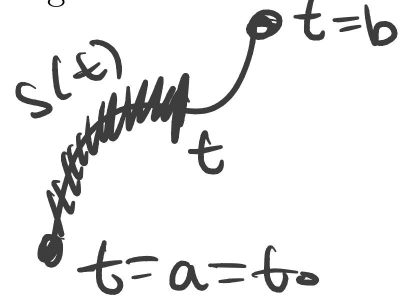
(0) Find arc-length parameterization.

$r_2(s) = r_1(s/5) = \langle 4\cos s/5, 4\sin s/5, \frac{3s}{5} \rangle$

Example 22. *You try it!* Find (a) an arc length parameter $s(t)$ of the curve $\mathcal{C}$, the portion of the helix of radius 4 in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$, and (b) use $s(t)$ to find L the length of $\mathcal{C}$

Formula: $s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$



$\vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 3\rangle$

$\|\vec{v}(t)\|^2 = \underbrace{16\sin^2 t + 16\cos^2 t}_{16(1)} + 9 = 16+9 = 25$ $\quad \|v(t)\| = \sqrt{25} = 5.$

$t_0 = 0$

(a) $s(t) = \int_0^t 5\, d\tau = 5\tau\Big|_0^t = \boxed{5t}$ arc-length parameter.

$b = \pi/2$

(b) $s(\pi/2) = 5 * \frac{\pi}{2} = \boxed{\frac{5\pi}{2}}$ is the length of $\mathcal{C}$.

Example 22. *You try it!* (a) Find an arc length parameter of the portion of the helix $\mathcal{C}$ of radius 4 in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$.
(b) use $s(t)$ to find L the length of $\mathcal{C}$.

① Find $s(t)$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -4\sin t,\ 4\cos t,\ 3\rangle$$

$|\vec{v}|^2 = 16 + 9 = 25$
so $|\vec{v}| = 5$

So,

$$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t 5\, d\tau = 5\tau \Big|_0^t$$

and $s = 5t$ ///

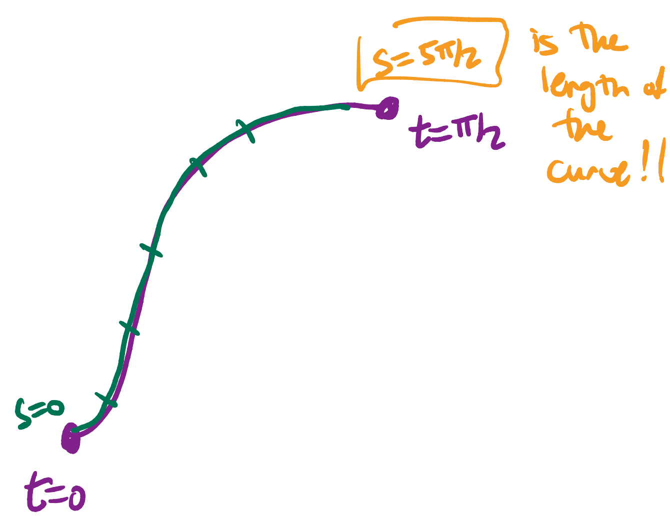
② Solve for t:

$$s = s(t) = 5t \quad \text{so} \quad t = f(s) = s/5$$

Sub into $\vec{r}$: $\vec{r}_2(s) = r(f(s))$

③ $$\boxed{\vec{r}_2(s) = \left\langle -4\sin(s/5),\ 4\cos(s/5),\ \frac{3s}{5}\right\rangle, \quad 0 \le s \le 5\pi/2}$$

$\begin{cases} t = 0 \\ t = \pi/2 \end{cases} \Rightarrow \begin{matrix} s = 0 \\ s = 5\pi/2 \end{matrix}$

