§15.1 Double Integrals, Iterated Integrals, Change of Order

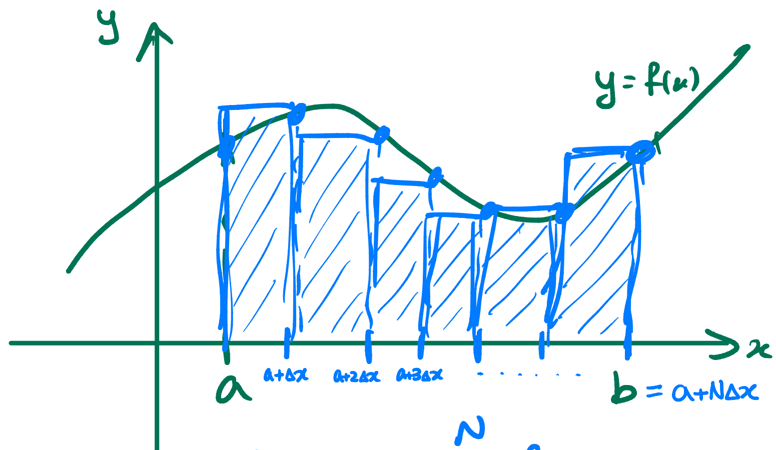
Recall: Riemann sum and the definite integral from single-variable calculus.

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k) * \Delta x$$

* test point in each sub-interval

$x_k^* \in [a + k\Delta x, a + (k+1)\Delta x]$

* area of rectangle $f(x_*) * \Delta x$



$$\text{Area} \approx \sum_{k=1}^{N} f(x_k)\Delta x$$

In the limit you get EXACT value of area

Also, same area w/ other choices of how to pick height so long as $x_k^* \in [a+k\Delta x, a+(k+1)\Delta x]$

such as :
* Right end point
* left endpoint
* midpoint

even other ways eg.
* average height (trapezoid rule)
* Simpson's rule (more complicated), etc. etc.

8:10 (5 min) 8:15

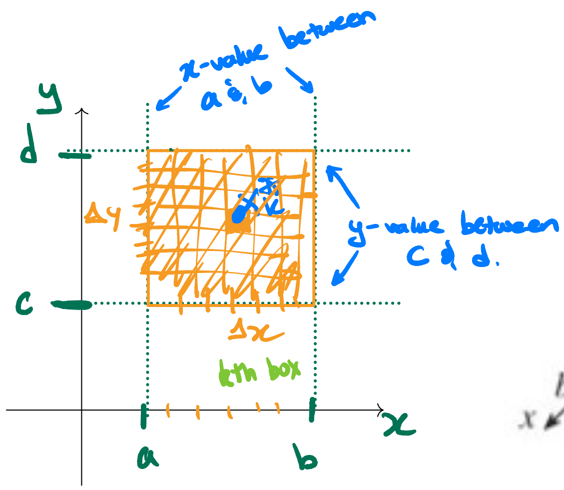
Double Integrals

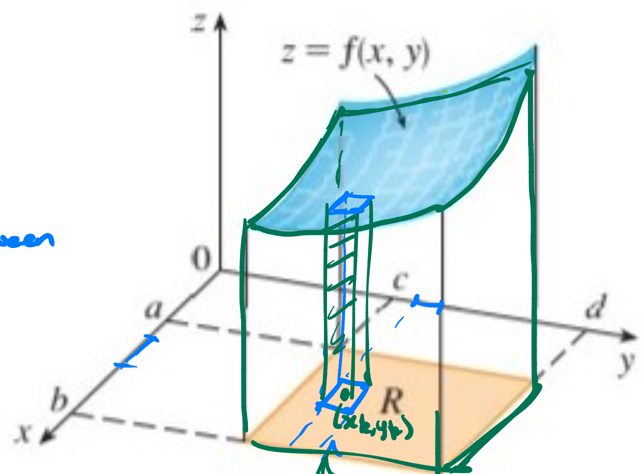
Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x,y)$ be a function defined on R such that $f(x,y) \geq 0$. Let S be the solid that lies above R and under the graph f.

