§15.3 Area & Average Value

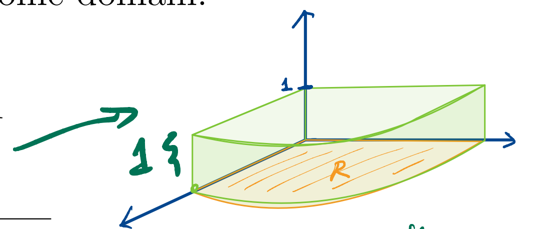
Exam 2 in one week (in lecture) on Monday

Two other applications of double integrals are computing the area of a region in ~~the~~ plane and finding the average value of a function over some domain.

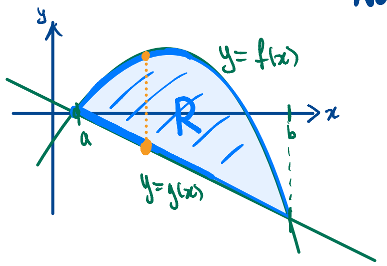
Area: If R is a region bounded by smooth curves, then

From Calc II,

$$\text{Area}(R) = \iint_R 1\, dA$$



Area R * 1 = $\iint_R 1\, dA$
base height "volume"

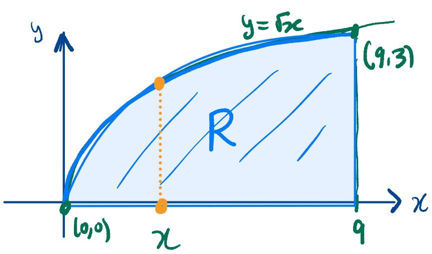


NOTE:

$$\int_a^b \int_{g(x)}^{f(x)} 1\, dy\, dx = \int_a^b y \Big|_{g(x)}^{f(x)} dx$$

$$= \int_a^b f(x) - g(x)\, dx \quad \text{"top - bot"}$$

Example 89. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.



$x \in [0, 9]$
$y \in [0, \sqrt{x}]$

$$\text{Area}(R) = \int_0^9 \int_0^{\sqrt{x}} 1\, dy\, dx$$

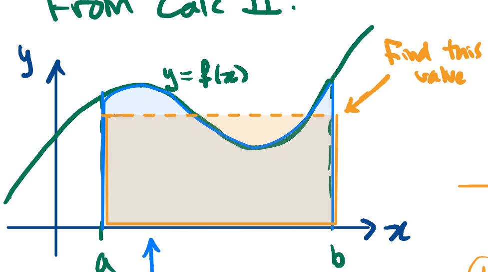
$$= \int_0^9 y \Big|_0^{\sqrt{x}} dx = \int_0^9 \sqrt{x} - 0\, dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^9 = \frac{2}{3}\left(9^{3/2} - 0\right) = \frac{2}{\cancel{3}} \cdot \cancel{27}^{\,9}$$

$$= \boxed{18}$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

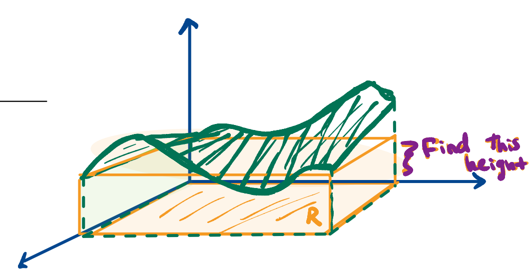
From Calc II.



blue area should match orange area

$$f_{avg} = \frac{1}{\text{Area } R} \iint_R f(x, y)\, dA$$

$(b-a) * \text{avg value} = \int_a^b f(x)\, dx$

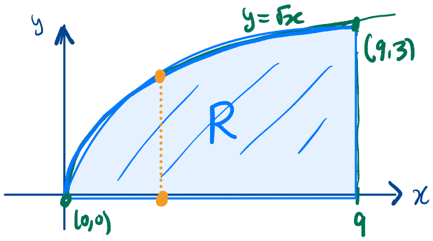


avg value of f * area R

$$= \iint_R f(x,y)\, dA.$$

$$\frac{1}{b-a} \int_a^b f(x)\, dx = \text{avg value of } f \text{ over } [a,b]$$

Example 90. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.