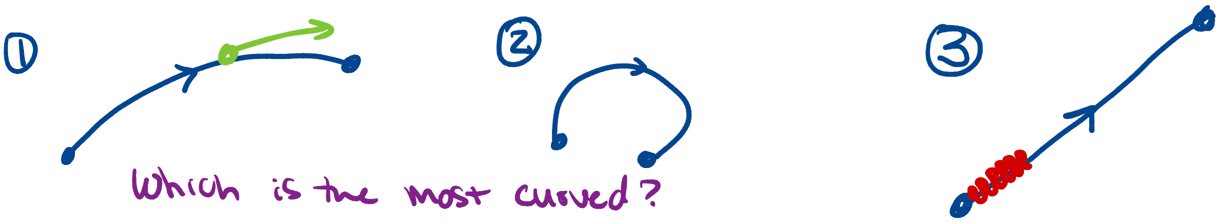


(b)
$$L = \int_0^{\pi/2} |\vec{v}|\, d\tau = s\Big|_{t=\pi/2} = \boxed{\frac{5}{2}\pi}$$

§13.3 & 13.4 - Curvature, Tangents, Normals

②>①>③

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



Which is the most curved?

Idea: scale 2nd derivative appropriately

First, we need the **unit tangent vector**, denoted **T**:

$T(s) = r'(s)$

- In terms of an arc-length parameter s: $r'(s)$ b/c $|r'(s)| \equiv 1$
- In terms of any parameter t: $\frac{r'(t)}{\|r'(t)\|}$

$T(t) = \frac{r'(t)}{\|r'(t)\|}$

This lets us define the **curvature**, $\kappa(s) =$ $\|T'(s)\|$

Example 23. In Example 21 we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

$$\vec{T}(s) = \vec{r}\,'(s) = \left\langle 4 \cdot \frac{-1}{4}\sin\left(\frac{s}{4}\right), 4 \cdot \frac{1}{4}\cos\left(\frac{s}{4}\right)\right\rangle$$

$$= \boxed{\left\langle -\sin(s/4), \cos(s/4)\right\rangle}$$

sanity check? $\|T(s)\| = 1$?

So $\kappa = \|T'(s)\|$

$$T'(s) = \left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin(s/4)\right\rangle$$

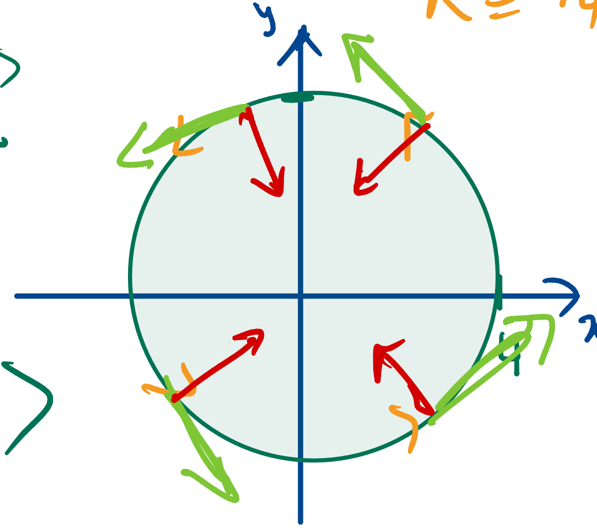
So $\|T'(s)\|^2 = \frac{1}{16}\cos^2\left(\frac{s}{4}\right) + \frac{1}{16}\sin^2\left(\frac{s}{4}\right) = \frac{1}{16}$

$$\Rightarrow \|T'(s)\| = \sqrt{1/16} = \boxed{\frac{1}{4}}$$

$\kappa = 1/4$

$$\vec{N} = \frac{T'(s)}{\|T'(s)\|} = \frac{\left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin(s/4)\right\rangle}{\left(\frac{1}{4}\right)}$$

$$= \left\langle -\cos\left(\frac{s}{4}\right), -\sin\left(\frac{s}{4}\right)\right\rangle$$



Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) = \vec{N} = \dfrac{T'(s)}{\|T'(s)\|}$

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \vec{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) = \|T'(t)\| / \|\vec{r}'(t)\|$ or $\dfrac{\|r(t) \times r'(t)\|}{\|r'(t)\|^3}$

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$, $t \in \mathbb{R}$.

