MATH 2550 G/J w/ Dr. Sal Barone

- Dr. Barone, Prof. Sal, or just Sal, as you prefer

Daily Announcements & Reminders:

Goals for Today: Sections 12.1, 12.4, 12.5

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
-
-

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

1551 Differential Calculus	1552 Integral Calculus
limits	integrals
deriv. (single var)	U-Sub
PROD/QUOT/chain	IBP
Riemann sum	~~trig~~
continuity.	partial ~~frac~~
	Power red.

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

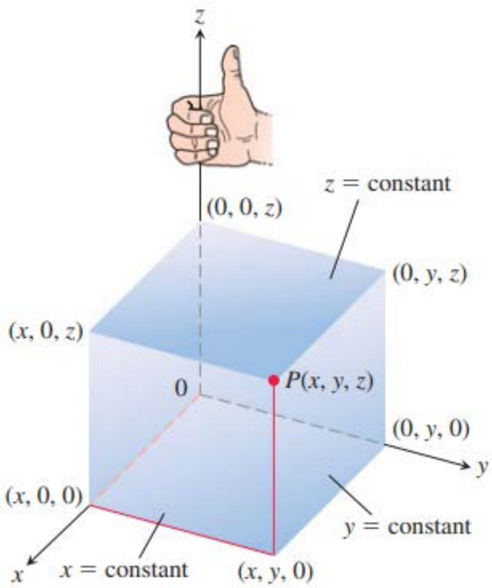
Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

$f : \mathbb{R} \to \mathbb{R}^3$ $\quad f(t) = \langle t, t, t \rangle, \ t \in \mathbb{R}.$

$g(t) = \langle t, t^2, t^3 \rangle, \ t \in \mathbb{R}$

$f : \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = x^2 + y^2 + z^2, \ x,y,z \in \mathbb{R}.$

§12.1: Three-Dimensional Coordinate Systems

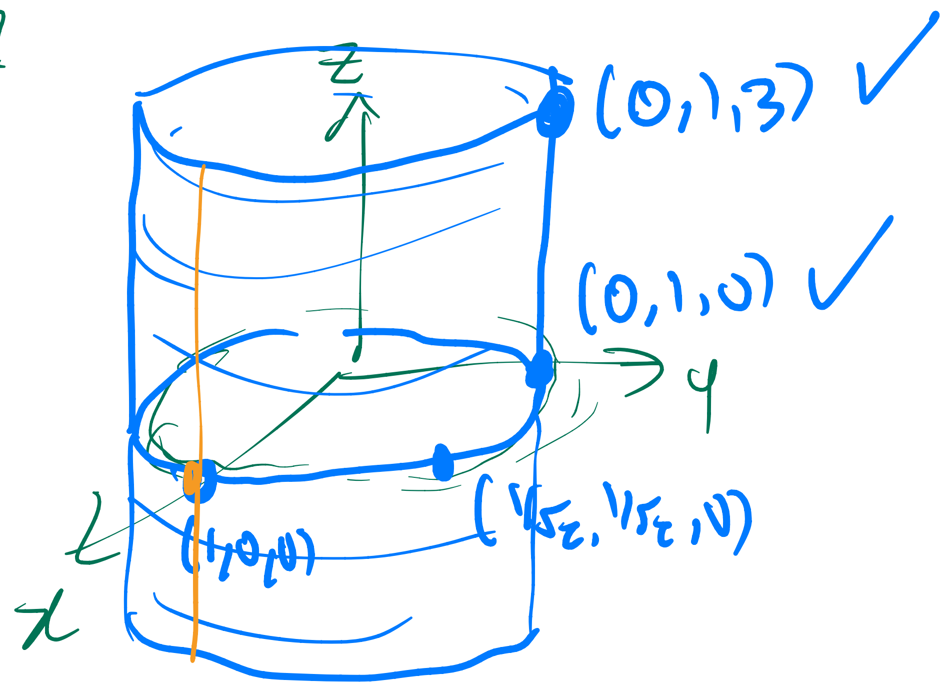


$\mathbb{R}^3$ (a,b,c) point location in space.

$\langle a,b,c\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Vector pointing in direction of (a,b,c) starting at $\vec{0}$.

Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2 + y^2 = 1$?

Cylinder? plane? circle?



