$f: \mathbb{R}^2 \to \mathbb{R}^3$

Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

Idea. $f(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$x(s,t) = s^2+t$

$y(s,t) = 2s-t$

$z(s,t) = st$

For first component

$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} x(s,t) = \frac{\partial}{\partial s}(s^2+t) = 2s$

$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(s^2+t) = 1$

$$Df = \begin{array}{c} x \to \\ y \to \\ z \to \end{array} \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

$\uparrow \frac{\partial}{\partial s}$ $\quad \uparrow \frac{\partial}{\partial t}$

3x2 matrix

entries are partial deriv. of x, y, z wrt. s or t.

For y-component

$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(2s-t) = 2$

$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(2s-t) = -1$

For z

$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}(st) = t$

$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}(st) = s$

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1,\ldots,x_n) = \begin{bmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{bmatrix}$,

we have __n__ inputs, __m__ output, and __mxn__ partial derivatives, which we can use to form the **total derivative**.

MATRIX.

This is a __linear.__ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an __mxn matrix__ with one column per input and one row per output.

ij-entry:

It has the formula $Df_{ij} = \frac{\partial}{\partial x_j} f_i(x_1,\ldots,x_n)$

$1 \le j \le n$ (# col)

$1 \le i \le m$ (# row)

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

$f: \mathbb{R} \to \mathbb{R}$

Df has size 1×1

$[2x]$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$r: \mathbb{R} \to \mathbb{R}^3$

Df has size 3×1

$r_1(t) = \cos t$

$r_2(t) = \sin t$

$r_3(t) = t$

$$Df = \begin{bmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

c) $f(x, y) = \sqrt{5x - y}$

$f: \mathbb{R}^2 \to \mathbb{R}$

Df is 1×2

$$Df = [f_x \ \ f_y] = \left[\frac{5}{2\sqrt{5x-y}} \ \ \frac{-1}{2\sqrt{5x-y}} \right]$$

d) $f(x, y, z) = 2xyz - z^2 y$

$f: \mathbb{R}^3 \to \mathbb{R}$

Df is 1×3

$$Df = [f_x \ f_y \ f_z] = [2yz \ \ 2xz - z^2 \ \ 2xy - 2yz]$$

e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{x}, \underbrace{2s - t}_{y}, \underbrace{st}_{z} \rangle$

$F: \mathbb{R}^2 \to \mathbb{R}^3$

Df size 3×2

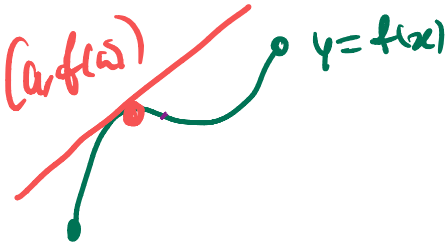
$$Df = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

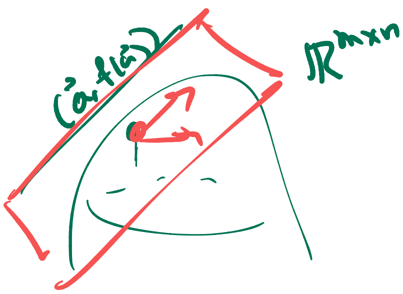
Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) \approx f(a) + f'(a)(x - a) = L(x)$



Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is



$$L(\mathbf{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

$f(1,1) = \sqrt{5(1)-1} = \sqrt{4} = 2$

Example 56. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Step 1: $Df = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$ @ $\vec{a} = \langle 1, 1 \rangle$

$Df(1,1) = \begin{bmatrix} \frac{5}{2\sqrt{4}} & \frac{-1}{2\sqrt{4}} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix}$

So $L(\vec{x}) = 2 + \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

and $L(1.1, 1.1) = 2 + \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix} \left(\begin{bmatrix} 1.1 \\ 1.1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 2 + \begin{bmatrix} 1.25 & -.25 \end{bmatrix} \begin{bmatrix} .1 \\ .1 \end{bmatrix}$

Question: What do you notice about the equation of the linearization?