$\vec{X}_k^* = (x_k^*, y_k^*)$ the kth subinterval test input value.





vol of box is $f(x_k^*, y_k^*) * \Delta x \Delta y$

Question: How can we estimate the volume of S?

$$\text{Volume} \approx \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} f(x_i^*, y_j^*) \Delta y \Delta x$$

In the limit

$$\text{Volume} = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

differentials

$$\text{AKA} = \iint_R f(x,y)\, dA$$

area measure

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Definition 79. The double integral of $f(x, y)$ over a rectangle R is

$$\iint_R \underbrace{f(x,y)}_{\text{integrand}}\, dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k \quad (*)$$

R: region of integration

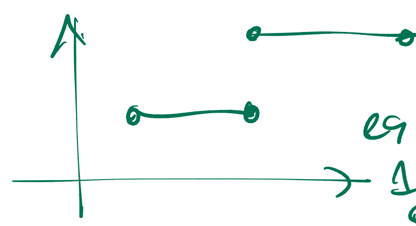
P is the "mesh size", largest size among all of the bases of the tall boxes.

if this limit exists.

When (*) actually exists. Then we say "f is integrable over R"

Eg.

- f is integrable over R if f is continuous over R
- possible f NOT continuous over R but still integrable.



eg. 1-var calc.

* Note: $\iint_R f(x,y)\, dA$ is a signed volume. (e.g. negative if $f(x,y) < 0$ for all $(x,y) \in R$.)

represents total volume w/ vol. above $z=0$ counts as $+$

and vol. below $z=0$ counts as $-$.

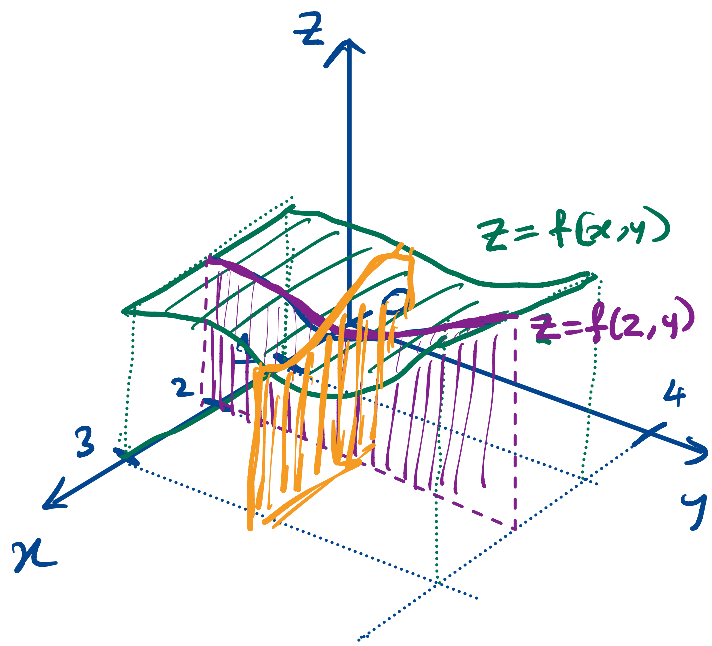
Question: How can we compute a double integral?

Answer: Integrate one variable at a time. (iterate!)

Let $\boxed{f(x,y) = 2xy}$ and lets integrate over the rectangle $R = [1,3] \times [0,4]$.

We want to compute $\int_1^3 \int_0^4 f(x,y)\,dy\,dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2,y)\,dy$ represent here?



