Main idea $\frac{\partial}{\partial x}(x^2 y) = 2xy$ ✓

§14.3: Partial Derivatives

idea treat the other var like a constant

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 47. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

idea @ (2,1) how does y-change affect the height.

@ $x=2$ Find $\frac{d}{dy} h(2,y) = ?$ plug in $y=1$.

@ $x=2$ $h(2,y) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}y^2 = 3 - \frac{1}{4}y^2$

$\frac{\partial}{\partial y} h\Big|_{x=2} = \frac{d}{dy}\left(3 - \frac{1}{4}y^2\right) = -\frac{1}{2}y$ plug $y=1$ $\frac{\partial}{\partial y} h\Big|_{(2,1)} = \boxed{-\frac{1}{2}}$

@ (2,1) how does x-change affect the height

@ $y=1$ $h(x,1) = 4 - \frac{1}{4}x^2 - \frac{1}{4}(1)^2 = 3.75 - \frac{1}{4}x^2$

$\frac{\partial}{\partial x} h\Big|_{y=1} = \frac{d}{dx}\left(3.75 - \frac{1}{4}x^2\right) = -\frac{1}{2}x$ @ $x=2$

$\frac{\partial}{\partial x} h\Big|_{(2,1)} = \boxed{-1}$

Let's investigate graphically.

Definition 48. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

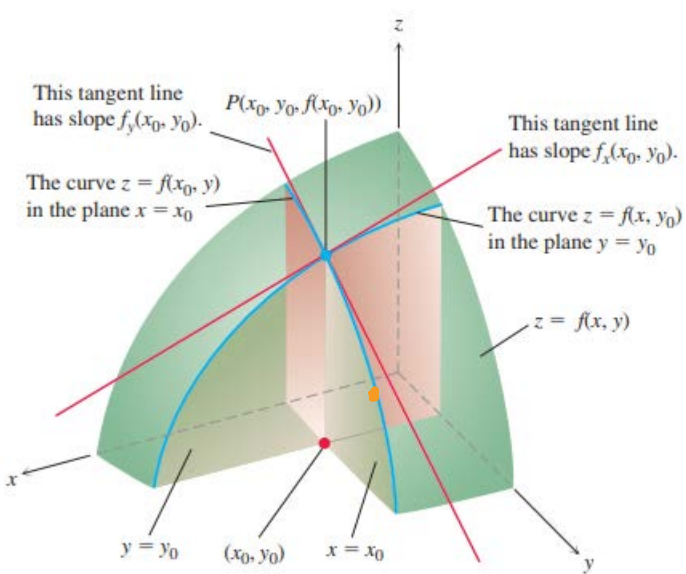
$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations:

$$f_x = \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} \leftarrow$$ The partial derivative of f wrt x.

$$f_y = \frac{\partial}{\partial y} f = \frac{\partial f}{\partial y} \leftarrow$$ partial of f wrt y.

Interpretations:



Example 49. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x,y)=\sqrt{5x-y}$

$$f_x = \frac{\partial}{\partial x}\left((5x-y)^{1/2}\right) = \frac{1}{2}(5x-y)^{-1/2} * 5 = \frac{5}{2(5x-y)^{1/2}}$$

$$f_y = \frac{\partial}{\partial y}\left((5x-y)^{1/2}\right) = \frac{1}{2}(5x-y)^{-1/2} * (-1) = \frac{-1}{2(5x-y)^{1/2}}$$

$$f_x(1,2) = \frac{5}{2(5-2)^{1/2}} = \boxed{\frac{5}{2\sqrt{3}}}$$

$$f_y(1,2) = \frac{-1}{2(5-2)^{1/2}} = \boxed{\frac{-1}{2\sqrt{3}}}$$

b) $f(x,y)=\tan(xy)$

$$f_x = \sec^2(xy) * y$$

$$f_y = \sec^2(xy) * x$$

@ $(2,1)$

$$\boxed{\begin{aligned} f_x(2,1) &= \sec^2(2) * 1 \\ f_y(2,1) &= \sec^2(2) * 2 \end{aligned}}$$

Question: How would you define the second partial derivatives?

