Definition 34. A <u>Function of three variables</u> is a rule that assigns to each <u>triple</u> of real numbers (x, y, z) in a set D a <u>unique output</u> denoted by $f(x, y, z)$.

Note: The graph of $w = f(x,y,z)$ lives... in $\mathbb{R}^4$!

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 35. Describe the largest possible domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

The only issue is $\div$ by 0. So need to avoid

$$4 - x^2 - y^2 - z^2 = 0$$

$$\Rightarrow \quad 4 = x^2 + y^2 + z^2$$

So D_f is all of $\mathbb{R}^3$ except the sphere of radius 2 centered at $(0,0,0)$.

Example 36. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

@ $k = 1$ $\quad 1 = 2x^2 + y^2 + z^2$ ellipsoid

@ $k = 0$ $\quad 0 = 2x^2 + y^2 + z^2$ a point

@ $k = -1$ $\quad -1 = 2x^2 + y^2 + z^2$ no solution.

$k < 0$ also no soln.

$k > 0$ ellipsoids that are bigger/smaller.

§14.2 Limits & Continuity

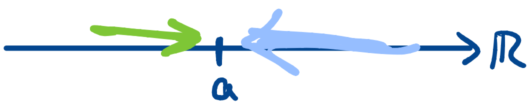
Exam 1 in one week. In this room @ 2pm.

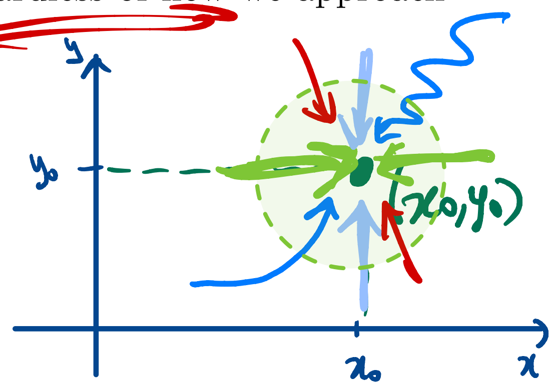
Definition 37. What is a limit of a function of two variables?

> **DEFINITION** We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write
>
> $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$
>
> if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,
>
> $$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ is close to L regardless of how we approach the point (x_0, y_0).

From Calc I. $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$





Definition 38. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ should exist
2. $f(x_0,y_0)$ exists
3. The two values should be equal.

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example: $f_1(x,y) = x$, $f_2(x,y) = y$ both continuous

e.g. $f(x,y) = x+y$, $f(x,y) = xy$, $f(x,y) = (x+y)xy$, $f(x,y) = \frac{x}{y}$ Cont. in its domain.

$f(x,y)$

Example 39. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \boxed{\frac{\sqrt{2x-y}-2}{2x-y-4}}$, if it exists. Check @ $(x,y)=(2,0)$

$$f(2,0)=\frac{\sqrt{4-0}-2}{4-0-4}=\frac{2-2}{4-0}=\frac{0}{0}\quad \text{DNE.}$$

tangent

① L'Hop?
② use alg.

$$\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to4}\frac{\frac{1}{2\sqrt{x}}-0}{1-0}=\lim_{x\to4}\frac{1}{2\sqrt{4}}=\frac{1}{4}\ \checkmark$$

$$\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}*\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to4}\frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$=\lim_{x\to4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}\ \checkmark$$

Try ② since not sure how to generalize ①.

$$\lim_{(x,y)\to(2,0)}\frac{\sqrt{2x-y}-2}{2x-y-4}*\frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}$$

$$=\lim_{(x,y)\to(2,0)}\frac{\cancel{2x-y-4}}{\cancel{(2x-y-4)}(\sqrt{2x-y}+2)}=\frac{1}{\sqrt{4-0}+2}=\frac{1}{4}$$

Example 40. *You try it!* Evaluate $\lim_{(x,y)\to(\frac{\pi}{2},0)} \dfrac{\cos y + 1}{y - \sin x}$, if it exists.

Example 40. *You try it!* Evaluate $\displaystyle\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

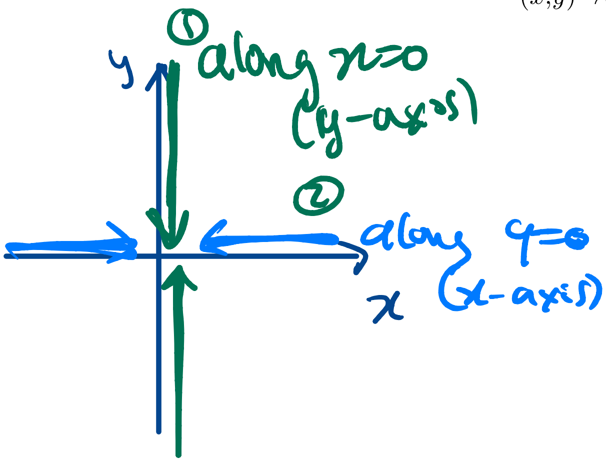
$$\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x} = \frac{\cos(0) + 1}{0 - \sin(\pi/2)} = \frac{1+1}{0-1} = \frac{2}{-1}$$

$$= \boxed{-2}$$

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 41. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.



@ x=0

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0 \checkmark$$

@ y=0

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0} = 1 \checkmark$$

So

> by the TWO-PATH test
> the limit is DNE

This idea is called the **two-path test:**

If we can find two paths approaching to (x_0, y_0) along which limit of f(x,y) takes on two different values, then the limit is DNE.

Exam 1 on Monday 9/22 @ 2pm Kendeda 152 (section 6)
@ 3:30 Phelps 109 (section J)

QUIZ 2 due Friday on Gradescope.