Purple slice has area @ $x=2$

$$\text{Area} = \int_0^4 f(2,y)\,dy$$

$$= \int_0^4 4y\,dy = 2y^2\Big|_0^4$$

$$= 2*4^2 - 2*0^2 = \boxed{32} - 0$$

If we move slice along x-direction & add up all the areas we get, then get volume of the whole thing.

$$\text{Vol} = \int_1^3 \underbrace{\int_0^4 f(x,y)\,dy}_{\text{area of purple slice at } x = \text{const.}}\,dx = \int_1^3 \overbrace{\int_0^4 2xy\,dy}^{\text{do inside first}}\,dx$$

$$= \int_1^3 xy^2\Big|_0^4\,dx$$

$$= \int_1^3 (16x - 0)\,dx = 8x^2\Big|_1^3 = 72 - 8 = \boxed{64}$$

volume is

8:30 (7 min) 8:37

In general, if $f(x, y)$ is integrable over $R = [a, b] \times [c, d]$, then $\int_c^d f(2, y)\, dy$ represents:

area of slice @ $x=2$. (area purple slice)

What about $\int_c^d f(x, y) dy$? $= A(x)$

the area of the slice at $x=$ const.

$\iint_R f(x,y)\, dA$ (double integral)

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$\boxed{\text{Volume} = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x,y)\, dy\, dx}$$

This is called an iterated integral.

Example 80. Evaluate $\int_1^2 \underbrace{\int_3^4 6x^2 y\, dy}_{\text{do inside first}}\, dx.$ $= \int_1^2 3x^2y^2 \Big|_3^4 dx$

$= \int_1^2 3x^2 * 16 - 3x^2 * 9\, dx = \int_1^2 7 * 3x^2\, dx = \int_1^2 21x^2\, dx$

$= 7x^3 \Big|_1^2 = 7(2^3) - 7(1^3) = 7 * 7 = \boxed{49}$

Theorem 81 (Fubini's Theorem). *If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then*

$$\int_a^b \underbrace{\int_c^d f(x,y)\, dy}_{\text{first } y}\, dx = \int_c^d \underbrace{\int_a^b f(x,y)\, dx}_{\text{first } x}\, dy$$

"order doesn't matter"

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

8:37 (13 min) 8:50

Example 82. *You try it!* Integrate:

a) $\displaystyle\int_0^2 \int_{-1}^1 x - y \; dy \; dx$ **easy**

b) $\displaystyle\cancel{\int_0^1} \int_0^1 \frac{\overset{2}{\cancel{y}}}{1 + x\overset{2}{\cancel{y}}} \; dx \; \cancel{dy}$ **medium**

c) $\displaystyle\int_1^4 \int_1^e \frac{\ln x}{xy} \; dx \; dy$ HARD!

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

$$= \int_0^2 xy - \tfrac{1}{2}y^2 \Big|_{-1}^{1} dx$$

$$= \int_0^2 x - (-x) - 0 \, dx$$

$$= \int_0^2 2x \, dx = x^2 \Big|_0^2 = \boxed{4}$$

b) $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy$ **medium**

$$\int_0^1 \int_0^1 \frac{y}{1+xy} \underbrace{dx}_{du} dy = \int_0^1 \ln(1+xy) \Big|_0^1 dy = \int_0^1 \ln(1+y) - \cancel{y \ln(1)}^{\,0} \, dy$$

u-sub

$$\boxed{\begin{array}{l} u = 1+xy \\ du = y\,dx \end{array}}$$

IBP $\int u \, dv = uv - \int v \, du$

$$\boxed{\begin{array}{ll} u = \ln(t) & dv = 1\,dt \\ du = \frac{1}{t}dt & v = t \end{array}}$$

So $\int \ln t \, dt = t \ln t - \int dt$

$= t \ln t - t + C$

$$= (1+y)\ln(1+y) - (1+y) \Big|_0^1$$

$$= [2\ln 2 - 2] - [1 \cancel{\ln(1)}^{\,0} - 1] = \boxed{2\ln 2 - 1}$$

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ HARD!

$$= \int_1^4 \underbrace{\tfrac{1}{2}(\ln x)^2}_{*} * \tfrac{1}{y} \Big|_1^e dy = \int_1^4 \left[\tfrac{1}{2}(\cancel{\ln e}^{\,1})^2 - \tfrac{1}{2}(\cancel{\ln 1}^{\,0})^2\right] * \tfrac{1}{y} \, dy$$

