\vec{v} = \frac{1}{\|\nabla f(P)\|}\nabla f(P)$

$= \boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

(d) See graph

6. (10 points) Evaluate $\frac{\partial W}{\partial t}$ at the point $(s,t) = (2, \frac{\pi}{4})$. [AJN]

$$W = xy + \ln y,\ x = s + t,\ y = \cos t.$$

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial t}$$

$\frac{\partial W}{\partial x} = y$ @ $(s,t) = (2, \pi/4)$ then $x = 2 + \pi/4$, $y = \cos(\pi/4) = \sqrt{2}/2$

So $\frac{\partial W}{\partial x} = \sqrt{2}/2$

$\frac{\partial W}{\partial y} = x + \frac{1}{y}$ So $\frac{\partial W}{\partial y} = 2 + \frac{\pi}{4} + \frac{2}{\sqrt{2}}$ @ $(s,t) = (2, \pi/4)$

$(x,y) = (2 + \frac{\pi}{4}, \sqrt{2}/2)$

$= 2 + \frac{\pi}{4} + \sqrt{2}$

$\frac{\partial x}{\partial t} = 1$ and $\frac{\partial y}{\partial t} = -\sin t$ @ $(s,t) = (2, \pi/4)$

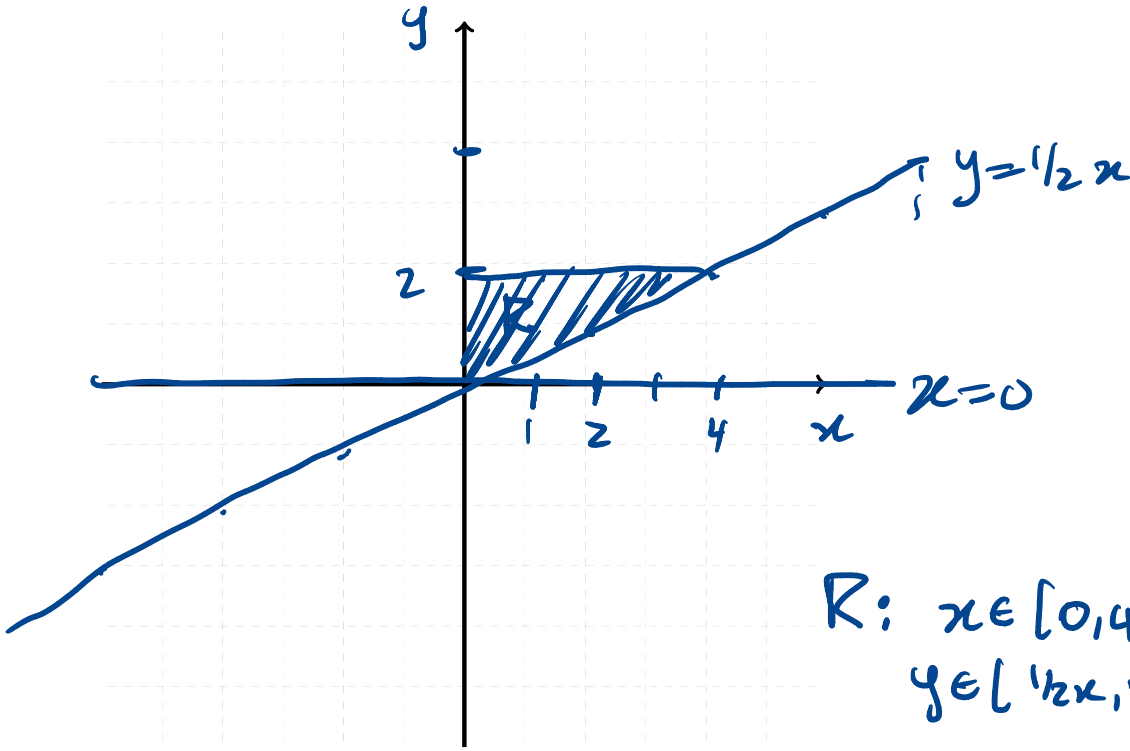
$\frac{\partial y}{\partial t} = -\sin(\pi/4) = -\frac{\sqrt{2}}{2}$.

So $\frac{\partial W}{\partial t} = \left(\frac{\sqrt{2}}{2}\right)(1) + \left(2 + \frac{\pi}{4} + \sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{8} - 1 = \boxed{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{8} - 1}$

7. (8 points) Sketch the region R including all labels for each of the axes, each boundary curve, and each intersection point between curves and axes. Then, compute the average value of $f(x, y) = xy$ over the region R in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = \frac{1}{2}x$, and $y = 2$.

[AJN]



$R: x \in [0,4)$
$y \in [\frac{1}{2}x, 2]$

Area $R = \frac{1}{2}(4)(2) = 4$ half rectangle.

$$\text{Vol} = \int_0^4 \int_{\frac{1}{2}x}^{2} xy \, dy \, dx$$

$$= \int_0^4 \frac{1}{2}xy^2 \Big|_{\frac{1}{2}x}^{2} dx = \int_0^4 \frac{1}{2}x\left(4 - \frac{1}{4}x^2\right) dx$$

$$= \int_0^4 2x - \frac{1}{8}x^3 \, dx = x^2 - \frac{1}{32}x^4 \Big|_0^4 = 16 - \frac{4^4}{32} = 16 - \frac{4^2}{2}$$

$$= 16 - 8 = 8$$

So Avg value is $\frac{\text{Vol}}{\text{Area } R} = \frac{8}{4} = \boxed{2}$

8. (8 points) Find all critical points and use the Hessian matrix to classify the critical points of the function. [AJN]

$$f(x, y) = x^2 + xy + 3x + 2y + 5$$

Solve

$\nabla f = 0$, $\nabla f = \begin{bmatrix} 2x+y+3 \\ x+2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $x = -2$ and $(-4)+y+3=0$

so $y = 1$.

only soln is $(-2, 1)$.

$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

and $\det Hf = 0 - 1 < 0$ so

$\boxed{\text{Saddle @ } (-2, 1).}$

FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: If z is implicitly given in terms of x and y by $F(x, y, z) = c$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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