$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1\rangle$

$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1}$
$= \sqrt{4+1} = \sqrt{5}$

(a) $\boxed{\vec{T}(t) = \left\langle \frac{-2}{\sqrt{5}}\sin t, \frac{2}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}} \right\rangle}$.

$\vec{T}'(t) = \left\langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0\right\rangle$

$\|T'(t)\|^2 = \frac{4}{5}\cos^2 t + \frac{4}{5}\sin^2$
$= 4/5$

$N(t) = \frac{\sqrt{5}}{2}\left\langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0\right\rangle$

(b) $\boxed{N(t) = \langle -\cos t, -\sin t, 0\rangle}$

(c) $\kappa = \dfrac{\sqrt{4/5}}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{1}{\sqrt{5}} = \boxed{\dfrac{2}{5}}$

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ \ t \in \mathbb{R}.$$

IF time: (?)

T/F

$\vec{r}(s) = \langle 2s^2+1, s\rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola $x = 2y^2+1$.

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ t \in \mathbb{R}.$$

Step 1) Find $r'(t)$ and $\|r'(t)\|$.

$v = \frac{dr}{dt} = (-\cancel{\sin t} + t\cos t + \cancel{\sin t})\hat{i} + (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\hat{j} + 0\hat{k}$

$= t\cos t\,\hat{i} + t\sin t\,\hat{j} + 0\hat{k}$

??

$|v|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$ so $|v| = |t|$

$T(t) = \frac{r'(t)}{\|r'(t)\|}$

$\boxed{T = \frac{v}{|v|} = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}}$

$\frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$ & $\left|\frac{dT}{dt}\right| = 1.$

so $\boxed{N = \frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}}$

$N(t) = \frac{T'(t)}{\|T'(t)\|}$

$\boxed{\kappa = \frac{1}{|t|}}$

$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$

IF time: (?)

$r'(s) = \langle 4s, 1 \rangle$

$\|r'(s)\| = \sqrt{16s^2+1} \neq 1.$

$\boxed{\text{T/F}}$

$\vec{r}(s) = \langle 2s^2+1, s \rangle$, $s \in \mathbb{R}$ is an arc-length parametrization of the parabola

$x = 2y^2 + 1.$

§14.1 Functions of Multiple Variables

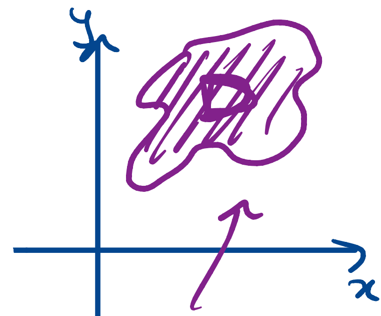
Definition 26. A function of 2 vars. is a rule that assigns to each pair of real numbers (x, y) in a set D a specified output. denoted by $f(x, y)$.

some subset of $\mathbb{R}^2$, domain of f

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

function name

codomain where outputs live.



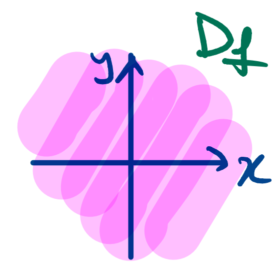
D domain set of all (x,y) where $f(x,y)$ is actually defined.

input space

Example 27. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x+y}}.$$

Example 28. Find the largest possible domains of f, g, and h.



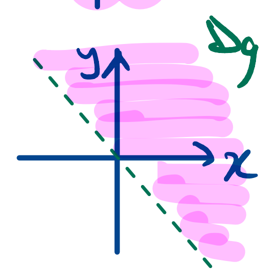
domain of f
$(x,y) \in \mathbb{R}^2$

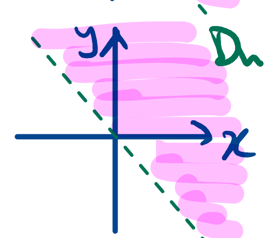
$$D_f = \{(x,y) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2$$

