	Fri Sep 12	Studio: 13.4, 14.1		13.4, 14.1	PA#3
Week 5	Mon Sep 15	Lecture: 14.1 (cont.), 14.2			
	Wed Sep 17	Lecture: 14.2 (cont.)			
	Fri Sep 19	Studio: 14.1, 14.2	Quiz 2	14.2	
Week 6	Mon Sep 22	Exam 1: 12.1-12.5, 13.1-13.4, 14.1-14.2			WeBWorK Set #1 deadline

§13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

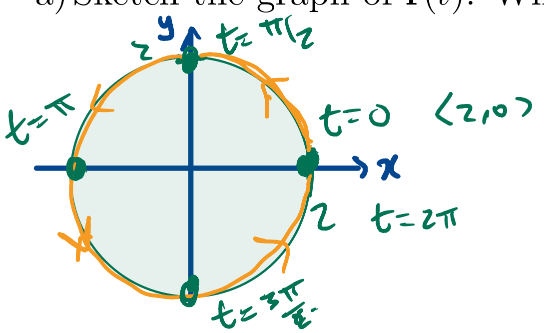
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

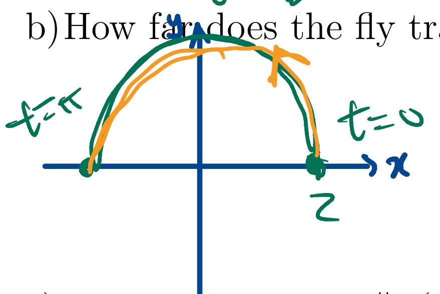
$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



b) How far does the fly travel between $t = 0$ and $t = \pi$?



$C = 2\pi r = 4\pi$ circumfrence

$\frac{1}{2} C = \boxed{2\pi}$

same value!!

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

$v(t) = r'(t) = \langle -2\sin t, 2\cos t\rangle$

$\|v(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

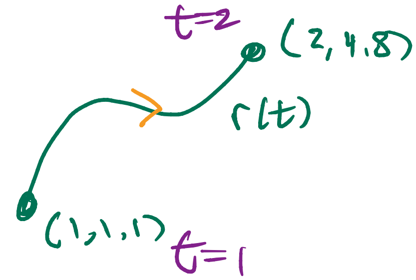
d) Compute the integral $\displaystyle\int_0^{\pi} \|\mathbf{v}(t)\|\, dt$. What do you notice?

$$L = \int_0^{\pi} 2\, dt = 2t\Big|_0^{\pi} = 2\pi - 0 = \boxed{2\pi}$$

Definition 17. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Example 18. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.



Step 0: Find values of a & b

When $t=0$ $\vec{r}(0) = \langle 0,0,0 \rangle$ doesn't work!

$t=1$ $\vec{r}(1) = \langle 1, 1^2, 1^3 \rangle = \langle 1,1,1 \rangle$ ✓

$t=2$ $\vec{r}(2) = \langle 2, 2^2, 2^3 \rangle = \langle 2,4,8 \rangle$ ✓ $\quad r(t) = \langle t, t^2, t^3 \rangle$

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$$L = \int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2}\,dt = \boxed{\int_1^2 \sqrt{1 + 4t^2 + 9t^4}\,dt}$$

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \le t \le 2\pi$.

Check your intuition

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \le t \le 2\pi$.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 12\cos 2t, -12\sin 2t, 5\rangle$$

$$\Rightarrow \quad |\vec{v}(t)|^2 = 144(\cos^2 2t + \sin^2 2t) + 25 = 169$$

$$\Rightarrow \quad |\vec{v}(t)| = 13$$

So,

$$L = \int_0^{2\pi} |\vec{v}(t)|\, dt = \int_0^{2\pi} 13\, dt = 13t\Big|_0^{2\pi} = \boxed{26\pi}$$

Check your intuition

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \leq t \leq 8.$

Check your intuition

8:30 (10 min) 8:40

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k}$, $0 \le t \le 8$.

