Example 71. *You try it!* Determine which of the functions below have a critical point at $(0, 0)$.

a) $f(x, y) = 3x + y^3 + 2y^2$

b) $g(x, y) = \cos(x) + \sin(x)$

c) $h(x, y) = \dfrac{4}{x^2 + y^2}$

d) $k(x, y) = x^2 + y^2$

Example 71. *You try it!* Determine which of the functions below have a critical point at $(0,0)$.

a) $f(x,y) = 3x + y^3 + 2y^2$

$Df = [3 \quad 3y^2 + 4y]$

NO CRIT POINTS since

$Df \neq [0 \quad 0]$ for

any $(x,y) \in \mathbb{R}^2$, so $\boxed{\text{NO}}$

b) $g(x,y) = \cos(x) + \sin(x)$

if $h(x,y)$ is scalar,

$\boxed{Dh^T = \nabla h}$

$Dg = [-\sin x + \cos x \quad 0] = [0 \quad 0]$

~~if $x = \frac{\pi}{4} + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.~~

So ~~but $Df(0,0) \neq [0 \quad 0]$~~

$\tan x = 1$

$x = \pi/4 + k\pi, k \in \mathbb{Z}$.

So $\boxed{\text{NO}}$

c) $h(x,y) = \dfrac{4}{x^2+y^2}$

(a,b) crit point of $h(x,y)$ if

⓪ (a,b) in domain of h and either

① $\nabla h(a,b) = \vec{0}$ or

② $\nabla h(a,b)$ is DNE.

$Dh = \left[\dfrac{-8x}{(x^2+y^2)^2} \quad \dfrac{-8y}{(x^2+y^2)^2}\right]$ is DNE @ $(0,0)$

BUT $(0,0)$ not in the DOMAIN of h!! So $\boxed{\text{NO}}$

d) $k(x,y) = x^2 + y^2$

$Dk = [2x \quad 2y]$ and

$Dk(0,0) = [0 \quad 0]$ so

$\boxed{\text{yes}}$

Sim to 2nd derivative test in Calc I.

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at (a, b) is

$f'' > 0$ ∪ min
$f'' < 0$ max ∩

Calc I.

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Symmetric
Same
$f_{xy} = f_{yx}$

Theorem 72 (2nd Derivative Test)**.** *Suppose (a, b) is a critical point of $f(x, y)$ and f has continuous second partial derivatives. Then we have:*

mult. version

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, $f(a, b)$ *is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, $f(a, b)$ *is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, f *has a saddle point at* (a, b)
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 73. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 70.

Step 1. Compute crit pts. Solve $\nabla f = \vec{0}$

Step 2 classify case $Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

Step 1: $\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (0,0)$ & $(1,1)$ only crit pts (from previous slide [J]).

Step 2: First compute ① f_{xx}, ② $f_{xy} = f_{yx}$, ③ f_{yy}

$f_{xx} = 6x$ $\quad f_{xy} = -3$

$f_{yx} = -3$ $\quad f_{yy} = 6y$

So $Hf = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

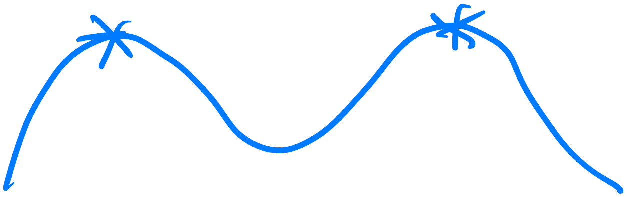
@ $(0,0)$ $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$, $\det Hf(0,0) = -9 < 0$