IBP $\int u \, dv = uv - \int v \, du$

$$\boxed{\begin{array}{ll} u = \ln x & dv = \frac{1}{x}dx \\ du = \frac{1}{x}dx & v = \ln x \end{array}}$$

$$= \int_1^4 \tfrac{1}{2} * \tfrac{1}{y} \, dy = \tfrac{1}{2}\ln(y) \Big|_1^4 = \tfrac{1}{2}\ln 4 - \tfrac{1}{2}\cancel{\ln 1}^{\,0} = \boxed{\ln 2}$$

$a \ln b = \ln(b^a)$

so

$\frac{1}{2}\ln 4 = \ln(\sqrt{4})$

$= \ln 2$

$$* = \int \frac{\ln x}{x} dx = (\ln x)^2 - \underbrace{\int \ln x \cdot \tfrac{1}{x} \, dx}_{*}$$

So $2* = (\ln x)^2 + C$

$* = \frac{1}{2}(\ln x)^2 + C$ ✓

Ans: use Fubini's Thm!

Example 83. Compute $\displaystyle\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.

Hint: Fubini's Theorem.

IDEA? $\displaystyle\int_{-1}^{1}\int_{0}^{4} x e^{e^{e^y}}\,dy\,dx = ?$

Better Idea? $\displaystyle\int_{0}^{4}\int_{-1}^{1} xe^{e^{e^y}}\,dx\,dy$

$$= \int_0^4 \frac{1}{2}x^2 e^{e^{e^y}} \Big|_{-1}^{1}\,dy$$

$$= \int_0^4 e^{e^{e^y}}\left[(1)^2 - (-1)^2\right]dy$$

$$= \int_0^4 0 * e^{e^{e^y}}\,dy = \boxed{0} \checkmark$$

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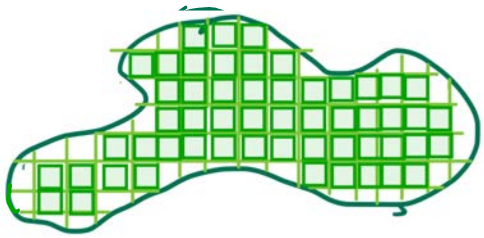
§15.2 Double Integrals on General Regions

Question: What if the region R we wish to integrate over is not a rectangle?