Area R = 18 from previous page, so

$$\text{Avg}_R f = \frac{1}{\text{Area}R} * \text{Vol}_R f$$

$$= \frac{1}{18} * \iint_R 4xy^2 \, dA$$

$$= \frac{1}{18} \int_0^9 \int_0^{\sqrt{x}} 4xy^2 \, dy \, dx$$

$$= \frac{1}{18} \int_0^9 \frac{4}{3} x y^3 \Big|_0^{\sqrt{x}} dx$$

$$= \frac{1}{18} \int_0^9 \frac{4}{3} x * x^{3/2} - 0 \, dx$$

$$= \frac{1}{18} \int_0^9 \frac{4}{3} x^{5/2} \, dx = \frac{1}{\cancel{18}_{\,9}} * \boxed{\frac{4}{3}} * \frac{\cancel{2}}{7} x^{7/2} \Big|_0^9$$

$$= \frac{4}{9*3*7} \left(9^{7/2} - 0\right) = \frac{4}{9*3*7} 3^7 = \frac{1}{7} * 3^4 = \frac{4*81}{7}$$

$$= \boxed{\frac{324}{7}}$$

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

$$\text{Vol} = \int_0^2\int_0^2 x^2+y^2\,dy\,dx = \int_0^2 x^2y+\tfrac{1}{3}y^3\Big|_0^2$$

$$= \int_0^2 2x^2+\tfrac{8}{3}\,dx = \tfrac{2}{3}x^3+\tfrac{8}{3}x\Big|_0^2 = \tfrac{16}{3}+\tfrac{16}{3} = 32/3$$

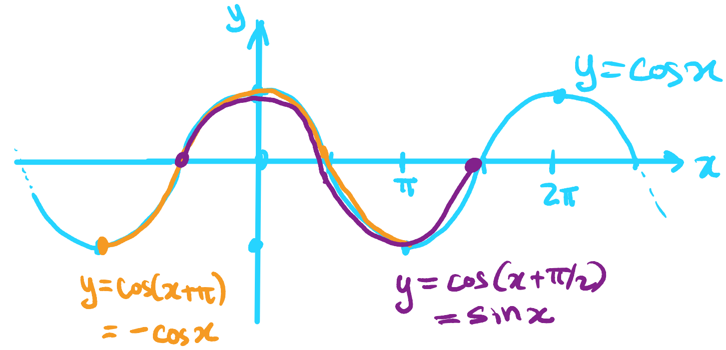
and Area $R = 4$

So $\text{Avg}_R(f) = \frac{1}{\text{Area}R} * \text{Vol} = \frac{1}{4} * \frac{32}{3} = \boxed{8/3}$

Example 92. Find the average value of the function $f(x, y) = \sin(x + y)$ on (a) the region $R_1 = [0, \pi] \times [0, \pi]$, and (b) the region $R_2 = [0, \pi] \times [0, \pi/2]$.

Hint: choose your order of integration carefully!

(a) $R_1 = [0,\pi] \times [0,\pi]$. ?



$$\text{Avg}_{R_1} f = \frac{1}{\pi^2}\boxed{\int_0^\pi}\int_0^\pi \sin(x+y)\,dy\,dx$$

$$= \frac{1}{\pi^2}\int_0^\pi -\cos(x+y)\Big|_0^\pi dx$$

$$= \frac{1}{\pi^2}\int_0^\pi \Big(\underbrace{-\cos(x+\pi)}_{\cos x} - \underbrace{(-\cos(x))}_{\cos x}\Big)dx = \frac{1}{\pi^2}\int_0^\pi 2\cos x\,dx$$

$$= \frac{1}{\pi^2}\, 2\sin x\Big|_0^\pi = \frac{2}{\pi^2}\big(\cancelto{0}{\sin\pi} - \cancelto{0}{\sin 0}\big) = \boxed{0}$$

(b) $R_2 = [0,\pi] \times [0,\pi/2]$

$$\text{Avg}_{R_2} f = \frac{2}{\pi^2}\boxed{\int_0^\pi}\int_0^{\pi/2} \sin(x+y)\,dy\,dx$$

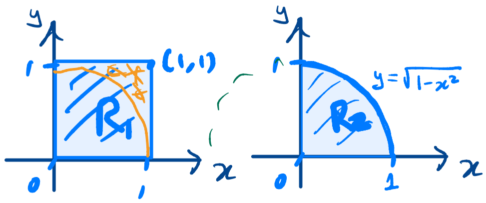
$$= \frac{2}{\pi^2}\int_0^\pi -\cos(x+y)\Big|_0^{\pi/2} dx = \frac{2}{\pi^2}\int_0^\pi \Big(\underbrace{-\cos(x+\pi/2)}_{\sin x} - \underbrace{(-\cos x)}_{\cos x}\Big)dx$$

$$= \frac{2}{\pi^2}\int_0^\pi \sin x + \cos x\,dx = \frac{2}{\pi^2}\big(-\cos x + \sin x\big)\Big|_0^\pi$$

$$= \frac{2}{\pi^2}\Big[\big(-\cos\pi + 0\big) - \big(-\cos 0 + 0\big)\Big] = \frac{2}{\pi^2}(1+1) = \boxed{\frac{4}{\pi^2}}$$

8:50 15 min 9:05

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.



BIGGER avg value.

$\mathrm{Avg}_{R_1}(f)$ or $\boxed{\mathrm{Avg}_{R_2}(f)}$?

guess b/c ÷ smaller area, so larger total?