$= 2 + (.125 - .025)$

$= 2 + .1 = \boxed{2.1}$

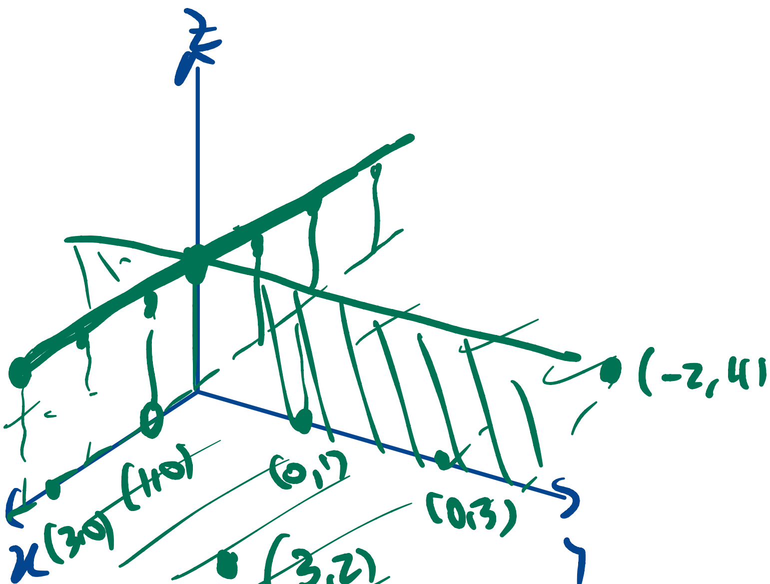
We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

No.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



$Df = ?$

f_x & f_y at $(0,0)$?

Notice $f(0,0) = 1$

@ $x=0$ $f(0,y) = 1$

So $f_y(0,y) = 0$.

@ $y=0$ $f(x,0) = 1$.

So $f_x(0,y) = 0$.

$Df(0,0) = [0 \quad 0]$,

$\vec{a} = (0,0)$

So $L(\vec{x}) = f(0,0) + Df(0,0)(\vec{x} - \vec{a})$

$= 1 + [0 \quad 0]\begin{bmatrix} x \\ y \end{bmatrix} = 1.$

but $f(\vec{x}) = 0$ for some $\vec{x} \approx \vec{0}$

§14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Sanity Check

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

$f \circ g$ from $\mathbb{R}^n$ to $\mathbb{R}^m$

Df is size $m \times p$

Dg is size $p \times n$

$\underset{m\times p}{Df} * \underset{p\times n}{Dg}$ is size $m \times n$.

Example 58. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

WANT $\cancel{\frac{d}{dt}h}$?? $Dh(1) = Dh\big|_{t=1}$

$D[h(\vec{r}(t))] = Dh(\vec{r}(t)) * D\vec{r}(t)$

$Dh = [h_x \;\; h_y] = [-\frac{1}{2}x \;\; -\frac{1}{2}y]$? @ $t=1$ $r(1) = \langle 2, 1 \rangle$

$Dr = \begin{bmatrix} r_1' \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$ @ $t=1$ $D\vec{r}(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$Dh\big|_{t=1} = Dh(1,-2)$
$= [-\frac{1}{2}(2) \;\; -\frac{1}{2}(1)]$
$= [-1 \;\; -\frac{1}{2}]$

So $Dh(r(t))\big|_{t=1} = [-1 \;\; -\frac{1}{2}]\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$= -1 + 1 = \boxed{0}$

$g(s,t) = \langle u(s,t), v(s,t) \rangle$

Example 59. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$
$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$
$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$
$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

Idea:

$DW = [W_s \ \ W_t]$

Find $W_s(1,0)$ and $W_t(1,0)$.

$$\boxed{D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x})}.$$

Formula for chain rule.

Sanity check

$W : \mathbb{R}^2 \to \mathbb{R}$ so DW is a 1×2 matrix.

$$DW = DF(g(s,t)) * Dg(s,t)$$

$$= \underbrace{\begin{bmatrix} F_u & F_v \end{bmatrix}}_{DF} \underbrace{\begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}}_{Dg}$$

$$= \begin{bmatrix} F_u u_s + F_v v_s & F_u u_t + F_v v_t \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s}} & \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} \end{bmatrix}$$