domain of g ($\ln(z)$ defined for $z>0$) $-x\}$

$(x,y) \in \mathbb{R}^2$ s.t. $x+y>0 \Leftrightarrow y>-x$

$$D_g = \{(x,y) \in \mathbb{R}^2 \mid x+y>0\}$$





$(x,y) \in \mathbb{R}^2$ s.t. $x+y>0$ b/c $\frac{1}{\sqrt{z}}$ defined of $z>0$

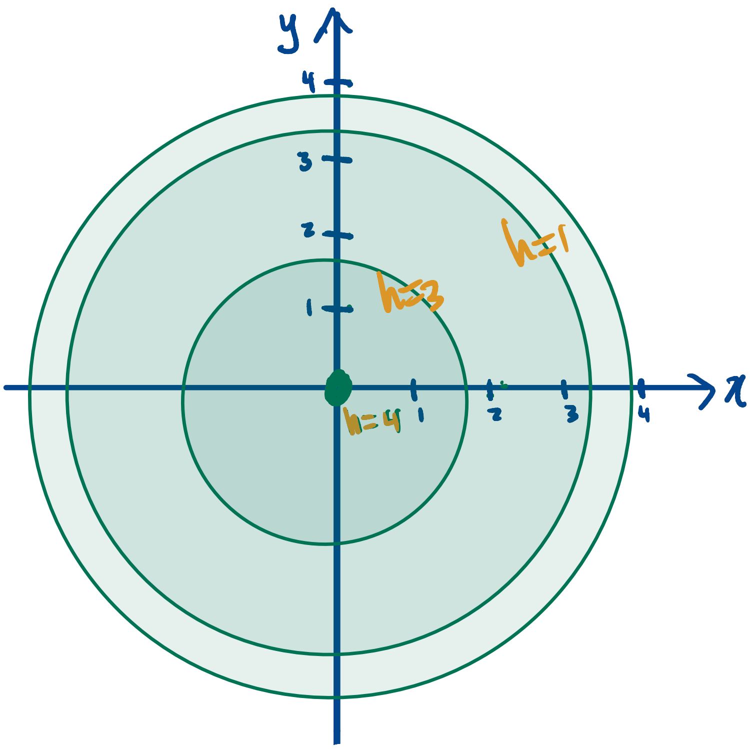
Definition 29. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 30. Suppose a small hill has height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x,y). How could we draw a picture that represents the hill in 2D?

@ $h=4$ $\quad \cancel{4} = \cancel{4} - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow \frac{1}{4}x^2 + \frac{1}{4}y^2 = 0$

only satisfied at $(0,0)$



@ $h=3$ $\quad 3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow \frac{1}{4}x^2 + \frac{1}{4}y^2 = 1$

$\Rightarrow x^2 + y^2 = 4$

@ $h=1$ $\quad 1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$\Rightarrow \frac{1}{4}x^2 + \frac{1}{4}y^2 = 3$