$\mathrm{Area}(R_1) = 1$ $\quad$ $\mathrm{Area}(R_2) = \pi * \frac{1}{4} = \pi/4 \approx 3/4$.

$$\mathrm{Vol}_{R_1} f = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2} x y^2 \Big|_0^1 dx = \int_0^1 \tfrac{1}{2} x \, dx = \tfrac{1}{4} x^2 \Big|_0^1 = \tfrac{1}{4}$$

So Avg value of f over R_1 $= \underbrace{\frac{1}{1}}_{\frac{1}{\text{Area}}} * \underbrace{\frac{1}{4}}_{\iint_R f(x,y)\,dA} = \boxed{1/4}$

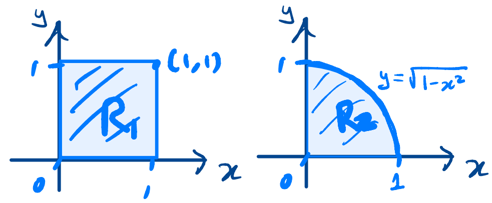
$$\mathrm{Vol}_{R_2} f = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \tfrac{1}{2} x y^2 \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \tfrac{1}{2} x \left(\sqrt{1-x^2}\right)^2 - 0 \, dx = \int_0^1 \tfrac{1}{2} x (1 - x^2) \, dx$$

$$= \int_0^1 \tfrac{1}{2} x - \tfrac{1}{2} x^3 \, dx = \tfrac{1}{4} x^2 - \tfrac{1}{8} x^4 \Big|_0^1 = \tfrac{1}{4} - \tfrac{1}{8} = 1/8$$

So avg value of f over R_2 $= \frac{1}{\pi/4} * \frac{1}{8} = \frac{4}{\pi * 8} = \boxed{\frac{1}{2\pi}} \approx 1/6$

Example 93. *You try it!* Which value is larger for the function $f(x,y) = xy$: the average value of f over the square $R_1 = [0,1] \times [0,1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.



$\mathrm{Avg}_{R_1}(f)$ or $\boxed{\mathrm{Avg}_{R_2}(f)}$?

guess b/c ÷ smaller area, so larger total?

$$V_{R_1} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^1 = \int_0^1 \tfrac{1}{2}x(1-0)\,dx = \int_0^1 \tfrac{1}{2}x \, dx$$

$$= \tfrac{1}{4}x^2 \Big|_0^1 = 1/4$$

So $\mathrm{Avg}_{R_1}(f) = \frac{1}{\text{Area } R_1} V_{R_1} = \frac{1}{1} * \frac{1}{4} = \boxed{\frac{1}{4}}$

$$V_{R_2} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1-x^2)\,dx$$

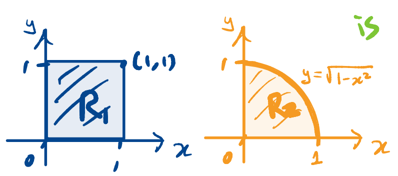
$$= \int_0^1 \tfrac{1}{2}(x - x^3)\,dx = \tfrac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} * \tfrac{1}{4} = 1/8$$

So $\mathrm{Avg}_{R_2}(f) = \frac{1}{\text{Area } R_2} V_{R_2} = \frac{1}{\pi/4} * \frac{1}{8} = \boxed{\frac{1}{2\pi}} \approx \boxed{\frac{1}{6}}$

Well I'll be...

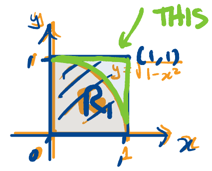
how's that work...

In retrospect, what seems to be happening



is that in the outer region of R_1, (the part not contained in R_2)

THIS BIT HERE



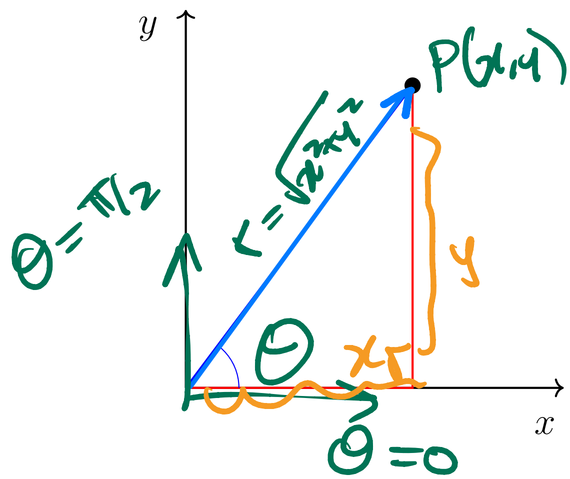
in the part of R_1 not contained in R_2 the value

who woulda think it.

of $f(x,y) = xy$ is "particularly large" since xy is small when x or y is small. And apparently this more than compensates for ÷ by larger area.