$$L = \int_a^b \|\vec{v}(t)\| \, dt$$

$x(t) = t$
$y(t) = 0$
$z(t) = \frac{2}{3}t^{3/2}$

$a = 0$
$b = 8$

$$\vec{v}(t) = \vec{r}'(t) = 1 * \hat{\imath} + \cancel{\tfrac{2}{3} * \tfrac{3}{2}}^{\,1} t^{1/2} \hat{k}$$

So $\|\vec{v}(t)\| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}$

and

$$L = \int_0^8 \sqrt{1+t} \, dt$$

u-sub Box

$$\boxed{\begin{array}{l} u = 1+t \\ du = dt \end{array}}$$

$t=0 \Rightarrow u=1$
$t=8 \Rightarrow u=9$

$$= \int_1^9 \sqrt{u} \, du = \frac{2}{3} u^{3/2} \Big|_1^9$$

$$= \frac{2}{3} 9^{3/2} - \frac{2}{3} 1^{3/2}$$

$$= \frac{2}{3} 27 - \frac{2}{3} = \frac{2}{3} * 26 = \boxed{52/3}$$

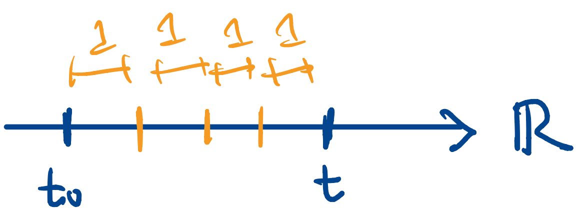
Check your intuition

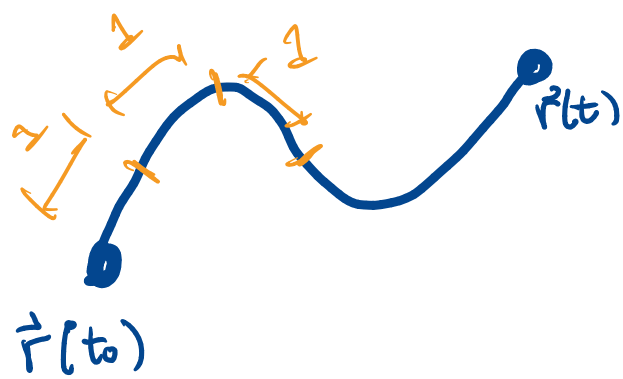
Arc length parametrization

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|v(\tau)\| \, d\tau$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.





Idea: why should we be measuring out units in the parameter space $\mathbb{R}$?

More natural to measure units in the codomain of $\vec{r}(t)$, $\vec{r}: \mathbb{R} \to \mathbb{R}^n$.

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

$\uparrow t_0 = 0$

META

1. Compute arc length function $s(t) = \int_{t_0}^{t} \|\vec{v}(t)\|\, dt$
2. Solve $s = s(t)$ for $t = f(s)$
3. Substitute back into $\vec{r}(t)$ to obtain $\vec{r}(s)$.

$\int_0^t 3t\, dt = \cancel{\frac{3}{2}t^2}\Big|_0^t$

1 Find $s(t)$:

$v(t) = r'(t) = \langle -4\sin t, 4\cos t\rangle$ so $\|v(t)\|^2 = 16\sin^2 t + 16\cos^2 t$
$= 16(\underbrace{\sin^2 t + \cos^2 t}_{1})$
$= 16$

so $\|v(t)\| = 4$. $\int_{t_0}^{t} \|v(t)\|\, dt$?? [N-deduction]

$s(t) = \int_{t_0}^{t} \|v(\tau)\|\, d\tau = \int_0^t 4\, d\tau = 4\tau\Big|_0^t = 4t - 0 = 4t.$

