§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?

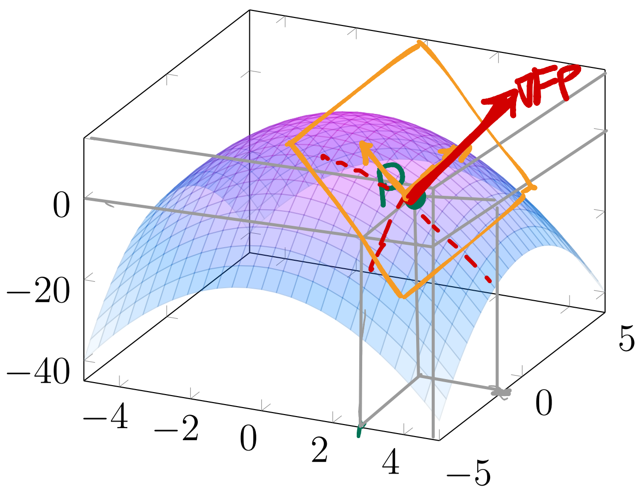
<u>Notice</u> $\nabla F(P)$ <u>is</u> normal to the surface/ tangent plane.

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

where $\vec{n}=\langle a,b,c\rangle$.

$\vec{n} = \nabla F(P) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$P(x_0, y_0, z_0)$



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

$F = x^2 + y^2 + z = 10 \quad (k = 10)$

$\nabla F = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}$ and $\nabla F(-1,3,0) = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$

So tangent plane eqn. is

$-2(x-(-1)) + 6(y-3) + 1(z-0) = 0$

$\Rightarrow \boxed{-2(x+1) + 6(y-3) + z = 0}$

Example 67. Find the equation of the tangent plane at the point $P(-2, 1, -1)$ to the surface given by

$$z = \boxed{4 - x^2 - y} \quad f(x,y)$$

$4-4-1$ $\quad f(-2,1) = -1$

① Identify F: $F(x,y,z) = 4 - x^2 - y - z = 0$ defines the surface.

② Find $\vec{n}$: $\vec{n} = \nabla F(P)$ $\quad \nabla F = \begin{bmatrix} -2x \\ -1 \\ -1 \end{bmatrix}$ $\quad \nabla F(P) = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$

③ Plane eq: use $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

get $$\boxed{4(x+2) - (y-1) - (z+1) = 0}$$

expand this term & move z.

$\nabla f = \begin{bmatrix} -2x \\ -1 \end{bmatrix}$ $\quad \nabla f(-2,1) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$\boxed{z = -1 + 4(x+2) - (y-1)}$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's <u>The linearization</u>

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7, \; P(1, -1, 3).$$

Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

Example 68. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

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Identity the function $F(x,y,z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. 4 Finally, find a parametric equation for the line normal to the surface at P.

surface is level set

1. $F(x,y,z) = x^2+2xy-y^2+z^2 = k$, $k=7$

2. $\nabla F = \begin{bmatrix} 2x+2y \\ 2x-2y \\ 2z \end{bmatrix}$ @ $(1,-1,3)$ $\nabla F(1,-1,3) = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} = \vec{n}$

3. plane eqn be comes $\cancel{0(x-1)} + \boxed{4(y+1)+6(z-3)=0}$

4. want line passing thru $P(1,-1,3)$ in the direction of $n = \langle 0,4,6 \rangle$

$$\boxed{\ell(t) = \langle 1,-1,3 \rangle + t\langle 0,4,6 \rangle,\ t \in \mathbb{R}}$$

(formula
$\ell(t) = \vec{OP} + t\vec{v}$ $t \in \mathbb{R}$)

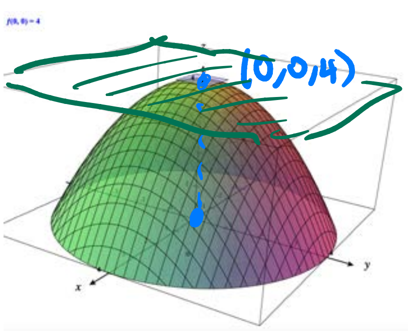
§14.7 Optimization: Local & Global

Gradient: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

$\nabla h = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$ @ $(0,0)$ $\nabla h(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

① What should we expect to get if we compute $\nabla h(0, 0)$? Why? ② What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



① meaning of $\nabla h(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is that @ $(0,0)$ we are already at a local max height value, so best direction to increase height is DON'T MOVE.