$\Rightarrow x^2 + y^2 = 12$

$r = 2\sqrt{3}$

@ $h=0$ $\quad \frac{1}{4}x^2 + \frac{1}{4}y^2 = 4$

$\Rightarrow x^2 + y^2 = 16$ $\quad r = 4$

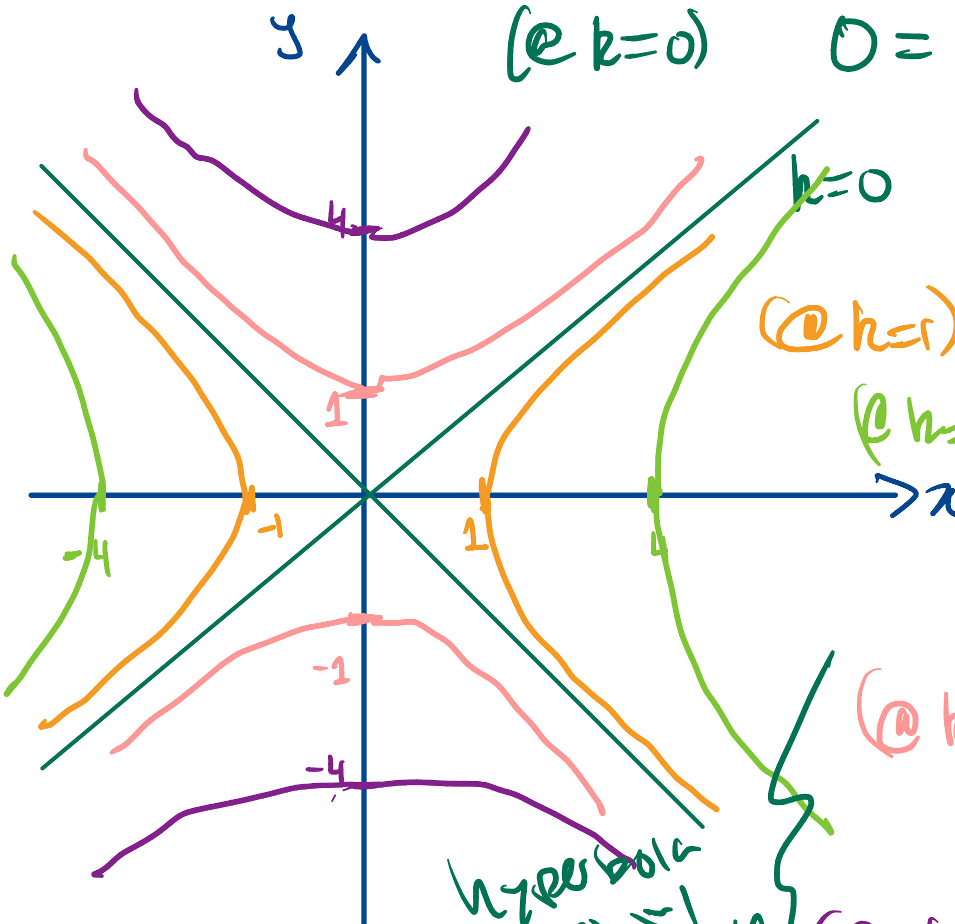
In 3D, it looks like this.

Definition 31. The Contours (also called level sets) of a function f of two variables are the curves with equations $k = f(x,y)$, where k is a constant (in the range of f). A plot of Contours for various values of z is a Contour map (or level curve plot).

Some common examples of these are:

- topography maps. e.g. hiking.
- elelectric field plots.
-

Example 32. Create a contour diagram of $f(x, y) = x^2 - y^2$ Try $k = 0, 1, 4$.



(@ $k=0$) $0 = x^2 - y^2 \Rightarrow x^2 = y^2$

$\Rightarrow |x| = |y|$

So $x = y$ or $x = -y$

(@ $k=1$) $1 = x^2 - y^2$

(@ $k=4$) $4 = x^2 - y^2$

hyperbolas opening sideways.