§15.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates



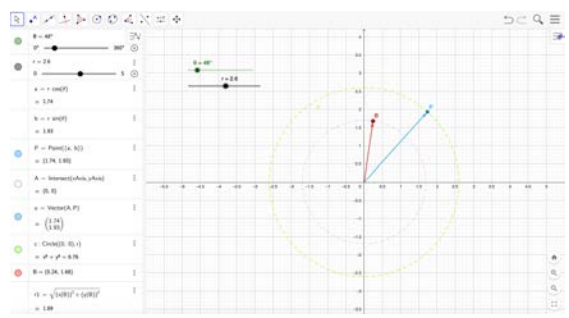
Cartesian coordinates: Give the distances in x and y directions from $(0,0)$

Polar coordinates:

- r = distance from (x,y) to $(0,0)$
- θ = angle between the ray $\langle x,y\rangle$ or $\overrightarrow{OP}$ and the positive x-axis.

We can use trigonometry to go back and forth.

https://www.geogebra.org/classic/thaxxzzp



Polar to Cartesian:

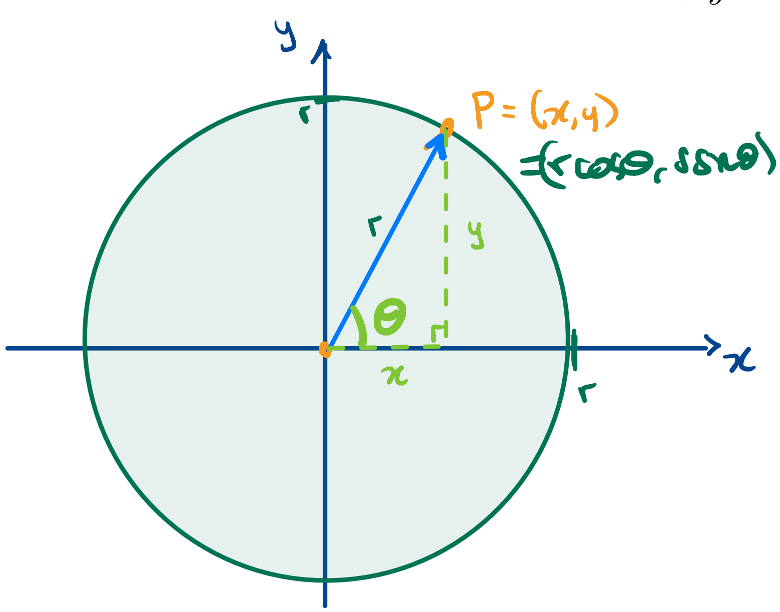
given (r,θ)

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar: given (x,y)

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

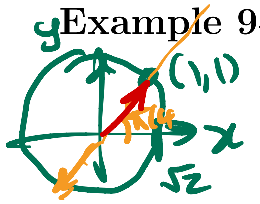
$\theta = \tan^{-1}(y/x)$ & quadrant.



Warning: many possible (r,θ) for one particular (x,y).

any of $(\sqrt{2}, \pi/4 + 2\pi k)$

Example 94. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.



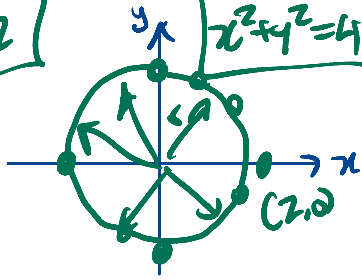
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\quad \theta = \tan^{-1}(1/1) = \tan^{-1}(1)$

$\Rightarrow \tan\theta = 1, \ \theta = \pi/4 + k\pi$ $\quad k \in \mathbb{Z}$

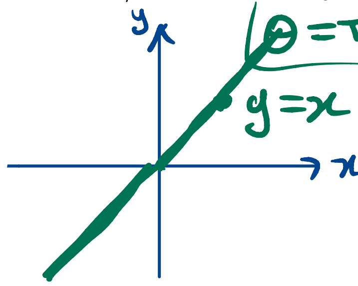
b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** xy**-plane.**

??

$\boxed{r=2}$ $\quad \boxed{x^2+y^2=4}$



$\boxed{\theta = \pi/4}$



$\boxed{y = x}$

idea $\iint_R \sqrt{x^2+y^2}\, dA$

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

approach?
$x = r\cos\theta$
$y = r\sin\theta$

So $\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r$

assume $r \geq 0$ and also $\theta \in [0, 2\pi]$ or $\theta \in [-\pi, \pi]$

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

Ideas ① $\sqrt{x^2+y^2} = r$

② $y = r\sin\theta$

$\Rightarrow \sin\theta = y/r = \dfrac{y}{\sqrt{x^2+y^2}}$

So $r = 2\sin\theta$

$\Leftrightarrow \sqrt{x^2+y^2} = 2 * \dfrac{y}{\sqrt{x^2+y^2}}$

$\Leftrightarrow x^2+y^2 = 2y \Rightarrow x^2+y^2-2y+1 = 1$

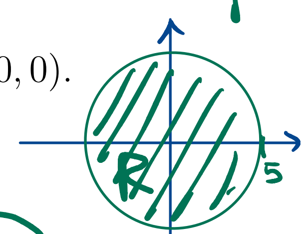
$\Rightarrow \boxed{x^2+(y-1)^2 = 1}$

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 95. Compute the area of the disk of radius 5 centered at $(0, 0)$.