②

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

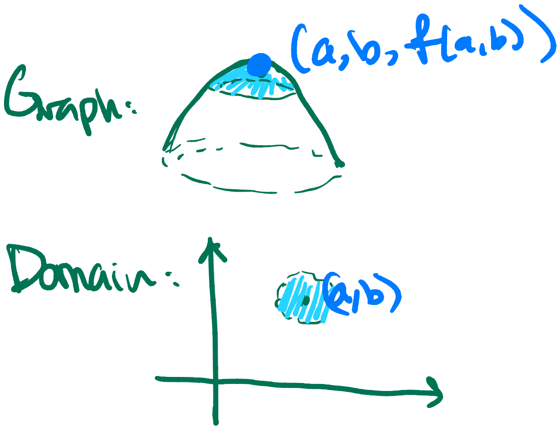
$\nabla h(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

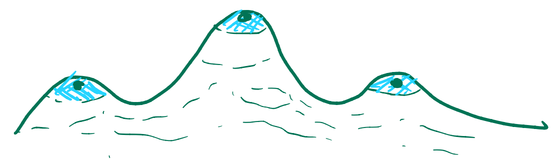
$z = h(0,0) + h_x(0,0)(x-0) + h_y(0,0)(y-0)$

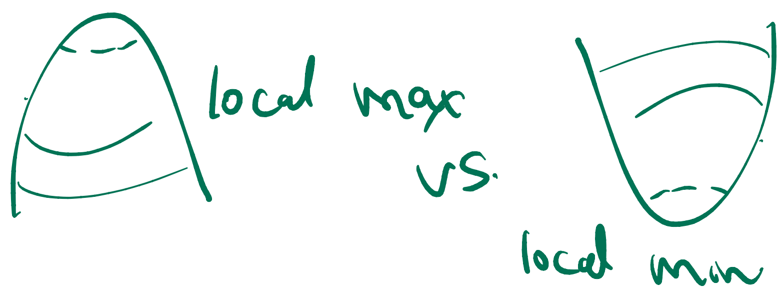
$= 4 + 0 + 0 \rightarrow \boxed{z = 4}$

Definition 68. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a <u>local max</u> value of f if $f(a, b)$ <u>$\geq$</u> $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a <u>local min</u> value of f if $f(a, b)$ ____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)

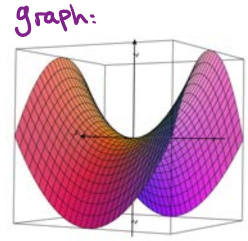




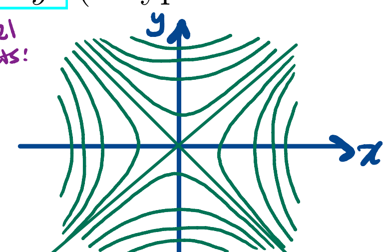


In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $\boxed{z = x^2 - y^2}$ (a hyperbolic paraboloid!) near $(0, 0)$.

graph:



level sets:



This is called a <u>Saddle point.</u>

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

(local) MAX/MIN occurs @ (a,b) if ① either $Df = [0\ 0]$ $\Leftrightarrow \nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e.g. $f(x,y) = \sqrt{x^2+y^2}$

$\nabla f = \left\langle \frac{2x}{2\sqrt{x^2+y^2}}, \frac{2y}{2\sqrt{x^2+y^2}} \right\rangle$

or ② $Df(a,b)$ is DNE

Definition 69. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ <u>$[0\ \ 0]$</u> or where $Df(a, b)$ <u>is DNE (undefined)</u> is called a <u>critical points</u> of f.

Example 70. Find the critical points of the function

① Set

$$f(x,y) = x^3 + y^3 - 3xy.$$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix} \overset{②}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

& Solve for x, y.

both work?

Need both to be true, so x,y is a soln

$(x,y) \in \{(1,1), (0,0)\}$.

$$\begin{cases} ① \; 3x^2 - 3y = 0 \\ ② \; 3y^2 - 3x = 0 \end{cases}$$

sub ② into ①

$$\Rightarrow \begin{cases} ① \; 3x^2 = 3y \\ ② \; 3y^2 = 3x \end{cases} \Rightarrow \begin{cases} ① \; y = x^2 \\ ② \; x = y^2 \end{cases} \Rightarrow \begin{cases} y = (y^2)^2 \\ x = y^2 \end{cases}$$

$$\Rightarrow \; ① \; y^4 - y = 0 \; \& \; ② \; x = y^2$$

$$\Rightarrow \; ① \; y(y^3 - 1) = 0 \; \& \; ② \; x = y^2$$

So ① $y = 0$ or $y = 1$ AND ② $x = y^2$

So If $y = 0$ then $x = 0$

If $y = 1$ then $x = 1$

So $\boxed{(0,0), (1,1)}$ are all crit pts.