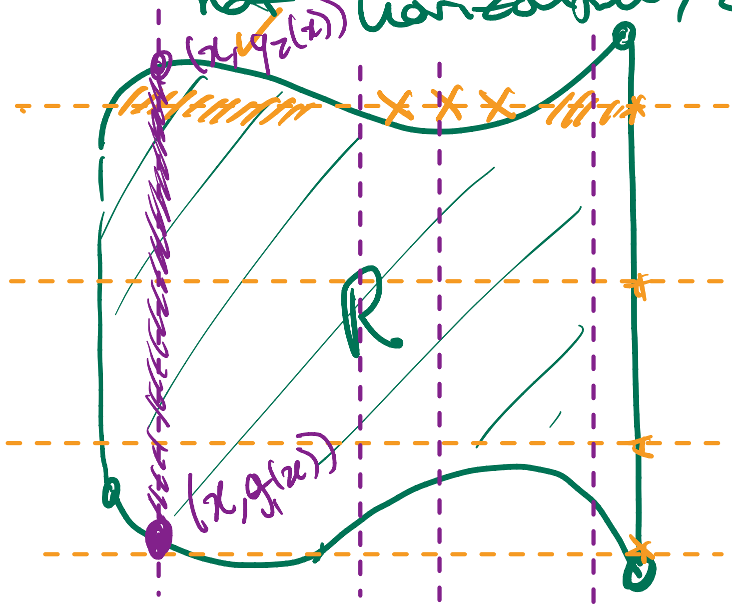
Need to sample (x,y)-values

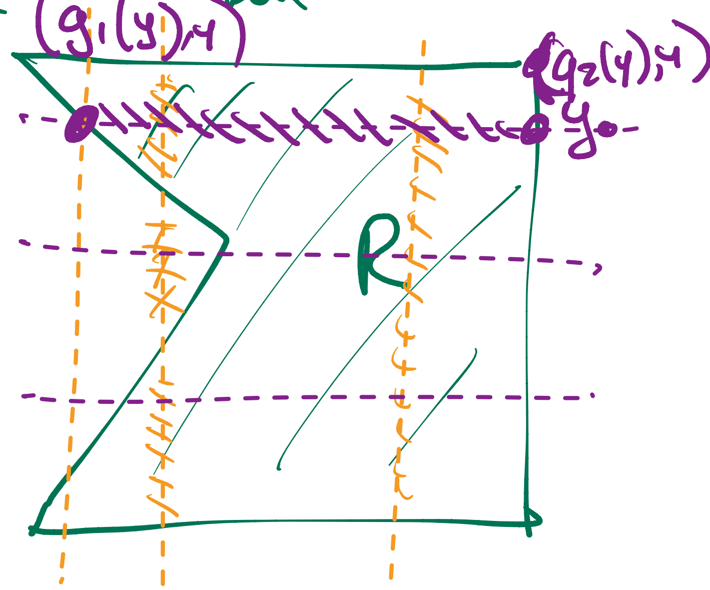
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

IDEA: We need to find some way to express the boundary curves like $y = g(x)$ or $x = g(y)$.

vertically simple but not horizontally simple

horizontally simple but not vert. simple

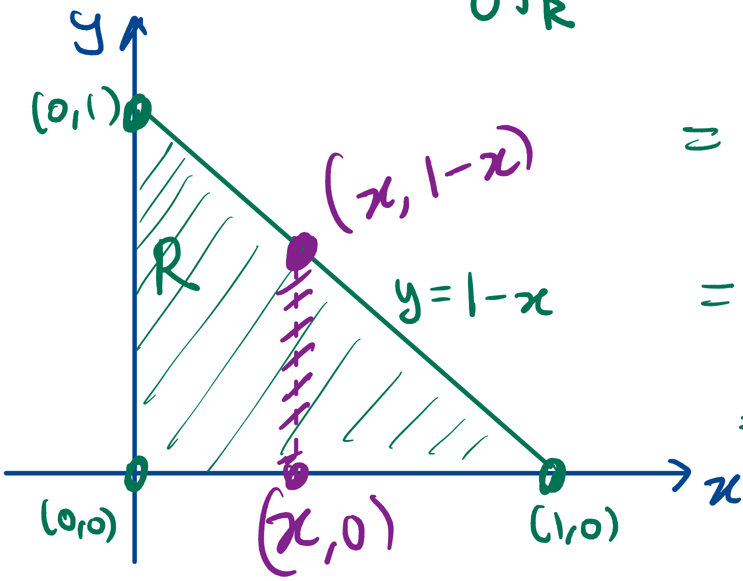




Example 84. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:

$$\text{Vol} = \iint_R 2-x-y \, dA = \int_0^1 \int_0^{1-x} 2-x-y \, dy \, dx$$



$$= \int_0^1 2y - xy - \tfrac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 \left[2(1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2\right] - 0 \, dx$$