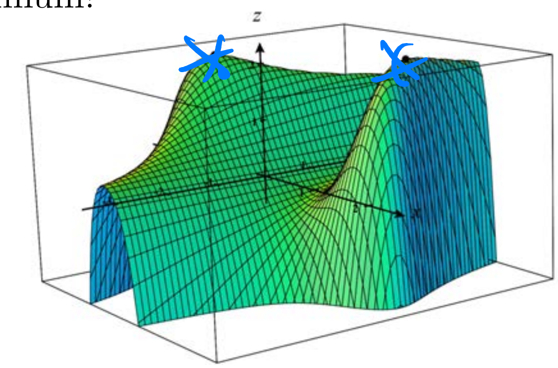
So $\boxed{\text{saddle @ } (0,0)}$

@ $(1,1)$ $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$, $\det Hf = 27 > 0$ & $f_{xx}(1,1) = 6 > 0$

So $\boxed{\text{MIN @ } (1,1)}$

Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!





A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 74. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

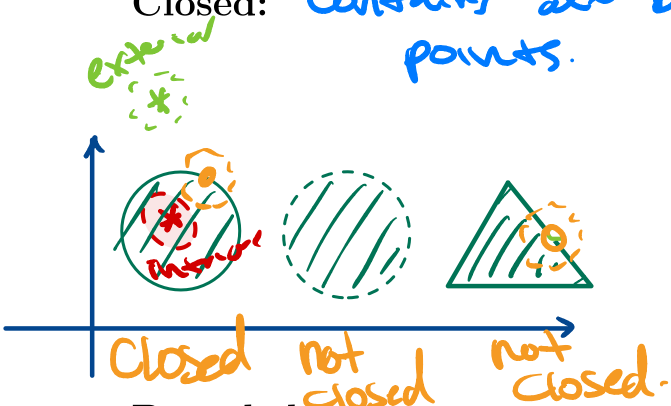
14.1 Functions of Several Variables **795**

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

(boundary)

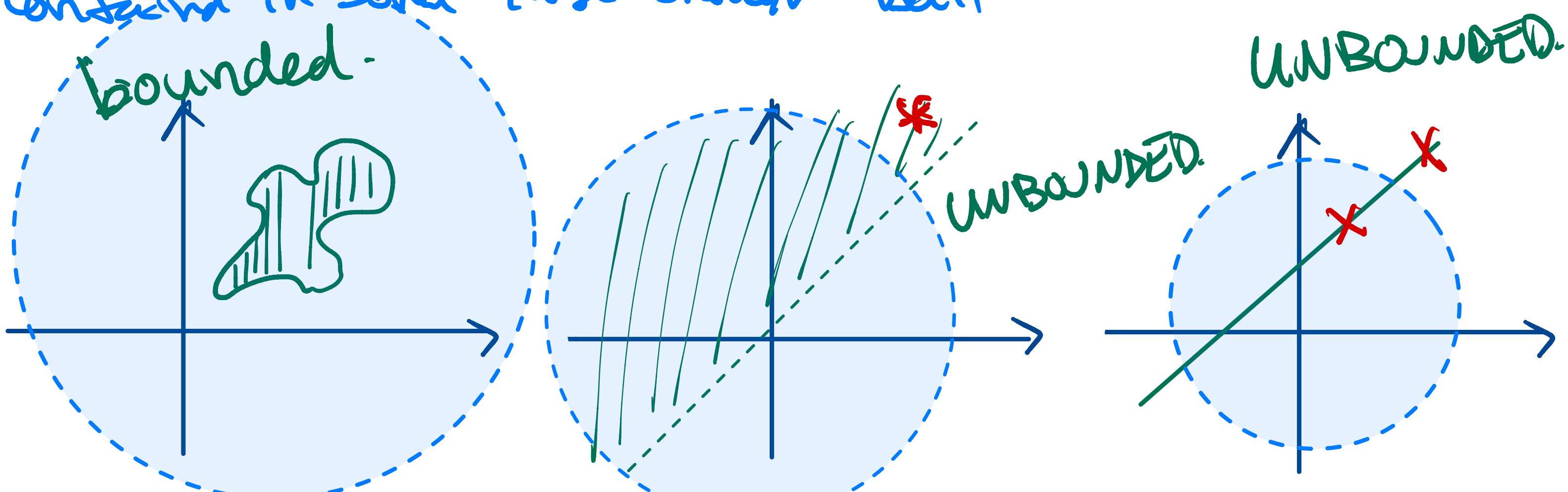
Closed: Contains all bdry points.



DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Bounded:

Contained in some large enough ball

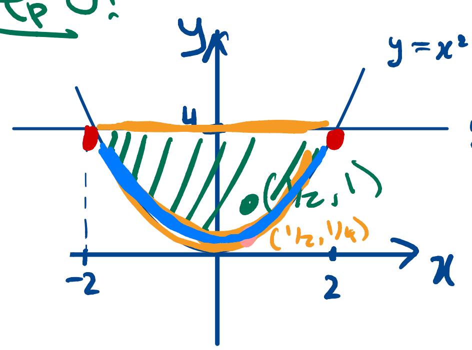


Strategy for finding global min/max of $f(x,y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $\boxed{f(x,y) = 4x^2 - 4xy + 2y}$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Step 0: Draw R.



Step 1: Find crit pts inside R

set $\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 8x - 4y \\ -4x + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Solve. $\begin{cases} 8x - 4y = 0 & ① \\ -4x + 2 = 0 & ② \end{cases}$ From ② $x = 1/2$

plug into ① $4 - 4y = 0$

so $y = 1$

So crit pt of f is $\boxed{(1/2, 1)}$.

Step 2 investigate the 1-dim'l boundary pieces.

@ $y = x^2$ $f(x, x^2) = 4x^2 - 4x \cdot x^2 + 2x^2 = -4x^3 + 6x^2$

Solve $f' = 0$ for x, $x \in (-2, 2)$ $f'(x) = -12x^2 + 12x = 0$

$\Rightarrow x(-x+1) = 0$

$\Rightarrow x = 0$ or $x = 1$

If $x = 0$ then $y = x^2 = 0$ So get $\boxed{(0,0)}$

If $x = 1$ then $y = x^2 = 1$ So get $\boxed{(1,1)}$

Example 76. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)

@ $y=4$ $\quad f(x,4) = 4x^2 - 16x + 8$ Set $f'=0$ & solve for $x \in (-2,2)$.

$f'(x) = 8x - 16 \Rightarrow \underline{\underline{x=2}}$ (ignore since on bdry of $(-2,2)$)

Step 3: investigate 0-dim'l bdry pts (intersect pts)

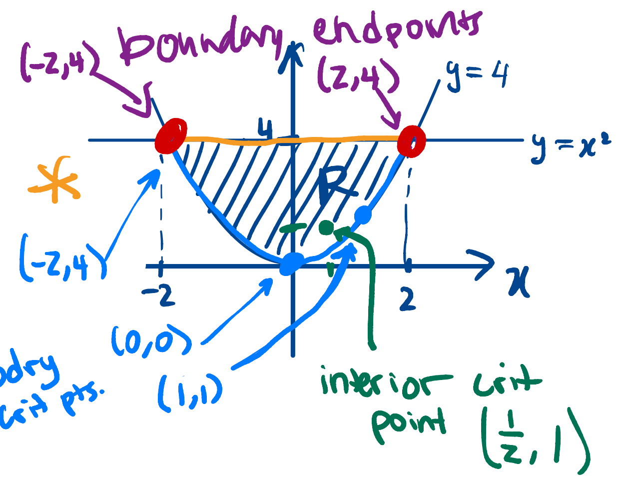
Get the intersection pts. by setting $y=4$ and $y=x^2$ equal $\Rightarrow 4 = x^2$

$\Rightarrow x = \pm 2$

$\Rightarrow \boxed{(2,4), (-2,4)}$

Step 3: Evaluate all points.