$$\text{Area } R \overset{?}{=} \int_0^5 \int_0^{2\pi} 1 \underbrace{d\theta\, dr}_{dA??} = \int_0^5 \theta \Big|_0^{2\pi} dr$$

$$= \int_0^5 2\pi\, dr = 2\pi r \Big|_0^5 = \boxed{10\pi}$$



r values in $[0,5]$
θ values in $[0,2\pi]$

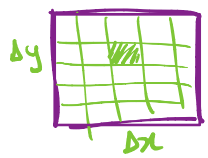
Remember: In polar coordinates, the area form $dA =$ $r\ d\theta\, dr$

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\, dA$.

Example 96. Compute the area of the disk of radius 5 centered at $(0,0)$.

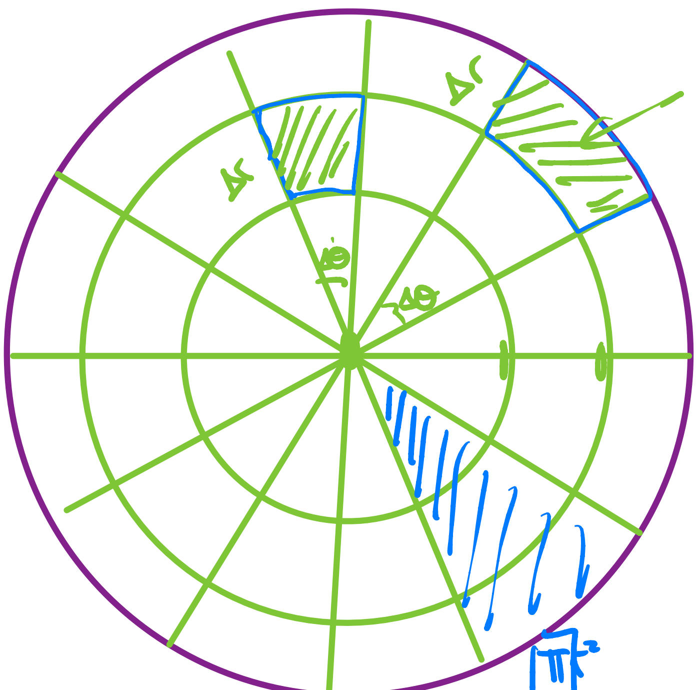
Cont.

recall.



$dA_k = \Delta x \Delta y$ so $\iint_R 1\, dA = \int_a^b \int_c^d 1\, dx\, dy$

but now.

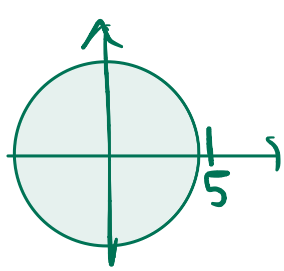


area $\Delta r * r\Delta\theta$

So $\iint_R 1\, dA = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} 1 \dots ?$

$\boxed{\pi r^2}$ Let's try it out!

Disk area $\iint_R 1\, dA = \int_0^5 \int_0^{2\pi} \cancel{1}\, r\, d\theta\, dr$



$= \int_0^5 r\theta \Big|_0^{2\pi} dr = \int_0^5 2\pi r\, dr$

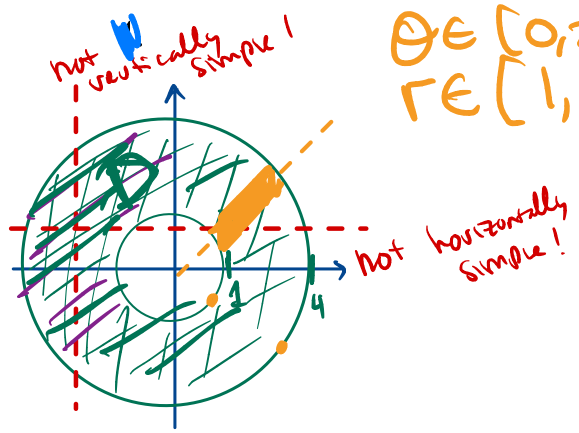
$= \pi r^2 \Big|_0^5 = \boxed{25\pi}$ ✓

Remember: In polar coordinates, the area form $dA =$ $r\, d\theta\, dr$

don't forget integration factor for polar coord. or $r\, dr\, d\theta$

$f(x,y) = e^{-(x^2+y^2)}$ $x^2+y^2=r^2$ ✓

Example 96. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 16$



$\theta \in [0, 2\pi]$
$r \in [1, 4]$

$$\text{Vol} = \int_0^{2\pi}\int_1^4 e^{-r^2}\, r\, dr\, d\theta$$

?? r, θ ?