$W_s = \frac{\partial W}{\partial s}$

@ $(s,t) = (1,0)$

$$Dg\Big|_{(s,t)=(1,0)} = \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

@ $(s,t) = (1,0)$ so $(u,v) = (2,3)$

$$DF\Big|_{(u,v)=(2,3)} = \begin{bmatrix} -1 & 10 \end{bmatrix}$$

$$DW\Big|_{(s,t)=(1,0)} = \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+50 & -6+40 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 36 \end{bmatrix}$$

$$\boxed{\begin{aligned} W_s(1,0) &= 52 \\ W_t(1,0) &= 36 \end{aligned}}$$

General Formulas for Chain rule

Suppose $h(x,y): \mathbb{R}^2 \to \mathbb{R}$ and $r: \mathbb{R} \to \mathbb{R}^2$,

$r(t) = \langle x(t), y(t) \rangle$ so $r' = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

Total derivative:

partial der operator

d's for regular 1-var derivative

sample exams

$$Dh * Dr = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$

practice see textbook

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Idea: $F(x,y,z)=c$ then $DF = [F_x \;\; F_y \;\; F_z]$

Formula $\frac{\partial z}{\partial x} = \boxed{\frac{-F_x}{F_z}}$

$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$

Example 60. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

$$F(x,y,z) = x^2+y^2+z^2-4=0$$

$$\frac{\partial F}{\partial x} = 2x \qquad \frac{\partial z}{\partial x} = \frac{-2x}{2z} = \boxed{\frac{-x}{z}}$$

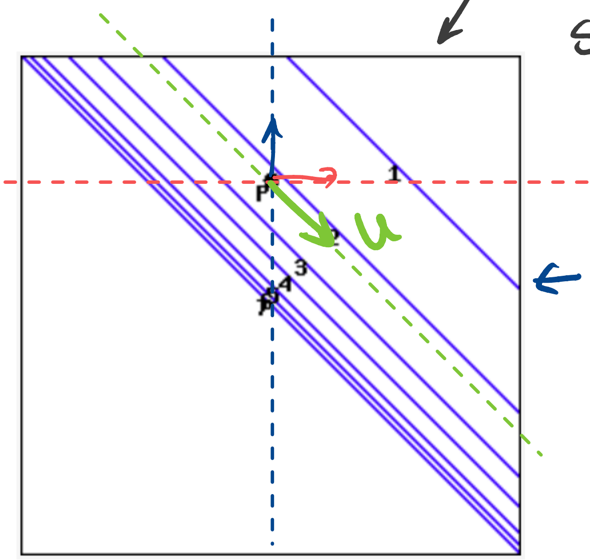
$$\frac{\partial F}{\partial y} = 2y$$

$$\frac{\partial F}{\partial z} = 2z \qquad \frac{\partial z}{\partial y} = \frac{-2y}{2z} = \boxed{\frac{-y}{z}}$$

§14.5 Directional Derivatives & Gradient Vectors

Example 61. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?

Level sets for $z=k$.
Shows all (x,y) pairs s.t. $f(x,y)=k$



$f_x<0$ $f_y<0$.

$f_u=0$

← blue lines are level sets w/ label is height

The directional derivative

$f_u(P) = D_u(P)$

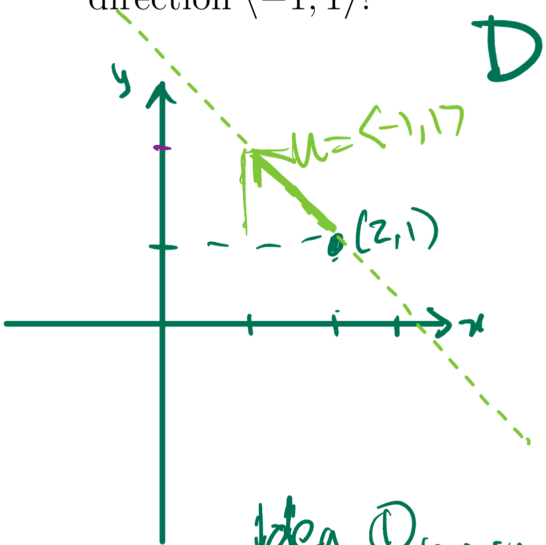
The rate of change of f as the inputs change in the direction of u.