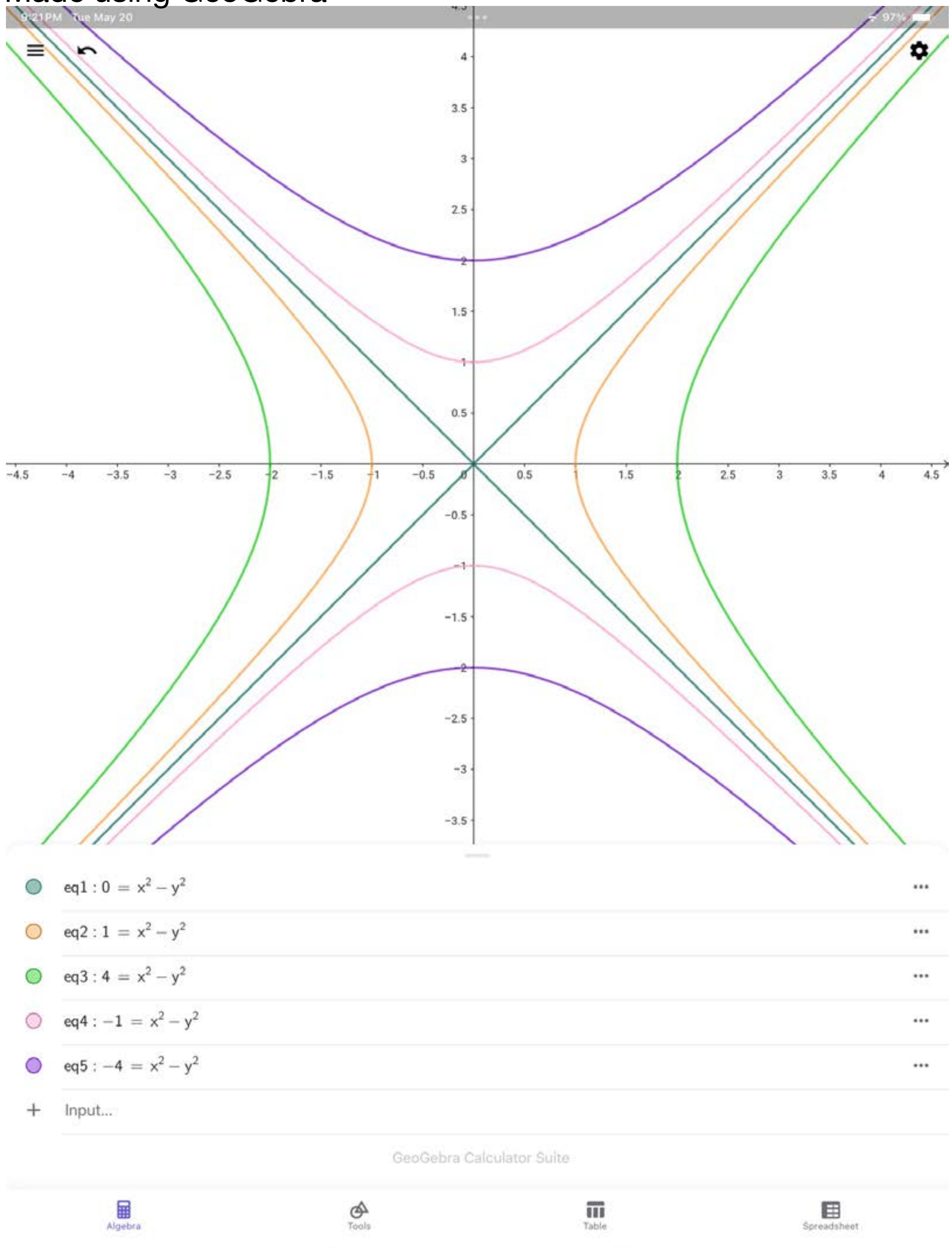
(@ $k=-1$) $-1 = x^2 - y^2$

$\Rightarrow 1 = y^2 - x^2$

hyperbola opening upsdown

(@ $k=-4$) $-4 = x^2 - y^2$

Made using GeoGebra



000
001
010
011

Definition 32. The <u>traces</u> of a surface are the curves of <u>intersect</u> of the surface with planes parallel to the <u>yz-plane, or xz-plane</u>.