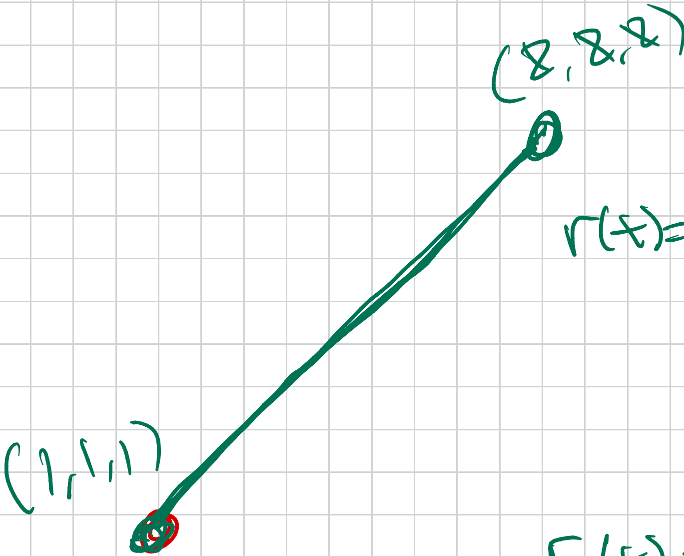
So ① $\boxed{s = 4t}$ So ② $\boxed{t = \frac{s}{4}}$ $f(s) = \frac{s}{4}$

3 $r_2(s) = r(t) = r\left(\frac{s}{4}\right) = \langle 4\cos(s/4), 4\sin(s/4)\rangle$

$$\boxed{r_2(s) = \langle 4\cos(s/4), 4\sin(s/4)\rangle,\ s \in [0, 8\pi]}$$

@ $t = 0$, $s = 4(0) = 0$
$t = 2\pi$, $s = 4(2\pi) = 8\pi$

Example 22. *You try it!* Find (a) an arc length parameterization $s(t)$ of the curve $\mathcal{C}$, the portion of the helix of radius 4 in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \leq t \leq \pi/2$, and (b) use $s(t)$ to find L the length of $\mathcal{C}$



$r(t) = \langle t, t, t \rangle$

$t=1$
$t=8$

$r_2(t) = (t^3, t^3, t^3)$

$t=1$
$t=2$

Example 22. *You try it!* (a) Find an arc length parameterization of the portion of the helix $\mathcal{C}$ of radius 4 in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$.

(b) use $s(t)$ to find L the length of $\mathcal{C}$.

① Find $s(t)$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -4\sin t, 4\cos t, 3\rangle$$

$|\vec{v}|^2 = 16 + 9 = 25$

so $|\vec{v}| = 5$

So,

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t 5\, d\tau = 5\tau \Big|_0^t$$

and $S = 5t$ ///

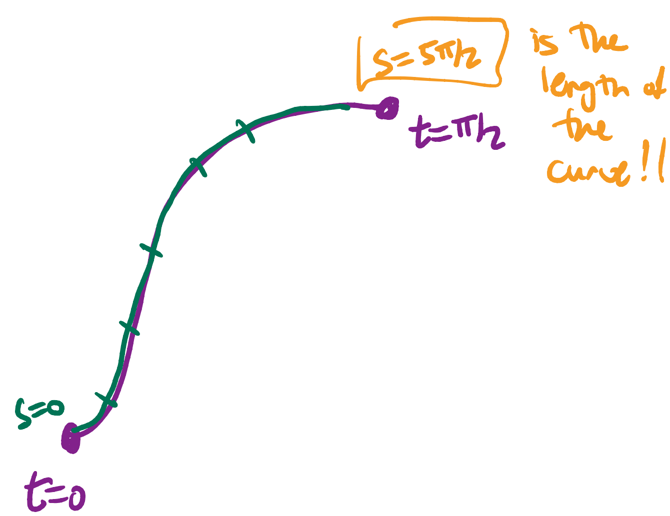
② Solve for t:

$$s = s(t) = 5t \quad \text{so} \quad t = f(s) = s/5$$

Sub into $\vec{r}$: $\vec{r}_2(s) = r(f(s))$

③
$$\boxed{\vec{r}_2(s) = \langle -4\sin(s/5), 4\cos(s/5), 3\rangle, \quad 0 \le s \le 5\pi/2}$$