	(x,y)	$f(x,y) = 4x^2 - 4xy + 2y$
interior	$(\frac{1}{2}, 1)$	1
on parabola	$(0,0)$	0
	$(1,1)$	2
endpt.	$(2,4)$	$\boxed{-8}$ MIN @ $(2,4)$
	$(-2,4)$	$\boxed{56}$ MAX @ $(-2,4)$

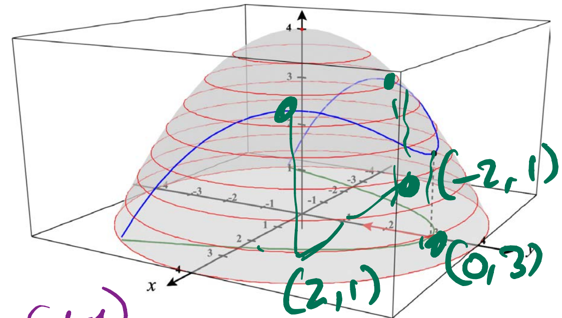


Idea: ① $\nabla f = \lambda \nabla g$ ② $g = k$.

§14.8 Constrained Optimization, Lagrange Multipliers

Goal: Maximize or minimize $f(x,y)$ or $f(x,y,z)$ subject to a *constraint*, $g(x,y) = c$.

Example 77. A new hiking trail has been constructed on the hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?



Objective function:

$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

(Function you want to maximize)

Constraint equation:

$g(x,y) = y + 0.5x^2 = 3$

$(k=3)$

(condition on (x,y) that must be satisfied)

---

① set up the Lagrange equations

$\begin{cases} ① \ \nabla h = \lambda \nabla g \\ ② \ g = k \end{cases}$

$\nabla h = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$ $\quad \nabla g = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}$

So L-G eqns are

$$① \begin{cases} -\frac{1}{2}x = \lambda x \\ -\frac{1}{2}y = \lambda \end{cases} \quad ② \ y + \frac{1}{2}x^2 = 3 \Rightarrow \begin{cases} -\frac{1}{2}x - \lambda x = 0 \\ -\frac{1}{2}y = \lambda \\ y - \frac{1}{2}x^2 = 3 \end{cases} \Rightarrow \begin{cases} ① \ x\left(-\frac{1}{2} - \lambda\right) = 0 \\ -\frac{1}{2}y = \lambda \\ y - \frac{1}{2}x^2 = 3 \end{cases}$$

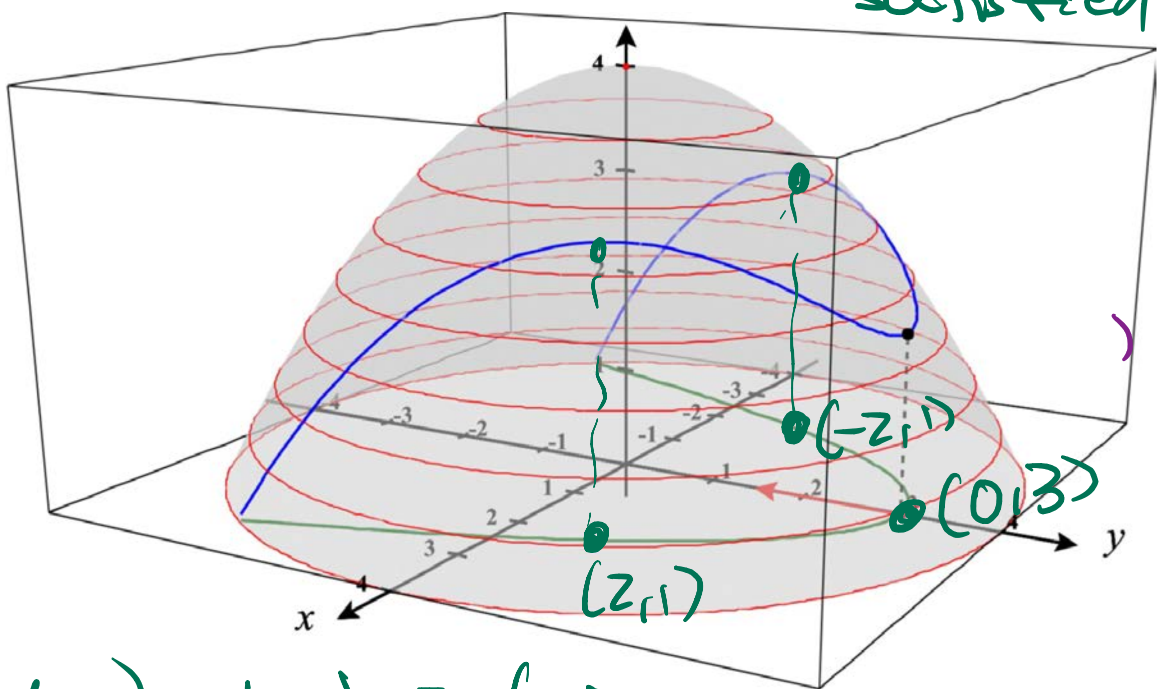
From ①
$x = 0$ or $\lambda = -\frac{1}{2}$