$u = r^2$
$du = 2r\,dr$
$\frac{1}{2}du = r\,dr$

$$= \int_0^{2\pi} -\frac{1}{2} e^{-u} \Big|_1^{16} d\theta$$

It is radially simple.

$$= \int_0^{2\pi} -\frac{1}{2}\left(e^{-16} - e^{-1}\right) d\theta = -\frac{1}{2}(e^{-16} - e^{-1})\theta \Big|_0^{2\pi}$$

$$= \boxed{-\pi\,(e^{-16} - e^{-1})}$$

Example 97. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

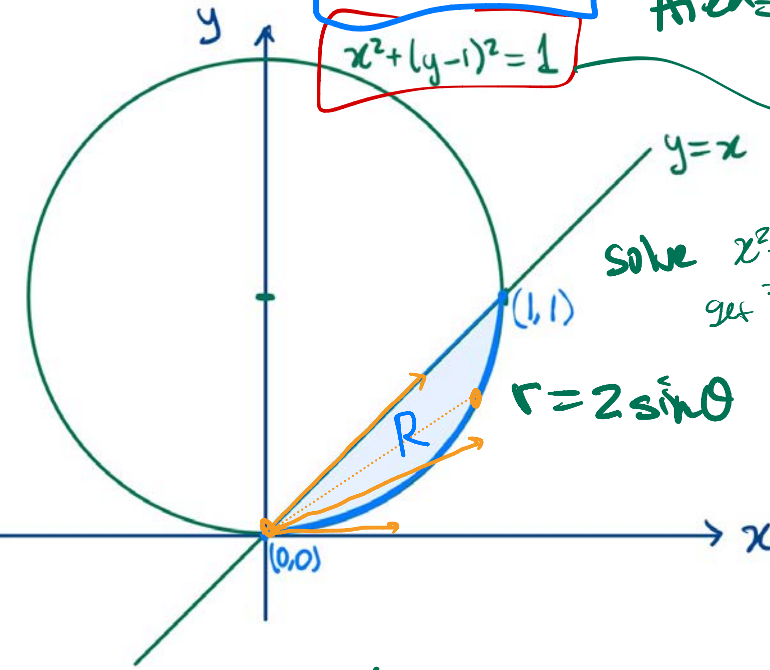
$\boxed{r = 2\sin\theta}$

$\boxed{x^2+(y-1)^2=1}$

$$\text{Area} = \iint_R 1\, dA$$

$$(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$$

$\Rightarrow \cdots \Rightarrow r = 2\sin\theta$



solve $x^2+(x-1)^2=1$
get $x=1$.

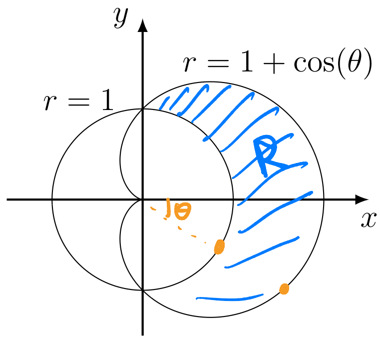
$\theta \in [0, \pi/4]$
$r \in [0, 2\sin\theta]$

$\sin^2 t = \frac{1-\cos 2t}{2}$

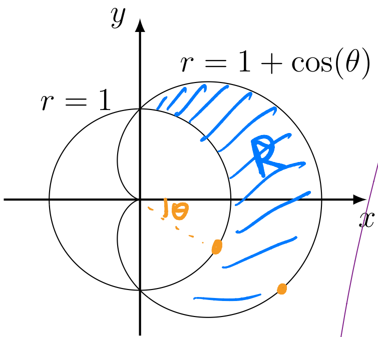
$$\text{Area} = \int_0^{\pi/4}\int_0^{2\sin\theta} 1 * \underbrace{r\, dr\, d\theta}_{dA} = \int_0^{\pi/4} \frac{1}{2} r^2 \Big|_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta\, d\theta = \int_0^{\pi/4} 1 - \cos 2\theta\, d\theta = \theta - \frac{1}{2}\sin 2\theta \Big|_0^{\pi/4} = \boxed{\frac{\pi}{4} - \frac{1}{2}} - (0 - 0)$$

Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$$\theta \in [-\pi/2, \pi/2]$$
$$r \in [1, 1+\cos\theta]$$

So $f(x,y) = x$
$$f(r,\theta) = r\cos\theta$$

don't forget r!!

$$\text{So } \iint_R f(x,y)\, dA = \boxed{\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\cos\theta * r\, dr\, d\theta}$$

Stop here ☺

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta \Big|_1^{1+\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\left[(1+\cos\theta)^3 \cos\theta - \cos\theta\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\left[(1+\cos\theta)^3 - 1\right] d\theta$$

oh lol.