Let's go back to our hill example again, $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1\rangle$?

$$Dh = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$$

$$h(2,1) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}(1)^2 = 11/4$$

$$Dh\Big|_{(2,1)} = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix}$$



Idea: ① normalize $\vec{u}$, $\frac{u}{\|u\|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

② take a limit!

$$\lim_{t\to 0} \frac{h(\vec{p} + t\vec{u}) - h(\vec{p})}{t} = \lim_{t\to 0} \frac{h(\langle 2,1\rangle + t\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle) - h(\langle 2,1\rangle)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\left(\cancel{4} - \frac{1}{\cancel{4}}\left(\cancel{4} - \frac{4t}{\sqrt{2}} + \frac{t^2}{(\sqrt{2})}\right) - \frac{1}{\cancel{4}}\left(1 + \frac{2t}{\sqrt{2}} + \frac{t^2}{2}\right) - \cancel{\frac{11}{4}} \right)$$?

$$= \lim_{t\to 0} \frac{1}{t}\left(\frac{t}{\sqrt{2}} - \frac{t^2}{8} - \frac{t}{2\sqrt{2}} - \frac{t^2}{8} \right) = \lim_{t\to 0} \left(\frac{1}{\sqrt{2}} - \frac{t}{8} - \frac{1}{2\sqrt{2}} - \frac{t}{8} \right) \stackrel{(?)}{=} \boxed{\frac{1}{2\sqrt{2}}}$$ ✓

Definition 62. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{t\to 0} \frac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{t}$$

if this limit exists.

E.g. for our hill example above we have: $D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle} h\Big|_{(2,1)} = \frac{1}{2\sqrt{2}}$

\nabla nabla ∇

Note that $D_{\mathbf{i}} f = f_x$ $\qquad D_{\mathbf{j}} f = f_y$ $\qquad D_{\mathbf{k}} f = f_z$

(The regular "standard" directional derivatives)

Definition 63. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = Df(\vec{p})^T$$

$$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \text{ then } \nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Ex.

$$Dh = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$$

$$\nabla h = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$$

and

$$\nabla h(2,1) = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) = \nabla f \cdot \vec{u}$$

Ex.

$$D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} h(2,1) = \nabla h(2,1) \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} + \frac{-1}{2\sqrt{2}} = \frac{2-1}{2\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x, y) = \ln(x^2 + y^2)$, $\mathbf{p} = \langle -1, 1 \rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = \langle 1, 2, 1 \rangle$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x,y) = \ln(x^2+y^2), \mathbf{p} = \langle -1, 1\rangle, \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2x/(x^2+y^2) \\ 2y/(x^2+y^2) \end{bmatrix} \quad @\ \vec{p} = \langle -1, 1\rangle \quad \nabla f(\vec{p}) = \begin{bmatrix} -2/2 \\ 2/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and $$D_{\vec{u}} f(\vec{p}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = -\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \boxed{1/\sqrt{5}}$$

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = \langle 1, 2, 1\rangle$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$$\nabla g = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 2x + 4y^2 \\ 8xy \\ 2z \end{bmatrix} \quad @\ \vec{p} = \langle 1,2,1\rangle \quad \nabla g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix}$$

$$v = \langle 1, 2, -1\rangle \Rightarrow u = \frac{1}{\|v\|} v = \langle 1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\rangle$$

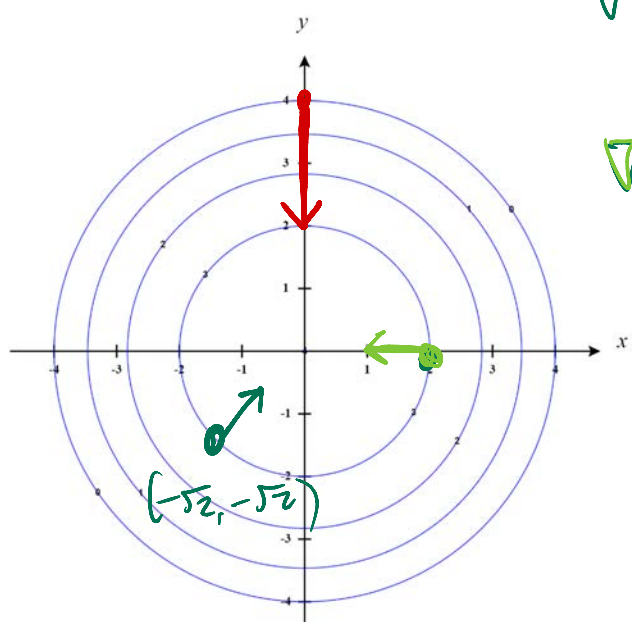
So $$D_{\vec{u}} g(\vec{p}) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}}(18 + 32 - 2)$$