$$= \int_0^1 2 - 2x - x + x^2 - \tfrac{1}{2}(1 - 2x + x^2) \, dx$$

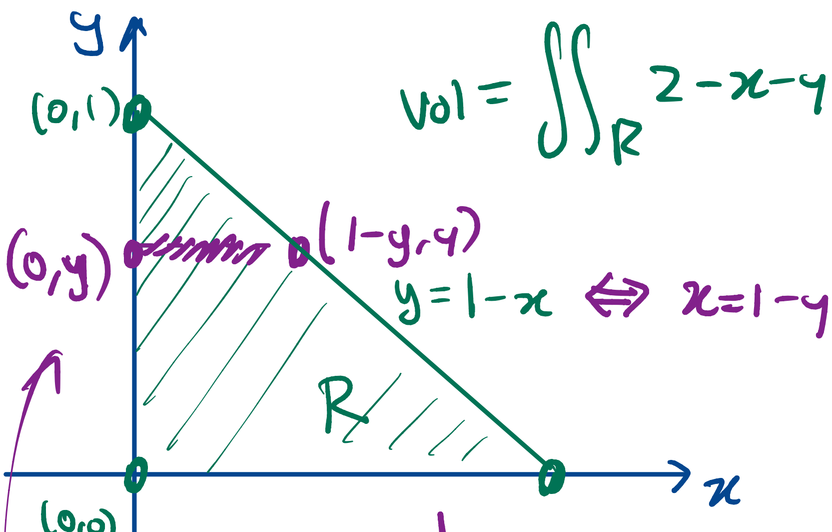
$$= \int_0^1 2 - 3x + x^2 - \tfrac{1}{2} + x - \tfrac{1}{2}x^2 \, dx$$

$$= \int_0^1 \tfrac{3}{2} - 2x + \tfrac{1}{2}x^2 \, dx = \tfrac{3}{2}x - x^2 + \tfrac{1}{6}x^3 \Big|_0^1 = \left(\tfrac{3}{2} - 1 + \tfrac{1}{6}\right) - 0 = \tfrac{1}{2} + \tfrac{1}{6}$$

$$= \tfrac{4}{6} = \boxed{2/3}$$

Horizontally simple:

$$\text{Vol} = \iint_R 2-x-y \, dA = \int_0^1 \int_0^{1-y} 2-x-y \, dx \, dy$$



$$= \dots = \boxed{2/3}$$

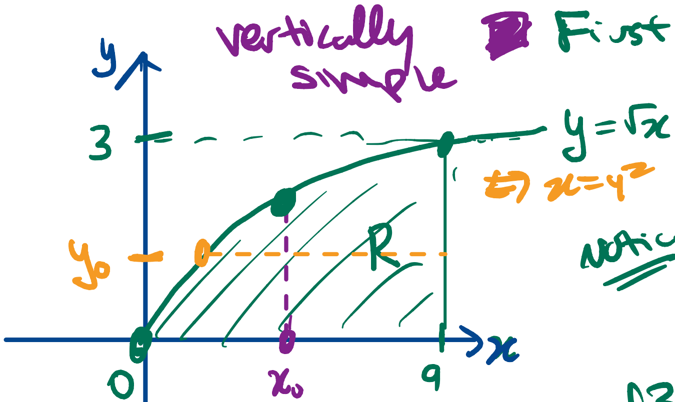
Same value.

For each fixed y-value the x-value ranges from 0 to 1-y.

Example 85. Write the two iterated integrals for $\iint_R 1\, dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

vertically simple

First pick x_0 then $y \in [0, \sqrt{x_0}]$



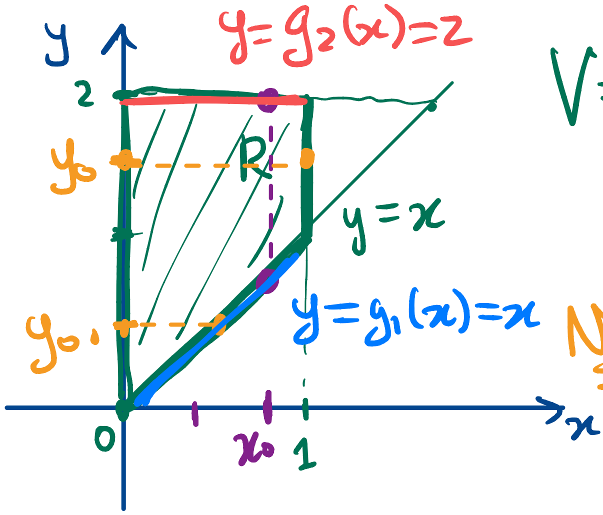
$V=\int_0^9 \int_0^{\sqrt{x}} 1 \, dy\, dx$

Notice $= \int_0^9 y \Big|_0^{\sqrt{x}} dx = \boxed{\int_0^9 \sqrt{x} - 0 \, dx}$

1

horizontally simple $V=\int_0^3 \int_{y^2}^9 1 \, dx\, dy$

Example 86. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\, dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.