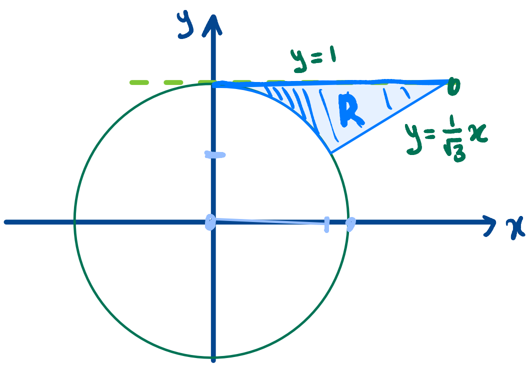
oops.

hmmm...

$$= \ldots = \boxed{5\pi/8}$$

SKIP.

Example 100. Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x,y)\, dA$.



$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta \; dr \; d\theta$$

Tips and tricks

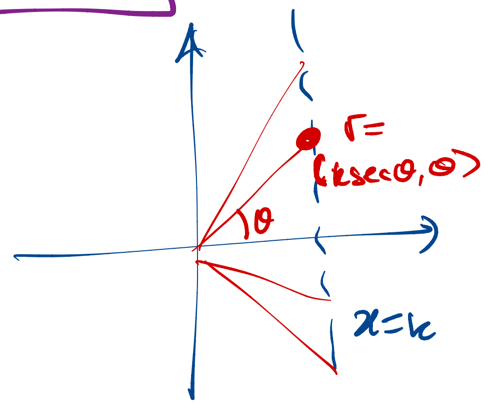
① $x = r\cos\theta$
② $y = r\sin\theta$

For horizontal lines such as $x = 2$:

use ① get $2 = r\cos\theta \Rightarrow \boxed{r = 2\sec\theta}$

in general

if $x = k \Rightarrow r = k\sec\theta$



For vertical lines such as $y = 1$ (e.g., Example 100):

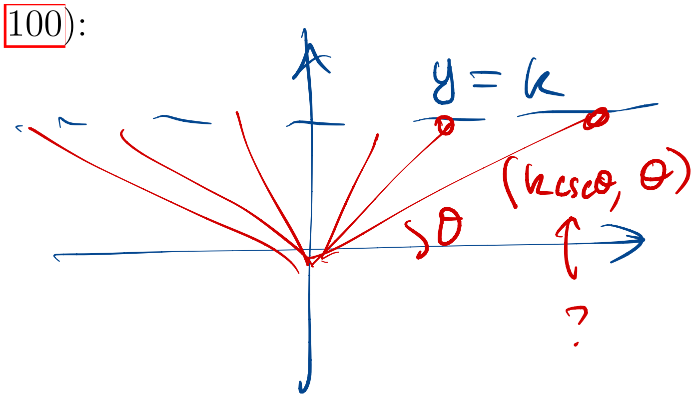
use ② get

$1 = r\sin\theta$

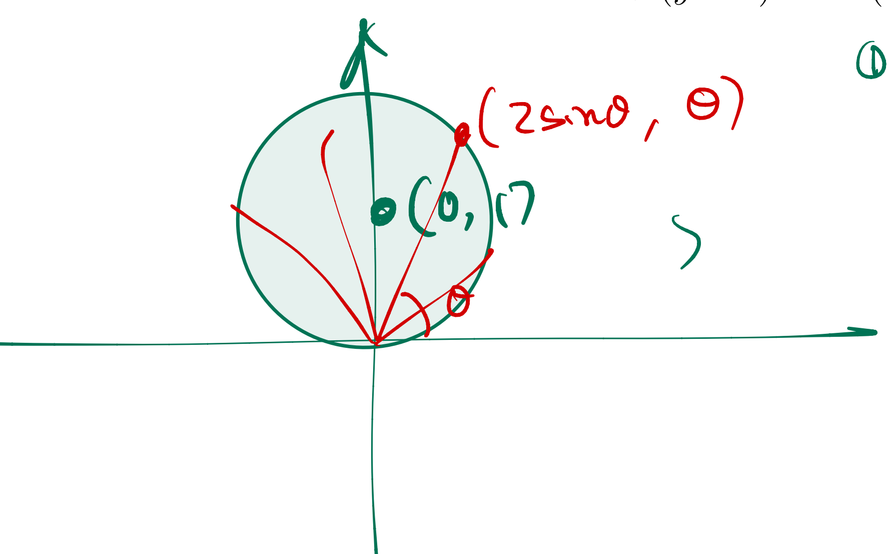
so $r = \csc\theta$

So in general $y = k$

$\Rightarrow r = k\csc\theta$



For off-set circles such as $x^2 + (y-1)^2 = 1$ (e.g., Example 98):