$$= \boxed{\frac{48}{\sqrt{6}}}$$

$4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 - z = 0$

Example 65. If $h(x, y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2, 0), (0, 4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2, 0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$$\nabla h = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$$



$$\nabla h(2,0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \nabla h(0,4) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\nabla h(-\sqrt{2}, -\sqrt{2}) = \begin{bmatrix} +\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

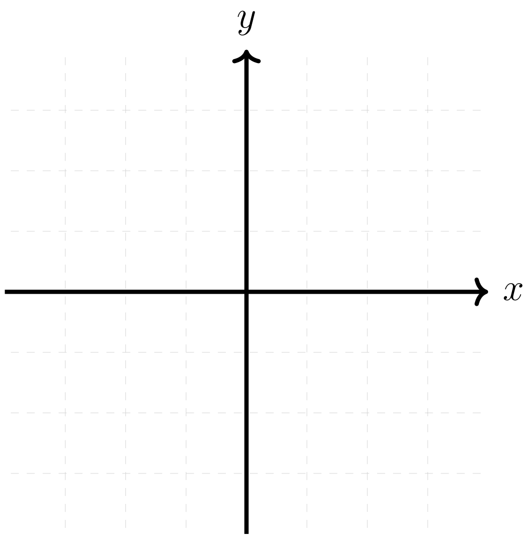
* contour lines are perp to $\nabla f(p)$

* $\nabla f(p)$ points "direction of greatest increase".

Note that the gradient vector ∇f is perp. to the level curves of the function $z = f(x,y)$

Similarly, for $f(x, y, z)$, $\nabla f(a, b, c)$ is orthogonal level surfaces.

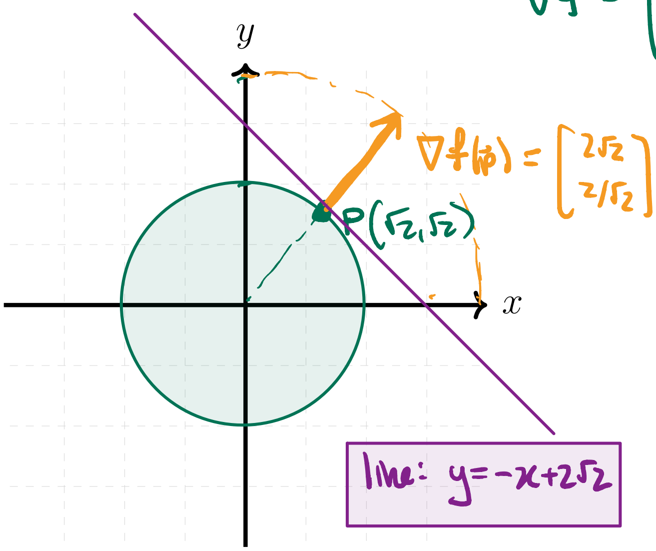
Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f\mid_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.



$f(x,y) = x^2 - y^2 - 4 = 0$

Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f\,|_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ @ $P(\sqrt{2},\sqrt{2})$ $\quad \nabla f(\vec{P}) = \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$



$8 + 8 = 16 = 4^2$

$\langle -1, 1 \rangle$ orth. to $\nabla f|_{\vec{p}}$, so.

$\ell(t) = \langle \sqrt{2}, \sqrt{2} \rangle + t\langle 1, -1 \rangle$

(so $m = -1$)

$y = -x + b$ and passes thru $(\sqrt{2}, \sqrt{2})$

so $\sqrt{2} = -\sqrt{2} + b$

$\Rightarrow 2\sqrt{2} = b$

Formula for tangent line

$a(x - x_0) + b(y - y_0) = 0$

where $\nabla f(p) = \begin{bmatrix} a \\ b \end{bmatrix}$

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what's the latex symbol \nabla

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

Tools