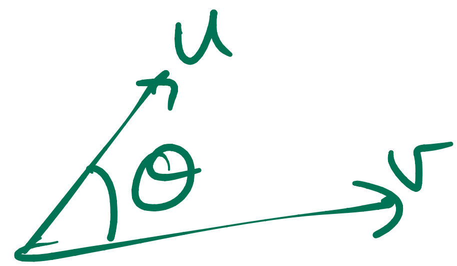
\langle \rangle

§12.3, 12.4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

This product tells us about angles & orthogonality.



$$\vec{u} \cdot \vec{v} = \|u\| * \|v\| * \cos\theta$$

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 2. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = (1)(-3) + (1)(-1) + (4)(1) = -3 - 1 + 4$$
$$= 0 \checkmark$$

So $\vec{u}$ & $\vec{v}$ are orthogonal.

$u \circ v$ scalar $\qquad$ $u \times v$ vector.

defined only in $\mathbb{R}^3$

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. play nice w/ scalars $\quad c\vec{u} \times \vec{v} = c(\vec{u} \times \vec{v})$

2. play nice w/ vector +

$$u \times (v_1 + v_2) = u \times v_1 + u \times v_2$$

$$\boxed{* \quad u \times v = -v \times u}$$?? why?

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

determinant of a 3x3 matrix

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$

Why is $\hat{j}$ neg?

Example 4. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle = \vec{w}$

$$\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= \hat{i}\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-1-6)$$

$$= 0\hat{i} - 0\hat{j} - 7\hat{k} = \boxed{-7\hat{k}}$$

either is ok.

$$= \boxed{\langle 0, 0, -7\rangle}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix}$$ also ok.

Example 5. *You try it!* Find $\langle 2, 1, 0\rangle \times \langle 1, 2, 1\rangle$. $= \vec{w}$

$$\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \boxed{\hat{i} - 2\hat{j} + 3\hat{k}}$$

$$= \langle 1, -2, 3\rangle$$ also ok.

Some common [AJN] things to look out for. specific to class.

[A] Accuracy

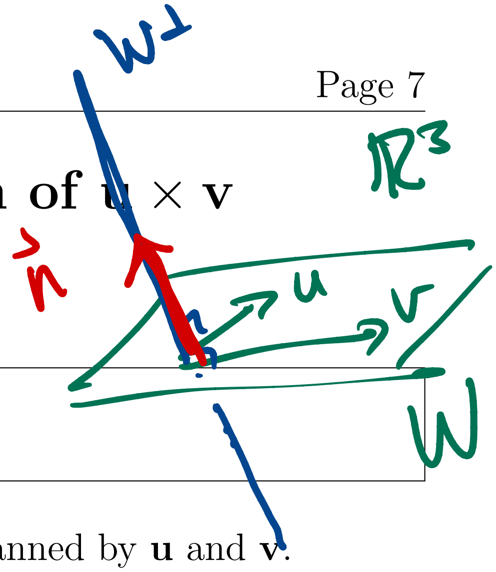
- simplify answer
- box answer

[J] Justification

- minus sign on $\mathbf{j}$ component
- show intermediate steps

[N] Notation

- use = sign for expressions that are equal
- vector notation vs. point notation



A Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$

The cross product $\mathbf{u} \times \mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin\theta)\mathbf{n}$$

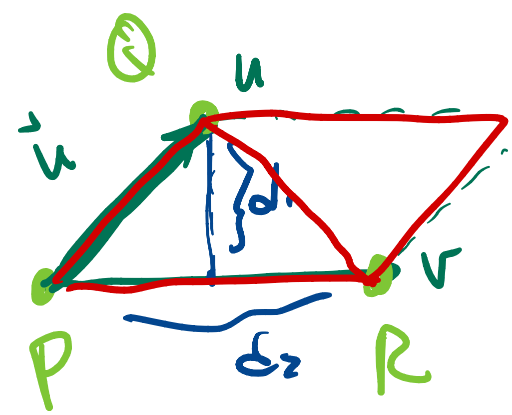
where $\mathbf{n}$ is a unit vector which is normal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

Example 5. Find the area of the parallelogram determined by the points P, Q, and R.

$$P(1,1,1),\ Q(2,1,3),\ R(3,-1,1)$$



$$\vec{u} = \vec{PQ} = \langle 2,1,3\rangle - \langle 1,1,1\rangle$$
$$= \langle 1,0,2\rangle$$

So

$$\vec{v} = \vec{PR} = \langle 3,-1,1\rangle - \langle 1,1,1\rangle$$

So

$$= \langle 2,-2,0\rangle$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}$$
$$= 4\hat{i} - (-4)\hat{j} - 2\hat{k} = \langle 4,4,-2\rangle$$

Now

$$|\vec{u}\times\vec{v}| = \sqrt{4^2+4^2+(-2)^2} = \sqrt{16+16+4} = \sqrt{36} = \boxed{6}$$