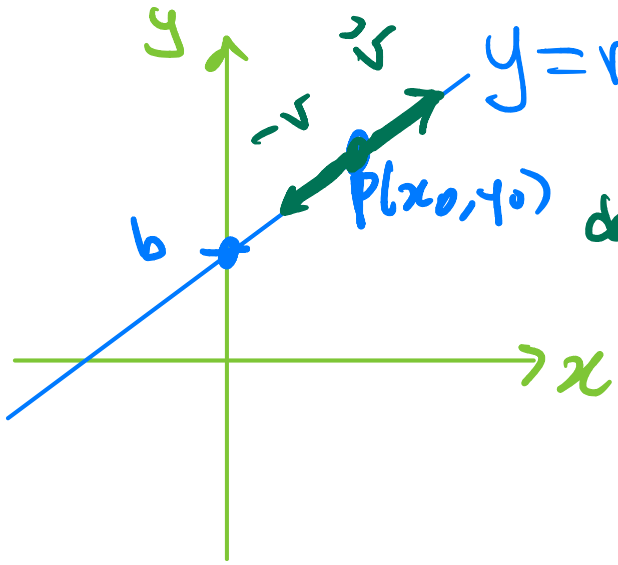
§12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

$m = \text{slope} = \frac{\text{rise}}{\text{run}}$

$b = y\text{-intercept at } x=0$



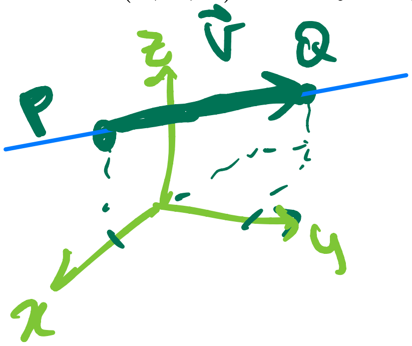
$y = mx + b$

describe the line as all points of the form

$$\boxed{\ell(t) = \overrightarrow{OP} + t\vec{v}, \ t \in \mathbb{R}}$$

vector equation of the line.

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.



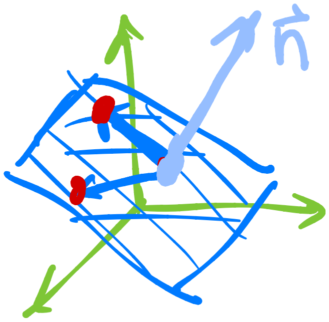
$$\vec{v} = \overrightarrow{PQ} = \langle -2, 1, 1 \rangle - \langle 1, 0, 2 \rangle$$

$$= \langle -3, 1, -1 \rangle$$

$$\boxed{\ell(t) = \langle 1, 0, 2 \rangle + t\langle -3, 1, -1 \rangle, \ t \in \mathbb{R}}$$

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.



Need: * point & two lin. ind vectors in the plane

* three points

* point & normal vector

* the equation of the plane.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$\boxed{ax+by+cz=d}$$ (two free vars)

Solns are (x,y,z) that satisfy

$$\langle x,y,z\rangle \cdot \langle a,b,c\rangle = d \qquad (1)$$

So given (x_0, y_0, z_0) on the plane then

$$\langle x_0,y_0,z_0\rangle \cdot \langle a,b,c\rangle = d \qquad (2)$$

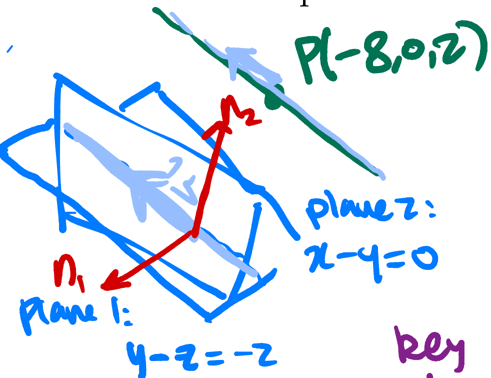
Subtracting (1) & (2) we get

$\vec{n} = \langle a,b,c\rangle$ is normal to the plane $ax+by+cz=d$

$$\left(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right) \cdot \langle a,b,c\rangle = 0$$

$$\Leftrightarrow \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Example 8. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.