Case 1: If $x = 0$ Then $\begin{cases} 0 = 0 \\ -\frac{1}{2}y = \lambda \\ y + 0 = 3 \end{cases}$ So $y = 3$ and get $(0,3)$

Case 2: If $\lambda = -\frac{1}{2}$ then $\begin{cases} -\frac{1}{2}x = -\frac{1}{2}x \\ -\frac{1}{2}y = -\frac{1}{2} \\ y + \frac{1}{2}x^2 = 3 \end{cases}$ So $y = 1$ → $1 + \frac{1}{2}x^2 = 3$ $\Rightarrow x^2 = 4,\ x = \pm 2$ get $(2,1)$, $(-2,1)$

Example 77. A new hiking trail has been constructed on the hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

(Cont.) So (x,y) either $(0,3)$, $(2,1)$, or $(-2,1)$ for the Lagrange eqns to all be satisfied



(x,y)	$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$
$(2,1)$	$h(2,1) = 4 - 1 - \frac{1}{4} = 2.75$ MAX
$(-2,1)$	$h(-2,1) = 4 - 1 - \frac{1}{4} = 2.75$ MAX
$(0,3)$	$h(0,3) = 4 - 0 - \frac{9}{4} = 1.75$ MIN??

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective function: X No $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ = "distance from (x,y,z) to $(0,0,0)$!"

yes ✓ $f(x,y,z) = x^2+y^2+z^2$

Constraint: $g(x,y,z) = z^2 - xy = 4$

$(k=4)$

IDEA FROM CALC 1
minimize instead
$d^2 = f(x,y,z) = x^2+y^2+z^2$
(no square root, easier w/ same answer)

Lagrange eqns.

① $\nabla f = \lambda \nabla g$ $\qquad \nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \qquad \nabla g = \begin{bmatrix} -y \\ -x \\ 2z \end{bmatrix}$

② $g = k$

Solve ① $\begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ 2z = \lambda 2z \end{cases}$ ② $z^2 - xy = 4$

From $2z - \lambda 2z = 0 \Rightarrow 2z(1-\lambda) = 0$
either $z=0$ or $\lambda = 1$.

Case $z=0$. $\begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ -xy = 4 \end{cases} \Rightarrow \begin{cases} x = \frac{-\lambda y}{2} \\ ⓐ\ 2y = -\lambda\left(\frac{-\lambda y}{2}\right) \\ ⓑ\ -\left(\frac{-\lambda y}{2}\right) y = 4 \end{cases}$

ⓐ $\Rightarrow 2y = \lambda^2 \frac{y}{2} \Rightarrow 2y - \frac{\lambda^2}{2} y = 0 \Rightarrow y\left(2 - \frac{\lambda^2}{2}\right) = 0$

either $y = 0$ or $\lambda = \pm 2$

Case $z=0$ (cont.)