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{(\partial x)^2}$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

Example 50. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

a) $f(x, y) = \sqrt{5x - y}$

$$f_x = \frac{5}{2\sqrt{5x-y}} = \frac{5}{2}(5x-y)^{-1/2}$$

$$f_y = \frac{-1}{2\sqrt{5x-y}} = -\frac{1}{2}(5x-y)^{-1/2}$$

$$f_{xx} = \frac{-5}{4}(5x-y)^{-3/2} * 5 = \boxed{\frac{-25}{4(5x-y)^{3/2}}}$$

pure partials (double partials)

$$f_{xy} = \frac{-5}{4}(5x-y)^{-3/2} * (-1) = \boxed{\frac{5}{4(5x-y)^{3/2}}}$$

mixed partials

$$f_{yx} = \frac{1}{4}(5x-y)^{-3/2}(5) = \boxed{\frac{5}{4(5x-y)^{3/2}}}$$

$$f_{yy} = \frac{1}{4}(5x-y)^{-3/2}(-1) = \boxed{\frac{-1}{4(5x-y)^{3/2}}}$$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

Then

$$\boxed{f_{yx} = f_{xy}}$$

Corollary $f_{xyx} = f_{xxy} = f_{yxx}$

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

First order

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem). *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

$$\boxed{f_{xy} = f_{yx}}$$

corollary: $f_{xyx} = f_{xxy} = f_{yxx}$ ~~Same~~?? why?

also $f_{yxy} = f_{yxy} = f_{xyy}$

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$$f_x = 2yz - 0 = \boxed{2yz}$$

$$f_y = \boxed{2xz - z^2}$$

$$f_z = \boxed{2xy - 2zy}$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

Idea. $f(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$x(s,t) = s^2+t$

$y(s,t) = 2s-t$

$z(s,t) = st$

For first component

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} x(s,t) = \frac{\partial}{\partial s}(s^2+t) = 2s$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(s^2+t) = 1$$

$$Df = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

For y-component

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(2s-t) = 2$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(2s-t) = -1$$

For z

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}(st) = t$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}(st) = s$$

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2 y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

$f: \mathbb{R} \to \mathbb{R}$

Df has size 1×1

$$[2x]$$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$

$r: \mathbb{R} \to \mathbb{R}^3$

Df has size 3×1

$r_1(t) = \cos t$

$r_2(t) = \sin t$

$r_3(t) = t$

$$Df = \begin{bmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

c) $f(x, y) = \sqrt{5x - y}$

$f: \mathbb{R}^2 \to \mathbb{R}$

Df is 1×2

$$Df = [f_x \;\; f_y] = \left[\frac{5}{2\sqrt{5x-y}} \;\; \frac{-1}{2\sqrt{5x-y}}\right]$$

d) $f(x, y, z) = 2xyz - z^2 y$

$f: \mathbb{R}^3 \to \mathbb{R}$

Df is 1×3

$$Df = [f_x \;\; f_y \;\; f_z] = [2yz \;\; 2xz - z^2 \;\; 2xy - 2yz]$$

e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{x}, \underbrace{2s - t}_{y}, \underbrace{st}_{z}\rangle$

$F: \mathbb{R}^2 \to \mathbb{R}^3$

Df size 3×2

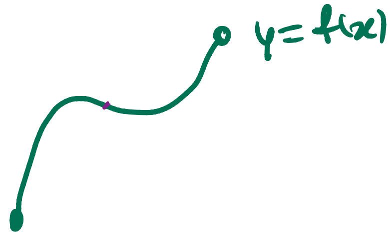
$$Df = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) \approx f(a) + f'(a)(x - a) = L(x)$



Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 56. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.