Plane 1: $0x + y - z = -2$ $\quad \vec{n}_1 = \langle 0, 1, -1 \rangle$

plane 2: $x - y + 0z = 0$ $\quad \vec{n}_2 = \langle 1, -1, 0 \rangle$

key idea:

$\vec{v}$ in the line of intersection of the two planes is perpendicular to both $\vec{n}_1$ & $\vec{n}_2$

① The planes intersect b/c they are not parallel, since $\vec{n}_1 \neq c\vec{n}_2$

② Use formula $l(t) = \vec{OP} + t\vec{v}$, $P(-8,0,2)$ & $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{v} = \langle 0,1,-1 \rangle \times \langle 1,-1,0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$= -\hat{i} - \hat{j} - \hat{k} = \langle -1,-1,-1 \rangle$ so the line eqn. is

③ $$\boxed{l(t) = \langle -8,0,2 \rangle + t\langle -1,-1,-1 \rangle \quad t \in \mathbb{R}}$$

Example 9. *You try it!* Find the plane containing the lines parameterized by

$$\ell_1(t) = \langle 1,1,1\rangle + t\langle 2,1,0\rangle, \qquad -\infty < t < \infty$$
$$\ell_2(s) = \langle 0,-1,0\rangle + s\langle 1,2,1\rangle, \qquad -\infty < s < \infty$$

Give your answer in the form $Ax+By+Cz = D$ or $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$.

Idea: use $\vec{v}_1 = \langle 2,1,0\rangle$ & $\vec{v}_2 = \langle 1,2,1\rangle$ both in plane

So $\vec{n} = \vec{v}_1 \times \vec{v}_2$ is normal to the plane.

$$\vec{n} = \langle 2,1,0\rangle \times \langle 1,2,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \langle 1,-2,3\rangle = \vec{n}$$

So eqn of plane

$$1 * x - 2 * y + 3 * z = D$$

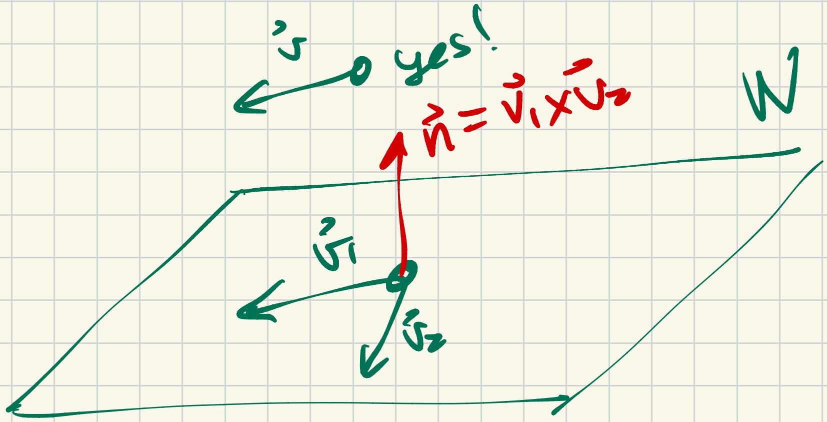
Notice that $l_1(0) = \langle 1,1,1\rangle$ is in the plane

So plug in $(x,y,z) = (1,1,1)$ & get D.

$$1(1) - 2(1) + 3(1) = 2 = D$$

So eqn of the plane is

$$\boxed{x - 2y + 3z = 2}$$



$\vec{v}$ is in W

$\cancel{\vec{v} \in W\,??}$

<u>No</u>.

§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

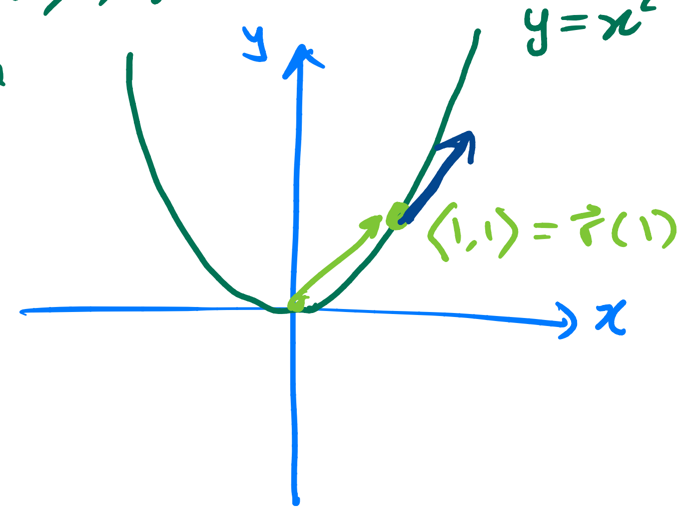
This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