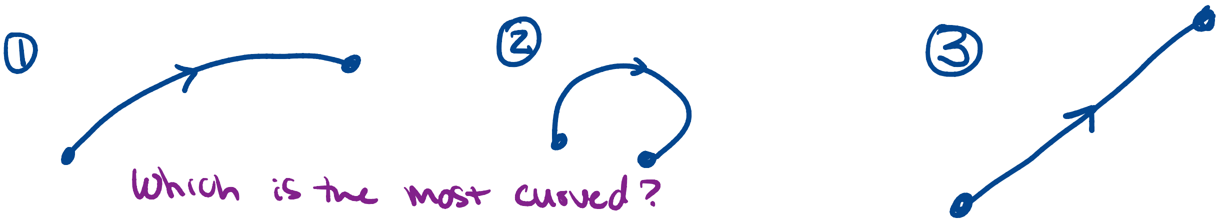
$\begin{cases} t = 0 \\ t = \pi/2 \end{cases} \Rightarrow \begin{array}{l} s = 0 \\ s = 5\pi/2 \end{array}$



(b)
$$L = \int_0^{\pi/2} |\vec{v}|\, d\tau = s\Big|_{t=\pi/2} = \boxed{\frac{5}{2}\pi}$$

§13.3 & 13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



Idea: scale 2nd derivative appropriately

First, we need the **unit tangent vector**, denoted **T**:

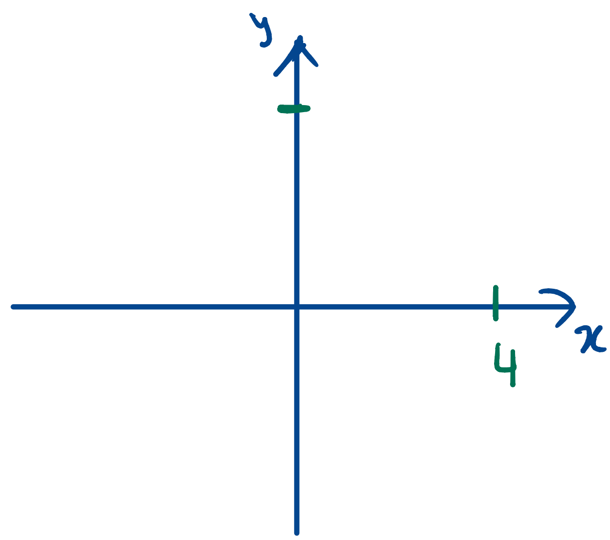
- In terms of an arc-length parameter s: ________________
- In terms of any parameter t: ________________

This lets us define the **curvature**, $\kappa(s) =$ ________________

Example 23. In Example 21 we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.



Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$________________

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ $\vec{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) =$ $\vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) =$ $\|T'(t)\| / \|\vec{r}'(t)\|$ or ______

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$, $t \in \mathbb{R}$.

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}, \ \ t \in \mathbb{R}.$$

IF time: (?)

T/F

$\vec{r}(s) = \langle 2s^2+1, s \rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola $x = 2y^2+1$.

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}, \ t \in \mathbb{R}.$$

Step 1 Find $r'(t)$ and $\|r'(t)\|$.

$$v = \frac{dr}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\hat{i} + (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\hat{j} + 0\hat{k}$$

$$= t\cos t\,\hat{i} + t\sin t\,\hat{j} + 0\hat{k}$$

$|v|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$ so $|v| = |t|$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\boxed{T = \frac{v}{|v|} = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}}$$

$\frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$ & $\left|\frac{dT}{dt}\right| = 1.$

so $\boxed{N = \frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$\boxed{\kappa = \frac{1}{|t|}}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

IF time: (?)

T/F

$\vec{r}(s) = \langle 2s^2+1, s\rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola $x = 2y^2+1$.