Example 33. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

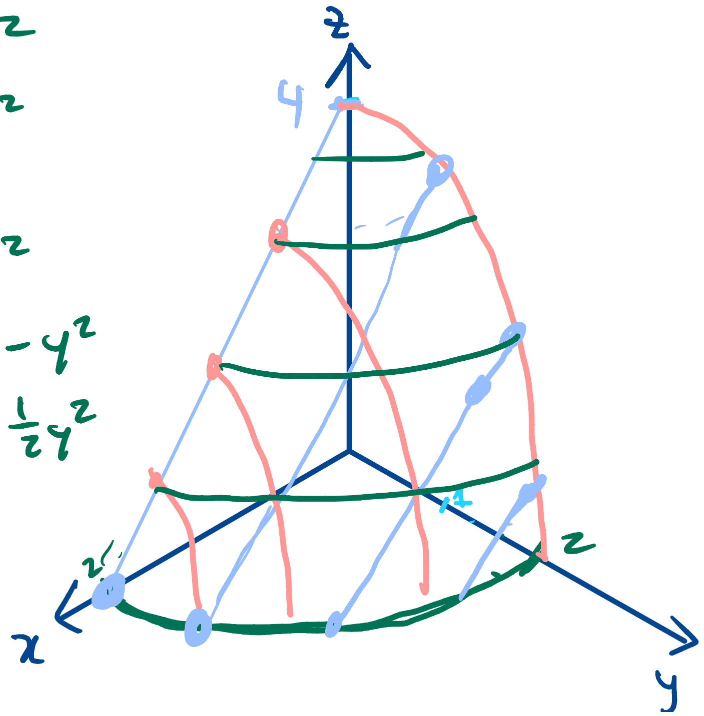
Contours @ $z=0$ $\quad 0 = 4 - 2x - y^2$

$\Rightarrow x = 2 - \frac{1}{2}y^2$

@ $z = k > 0$, then $k = 4 - 2x - y^2$

$\Rightarrow 2x = 4 - k - y^2$

$\Rightarrow x = \frac{4-k}{2} - \frac{1}{2}y^2$



traces w/ $y = k$

@ $y = 0$ then $\boxed{z = 4 - 2x - 0^2}$

@ $y = 1$ then $z = 4 - 2x - 1^2$

$\Rightarrow z = 3 - 2x$

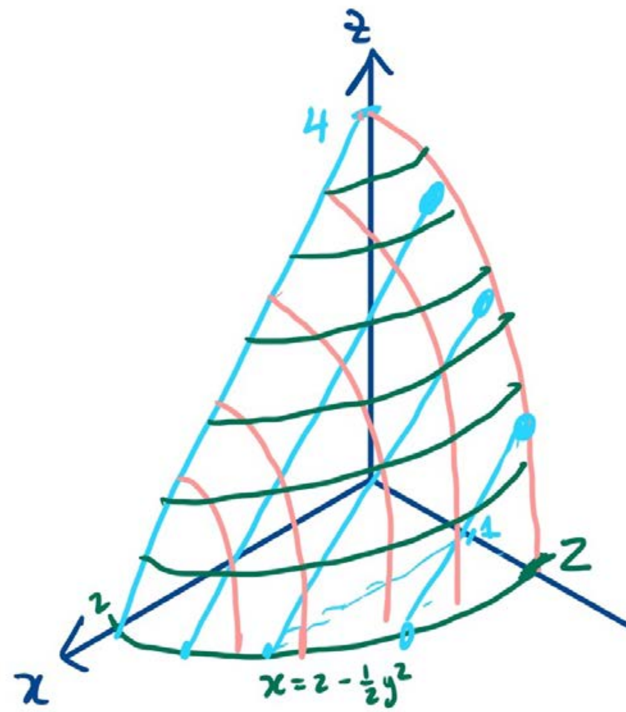
@ $y = k > 0$, then $z = 4 - 2x - k^2$

$\Rightarrow z = 4 - k^2 - 2x$

traces w/ $x = k$

@ $x = 0$ $\quad z = 4 - y^2$

@ $x = k > 0$ $\quad z = 4 - 2k - y$



Let's check our work: https://tinyurl.com/math

Definition 34. A Function of 3 variables is a rule that assigns to each triple of real numbers (x, y, z) in a set D a particular output denoted by $f(x, y, z)$.

Sad news: to graph f you need a picture in $\mathbb{R}^4$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 35. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Want to avoid the triples (x,y,z) s.t.

$4-x^2-y^2-z^2=0$

$\Rightarrow$ $4 = x^2+y^2+z^2$ Sphere of radius 2 centred at $(0,0,0)$ ~~in $\mathbb{R}^3$.~~

$D_f = \{(x,y,z) \in \mathbb{R}^3 \mid 4 \neq x^2+y^2+z^2\}$ all points not on the sphere.

Example 36. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.