* lines ① $\vec{r}(t) = \vec{OP} + t\vec{v}$, $t \in \mathbb{R}$
* circle ② $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $t \in \mathbb{R}$

but could be for example

③ $\vec{r}(t) = \langle t, t^2 \rangle$

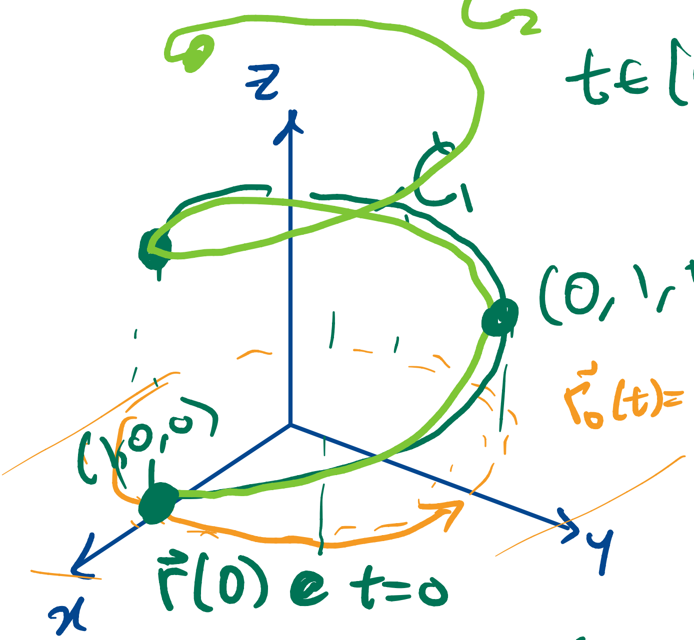


$\vec{r}'(t) = \langle 1, 2t \rangle$

hold on for more details.

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called parametrizing the curve C, and $\mathbf{r}$ is called a parametrization of C.

Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t\rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?



$t \in [0, 2\pi]$ one loop around the circle.

$(0, 1, \pi/2) = r_1(\pi/2) = r_2(\pi/4)$

$\vec{r}_0(t) = \langle \cos t, \sin t, 0\rangle$

$\vec{r}(0)$ @ $t = 0$

$\vec{r}_1(0) = \langle \cos 0, \sin 0, 0\rangle = \langle 1, 0, 0\rangle$

$r_1(\frac{\pi}{2}) = \langle \cos \pi/2, \sin \pi/2, \pi/2\rangle = \langle 0, 1, \pi/2\rangle$

Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits: $\vec{r}(t) = \langle t^2, 2, \ln(t) \rangle$, $t \in (0, \infty)$

Example 11. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$.

$$L = \left\langle \lim_{t \to e} t^2, \lim_{t \to e} 2, \lim_{t \to e} \ln(t) \right\rangle$$

@ $t = e$

$$= \langle e^2, 2, \ln(e) \rangle = \boxed{\langle e^2, 2, 1 \rangle.}$$

And with continuity:

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2 - 4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

$$\vec{r}(t) = \langle \underbrace{t}_{f(t)}, \underbrace{\frac{-1}{t^2-4}}_{g(t)}, \underbrace{\sin t}_{h(t)} \rangle$$

$$D_{\vec{r}} = \mathbb{R} \cap \big((-\infty, -2) \cup (-2, 2) \cup (2, \infty)\big) \cap \mathbb{R}$$

So get $\boxed{D_{\vec{r}} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$

b/c input t has to be domain for all of the component functions f, g, h.