$V=\int_0^1 \int_x^2 6x^2y \; dy\, dx$

↑ y?

NOT horizontally simple.

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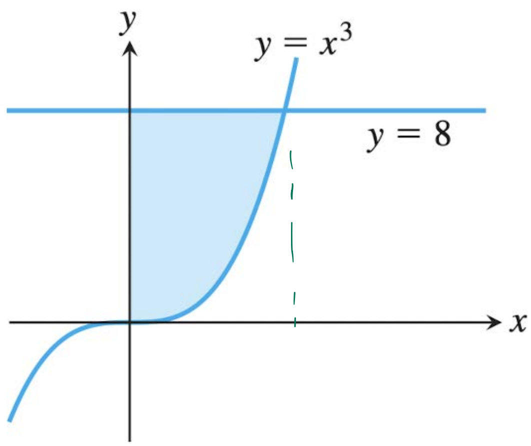
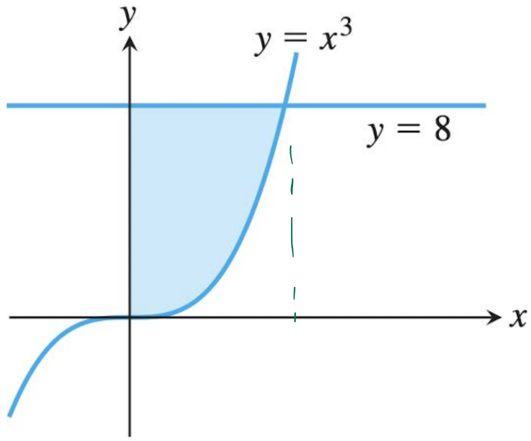
Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \; dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



(Vertical)

$$\text{volume} = \int_0^2 \int_{x^3}^8 1 \; dy \, dx$$

(horizontal)

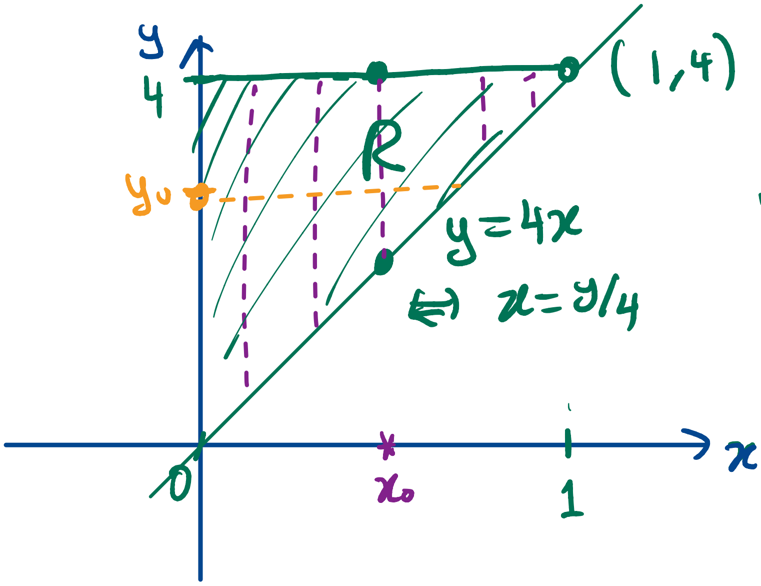
$$\text{volume} = \int_0^8 \int_0^{\sqrt[3]{y}} 1 \; dx \, dy.$$

Example 88. Sketch the region of integration for the integral

$$V = \int_0^1 \int_{4x}^4 f(x, y)\, dy\, dx.$$

Then write an equivalent iterated integral in the order $dx\, dy$.

For $(x,y) \in R$. The x-values are in $[0,1]$ and then the y-values are in $[4x, 4]$.



$$V = \int_0^4 \int_0^{y/4} f(x,y)\, dx\, dy$$

$$\text{Vol} = \iint_R 1\, dA = \text{Area}(R)$$