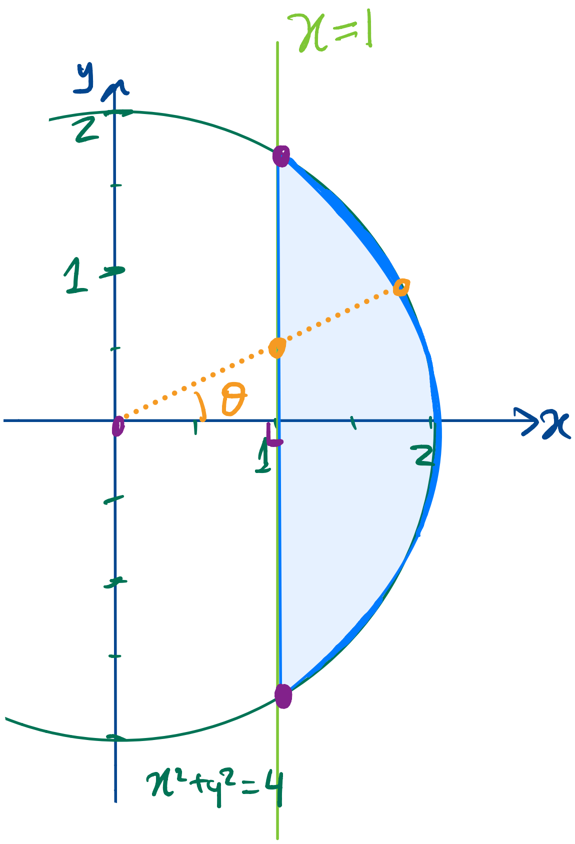
① & ② CTS.

$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$\Rightarrow r^2 - 2r\sin\theta = 0$

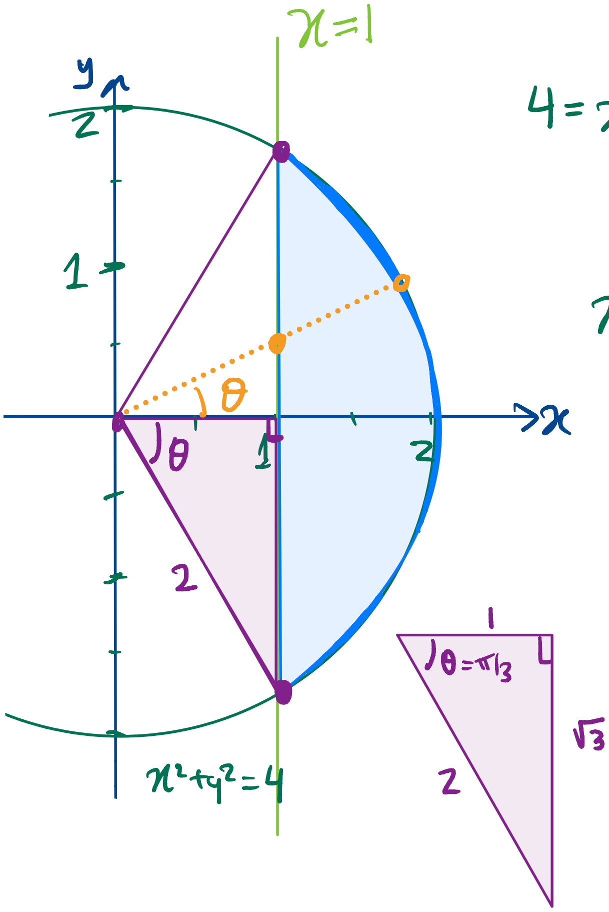
$\Rightarrow \boxed{r = 2\sin\theta}$

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.



Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

The θ-value is between $(1,-\sqrt{3})$ & $(1,\sqrt{3})$ so $\theta \in [-\pi/3, \pi/3]$.



$$4 = x^2 + y^2 = r^2 \Rightarrow r = \pm 2,\ r \geq 0$$

so $r = 2$ on circle.

$$x = r\cos\theta \Rightarrow r = \frac{x}{\cos\theta} = x\sec\theta$$

@ $x=1$ $\quad r = \sec\theta$ on line $x = 2$.

So $\theta \in [-\pi/3, \pi/3]$

and $r \in [\sec\theta, 2]$

So $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$$\text{Area} = \int_{-\pi/3}^{\pi/3}\int_{\sec\theta}^{2} 1 * r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}r^2\Big|_{\sec\theta}^{2}\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(\sec^2\theta - 4\right)d\theta$$

$$= \frac{1}{2}\tan\theta - 2\theta\Big|_{-\pi/3}^{\pi/3} = \tan(\pi/3) - 4 * \pi/3 = \boxed{\sqrt{3} - 4\pi/3}$$

Function is odd

Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.
   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
   (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
   (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.
   (d) At (x_0, y_0), the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.
2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.
3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.
6. Evaluate the integral $\displaystyle\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x, y) \geq 2$ for all (x, y), is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\dfrac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.