$f(t)$ $g(t)$

And with derivatives: $\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle$

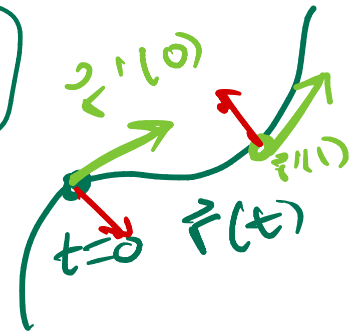
Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$, find $\mathbf{r}'(t)$.

$$\boxed{\vec{r}\,'(t) = \langle 2 - t, 1 \rangle}$$

@ $t=0$

$\vec{r}\,'(0) = \langle 2-0, 1 \rangle = \langle 2, 1 \rangle$

@ $t=1$ $\vec{r}\,'(1) = \langle 2-1, 1 \rangle = \langle 1, 1 \rangle$



Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity vector at time t (speed & direction)
- $|\mathbf{r}'(t)|$ gives speed (scalar) at time t.
- $\mathbf{r}''(t)$ gives accelleration vector

Check $\vec{r}\,''(t) = \langle -1, 0 \rangle$

Let's see this graphically

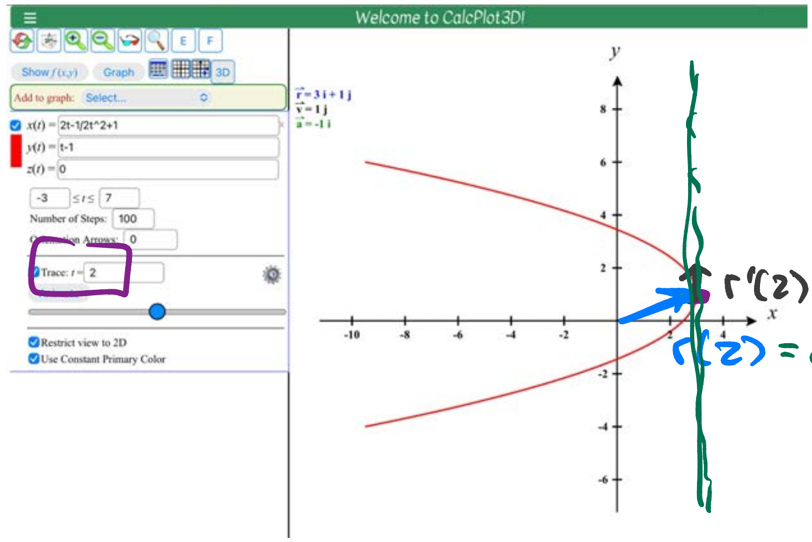
Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$ at time $t = 2$.

vector form $\boxed{\ell(t) = \vec{OP} + t\vec{v}}$

Example 14 (cont.) Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

$\ell(t) = \vec{OP} + t\,\vec{v}$ ← direction of line $\vec{r}\,'(2)$

point on line $r(2)$



$\vec{r}\,'(t) = \langle 2 - t, 1\rangle$

$\vec{r}(2) = \langle 2(2) - \frac{1}{2}(2)^2 + 1, 2 - 1\rangle$

$= \langle 3, 1\rangle$

$r'(2) = \langle 2 - 2, 1\rangle = \langle 0, 1\rangle$

$$\boxed{\ell(s) = \langle 3, 1\rangle + s\langle 0, 1\rangle,\quad s \in \mathbb{R}}$$

And with integrals:

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \; dt.$ $= \langle \frac{1}{2}t^2, \frac{1}{2}e^{2t}, \tan t \rangle \Big|_0^1$

$= \langle \frac{1}{2}, \frac{1}{2}e^2, \tan(1) \rangle - \langle 0, \frac{1}{2}, 0 \rangle$

$= \boxed{\langle \frac{1}{2}, \frac{1}{2}e^2 - \frac{1}{2}, \tan(1) \rangle}$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.

$\vec{s}(t) = \int \vec{v}(t)\,dt$ (integral of velocity vector is the position vector)

$= \int \langle -200\sin 2t, 200\cos t, 400 - \frac{400}{1+t} \rangle \, dt$

$= \langle 100\cos 2t + C_1, 200\sin t + C_2, 400t - 400\ln(1+t) + C_3 \rangle$

@ $t=0$ $\vec{s}(0) = \vec{0}$ $\vec{s}(0) = \langle 100 + C_1, 0 + C_2, 0 + C_3 \rangle = (0,0,0)$

$C_1 = -100$
$C_2 = C_3 = 0$

$\boxed{\vec{s}(t) = \langle 100\cos 2t - 100, 200\sin t, 400t - 400\ln(1+t) \rangle}$