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

b) $g(x,y) = \cos(x) + \sin(x)$

c) $h(x,y) = \dfrac{4}{x^2+y^2}$

d) $k(x,y) = x^2 + y^2$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

$$Df = \begin{bmatrix} 3 & 3y^2+4y \end{bmatrix}$$

NO CRIT POINTS since $Df \neq [0 \;\; 0]$ for any $(x,y) \in \mathbb{R}^2$, so $\boxed{\text{NO}}$

b) $g(x,y) = \cos(x) + \sin(x)$

$$Dg = \begin{bmatrix} -\sin x & \cos x \end{bmatrix} = [0 \;\; 0]$$

if $x = \frac{\pi}{4} + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.

but $Df(0,0) \neq [0 \;\; 0]$

so $\boxed{\text{NO}}$

c) $h(x,y) = \dfrac{4}{x^2+y^2}$

$$Dh = \begin{bmatrix} \frac{-8x}{(x^2+y^2)^2} & \frac{-8y}{(x^2+y^2)^2} \end{bmatrix}$$ is

<u>DNE</u> @ $(0,0)$

<u>BUT</u> $(0,0)$ not in the DOMAIN OF h!! So $\boxed{\text{NO}}$

d) $k(x,y) = x^2 + y^2$

$$Dk = \begin{bmatrix} 2x & 2y \end{bmatrix}$$ and

$Dk(0,0) = [0 \;\; 0]$ so

$\boxed{\text{yes}}$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a, b) =$$

Theorem 72 (2nd Derivative Test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and f has continuous second partial derivatives. Then we have:*

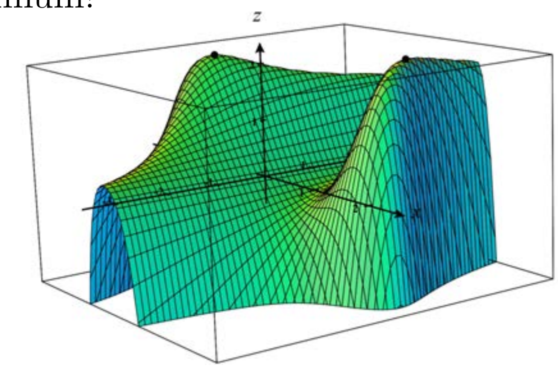
- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum*
- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum*
- *If $\det(Hf(a, b)) < 0$, f has a saddle point at (a, b)*
- *If $\det(Hf(a, b)) = 0$, the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 73. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 70.

Two Local Maxima, No Local Minimum: The function $\boxed{g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2}$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!



A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

Theorem 74. *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*

14.1 Functions of Several Variables 795

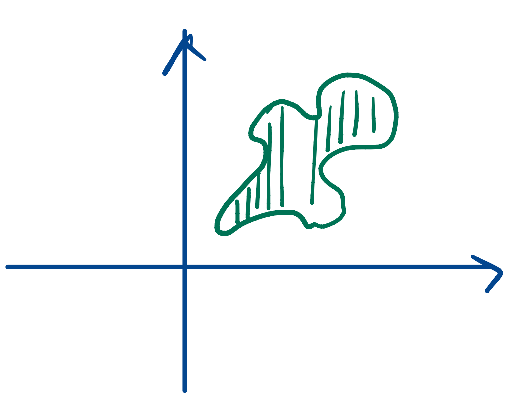
DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

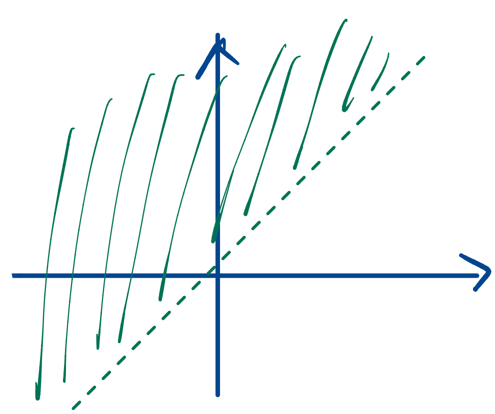
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

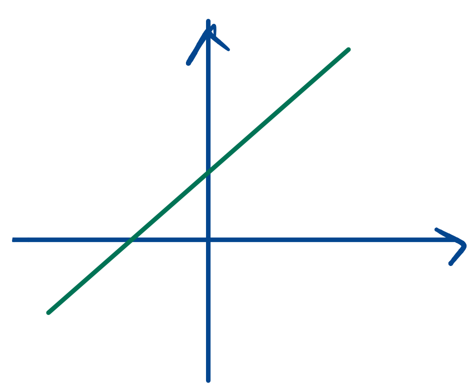
Closed:

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Bounded:





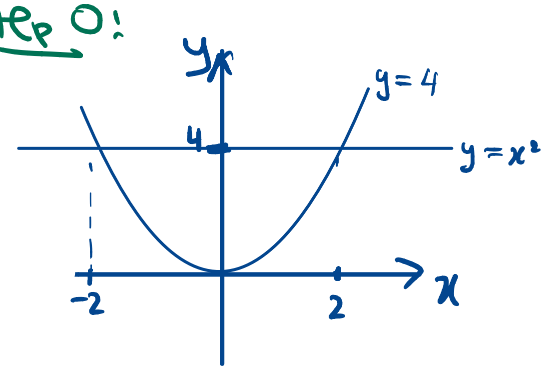


Strategy for finding global min/max of $f(x,y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Step 0!



Example 76. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)

Step 3: Eval

(x,y)	$f(x,y)$