If $z=0$ and $y=0$ then get $\boxed{(0,0,0).}$ X

If $z=0$ and $\lambda=2$ then $\begin{cases} 2x = -2y \\ 2y = -2x \\ -xy = 4 \end{cases} \Rightarrow \begin{cases} x = -y \\ x^2 = 4 \end{cases} \Rightarrow x = \pm 2,\ y = \mp 2$

get $(2,-2,0), (-2,2,0)$

If $z=0$ and $\lambda=-2$ then

$\begin{cases} 2x = 2y \\ 2y = 2x \\ -xy = 4 \end{cases} \Rightarrow \begin{cases} x = y \\ -x^2 = 4 \end{cases}$ X impossible.

Example 78. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

(Cont.) Last case $\lambda = 1$.

Case $\lambda = 1$ then solve $\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ 2z = \lambda 2\lambda \\ z^2 - xy = 4 \end{cases}$ becomes $\begin{cases} 2x = y \\ 2y = x \\ 2z = 2z \checkmark \\ z^2 - xy = 4. \end{cases}$

Plug in $y = 2x$ into $2y = x$ get $2(2x) = x \Rightarrow 4x = x$

$\Rightarrow x = 0$

So $y = 2(0) = 0$

constraint becomes $z^2 - 0 = 4 \Rightarrow z = \pm 2$

get $\boxed{(0,0,2) \text{ and } (0,0,-2)}$

(x,y,z)	$f(x,y,z) = x^2 + y^2 + z^2$
$(2,-2,0)$	$4+4+0 = 8$
$(-2,2,0)$	8
$(0,0,2)$	4
$(0,0,-2)$	4

max distance $\sqrt{8}$

min distance $\sqrt{4} = \boxed{2}$

Example 79. *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Example 79. *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Set up: $\nabla f = \lambda \nabla g$ $\quad d(x,y) = \sqrt{x^2+y^2}$ but do instead $f(x,y) = x^2+y^2$

$g(x,y) = x^2+xy+y^2$

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x+y \\ 2y+x \end{bmatrix}$

So Lagrange equations to solve are

① $2x = \lambda(2x+y)$

② $2y = \lambda(2y+x)$

③ $x^2+xy+y^2=1$

Case 1: If $x=0$ then ③ $y^2=1 \Rightarrow y=\pm 1$.

get $(0,1)$ & $(0,-1)$.

Case 2: if $y=0$ Then ③ $x^2=1 \Rightarrow x=\pm 1$ get $(1,0)$, $(-1,0)$.

Case 3: $x\neq 0$ and $y \neq 0$. Then ① & ② become $\lambda = \frac{2x}{2x+y} = \frac{2y}{2y+x}$

$\Rightarrow 2x(2y+x) = 2y(2x+y)$

$\Rightarrow 4xy + 2x^2 = 4xy+2y^2$

$\Rightarrow x^2 = y^2$

$\Rightarrow$ (a) $x=y$ or (b) $x=-y$.

(x,y)	$f(x,y)$
$(0,1)$	1
$(0,-1)$	1
$(-1,0)$	1
$(1,0)$	1
$(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$	$2/3$
$(1,-1)$	2
$(-1,1)$	2

In case $y=x$ then

③ $x^2+x^2+x^2=1$

$\Rightarrow x^2 = 1/3$

$\Rightarrow x = \pm 1/\sqrt{3}$

get $(1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, -1/\sqrt{3})$

In case $y=-x$ then

③ $x^2-x^2+x^2=1$

$\Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

get $(1,-1)$ and $(-1,1)$

So MIN value $\sqrt{2}$ @ $(1,-1)$ & $(-1,1)$

MAX value $\sqrt{2/3}$ @ $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$