Example 42. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Try @ $x=0$

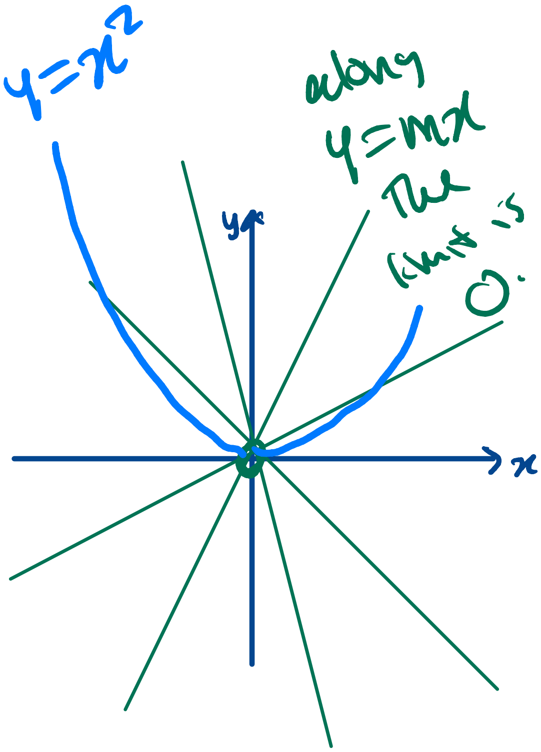
$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0 \checkmark$$

Try @ $y=0$ ditto 0 ✓

Try @ $y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,mx)\to(0,0)} \frac{x^2 * mx}{x^4+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^3} * \frac{m}{x+\frac{m^2}{x}} = 0 \checkmark$$



along $y=mx$ The limit is 0.

try @ $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^2 * x^2}{x^4+(x^2)^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2} \checkmark$$

by The two-path test the limit is DNE

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^4}{x^4+y^2}$ is DNE by using the two-path test. Hint: try two parabolas.

① try: @ $y=0$ The x-axis

② @ $y=x^2$ parabola.

① @ $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^4}{x^4+0} = 1. \checkmark$$

② @ $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^4}{x^4+(x^2)^2} = \lim_{x\to0} \frac{x^4}{2x^4}$$

$$= \frac{1}{2} \checkmark$$

by the two-path test The limit is DNE

Example 43. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test. Hint: try two parabolas.

$f(x,y) = \frac{x^4}{x^4+y^2}$ Let $y = mx^2$

Then $f(x, mx^2) = \frac{x^4}{x^4+m^2x^4} = \frac{1}{1+m^2}$.

So $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} f(x,y) = \frac{1}{1+m^2}$.

So, by Two-path test the limit is DNE. ///

doesn't happen

~~$\lim_{x\to 0^+} x + \frac{1}{x^2}$~~

goes to zero · goes to ∞.

Example 44. [Challenge:] Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^2}$$

does exist using the Squeeze Theorem.

Theorem 45 (Squeeze Theorem)**.** *If* $f(x,y) = g(x,y)h(x,y)$, *where* $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ *and* $|h(x,y)| \le C$ *for some constant* C *near* (a,b)*, then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.

Idea

$$f(x,y) = \frac{x^4y}{x^4+y^2} = \underbrace{y}_{\text{goes to zero}} \; \underbrace{\frac{x^4}{x^4+y^2}}_{\text{bounded}}$$

Show ① goes to zero ②

① $g(x,y) = y$ tend to zero as $y \to 0$ ✓

② notice $y^2 \geq 0$ so $x^4+y^2 \geq x^4$

$$\Rightarrow \quad 1 \geq \frac{x^4}{x^4+y^2}$$

so $h(x,y) \leq 1 = C$

$\boxed{\text{So } \lim_{(x,y)\to(0,0)} f(x,y) = 0 \text{ by squeeze theorem}}$

$\boxed{\text{by ① \& ② } f(x,y) = g(x,y) * h(x,y) \text{ tends to } 0 \text{ as } (x,y) \to (0,0).}$

Math 2551 Worksheet 8 - Review for Exam 1

1. Set up the integral to find the arc length of the curve $y = e^x$ from the point $(0, 1)$ to the point $(1, e)$. Focus on finding a parameterization, and on what values of t give these two points. Is this an integral you would want to compute? Why or why not?

2. Parameterize the line tangent to the curve
$$\mathbf{r}(t) = \langle \cos^2(t), \sin(t)\cos(t), \cos(t)\rangle$$
at the point where $t = \pi/2$.

3. Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ to the circle
$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle.$$
Before checking, should the normal vector be pointing into or out of the circle? Why?

4. We have seen that the curvature of a circle with radius a is $1/a$. Thinking about the geometry of a helix with radius a, do you think its curvature will be greater than or less than $1/a$? Why? Compute the curvature using the parameterization
$$\mathbf{r}(t) = \langle a\cos(t), t, a\sin(t)\rangle$$
to confirm or challenge your intuition.

5. The function $\ell(t)$ below describes a line. There is a particular plane that $\ell(t)$ is normal to at the point $t = 0$. Find an equation of this plane.
$$\ell(t) = \langle 3 - 3t, 2 + t, -2t\rangle.$$
Where does this line intersect the different plane $3x - y + 2z = -7$?

6. Find and sketch the domain of each of the following functions of two variables:
   (a) $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$
   (b) $\arcsin(x^2 + y^2 - 2)$
   (c) $\sqrt{16 - x^2 - 4y^2}$

7. Solve the differential equation below, together with its given initial conditions. Remember that this means finding all functions $\mathbf{r}(t)$ which satisfy the given equations.
$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}, \quad \mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}$$

8. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?