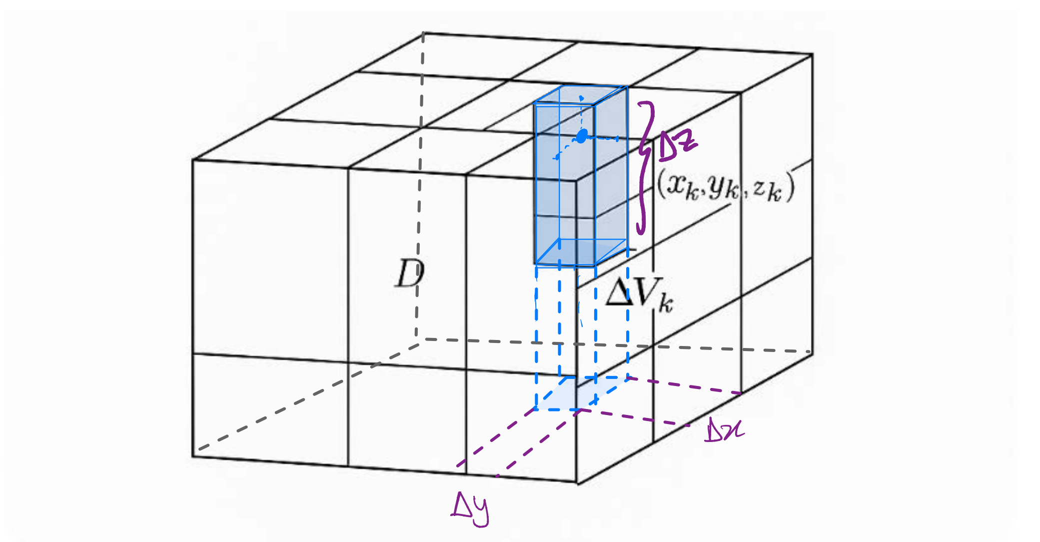
§15.5-15.6 Triple Integrals & Applications ??

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.



$\Delta V_k = \Delta x \Delta y \Delta z$

Taking the limit gives a double integrals $\iint_R f(x,y)\,dA$

density function

triple integral : $\iiint_D f(x,y,z)\,dV$ = mass of D

vol. element

Important special case:

$$\iiint_D 1\,dV = \text{volume of } D$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

$$\int_a^b\int_c^d\int_e^g f(x,y,z)\,dz\,dy\,dx = \int_c^d\int_e^g\int_a^b f(x,y,z)\,dx\,dz\,dy$$

doesn't seem to appear in any sample exams?

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta\,dV$ $\quad \delta = \delta(x,y,z)$ is the density at (x,y,z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\,dV, \quad M_{xz} = \iiint_D y\,\delta\,dV, \quad M_{xy} = \iiint_D z\,\delta\,dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta\,dA$ $\quad \delta = \delta(x,y)$ is the density at (x,y).

First moments: $M_y = \iint_R x\,\delta\,dA, \quad M_x = \iint_R y\,\delta\,dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

§15.6

Example 102. 1. **How to do the computation:**

Compute $\displaystyle\int_0^1\int_0^{2-x}\boxed{\int_0^{2-x-y} 1\, dz}\, dy\, dx. = \int_0^1\int_0^{2-x} \boxed{z\Big|_0^{2-x-y}}\, dy\, dx$

Start inside & work towards outside.

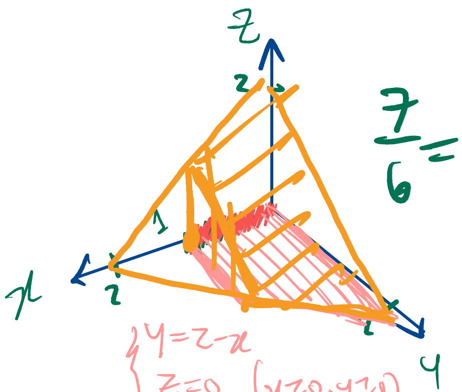
$$= \int_0^1 \boxed{\int_0^{2-x} 2-x-y\; dy}\, dx = \int_0^1 \boxed{2y - xy - \tfrac{1}{2}y^2\Big|_0^{2-x}}\, dx$$

$$= \int_0^1 2(2-x) - x(2-x) - \tfrac{1}{2}(2-x)^2\, dx = \int_0^1 4 - 2x - 2x + x^2 - \tfrac{1}{2}(4 - 4x + x^2)\, dx$$

$$= \int_0^1 4 - 4x + x^2 - 2 + 2x - \tfrac{1}{2}x^2\, dx = \int_0^1 2 - 2x + \tfrac{1}{2}x^2\, dx = 2x - x^2 + \tfrac{1}{6}x^3\Big|_0^1$$

$$= \left(2 - 1 + \tfrac{1}{6}\right) - 0 = 1 + \tfrac{1}{6} = \boxed{7/6}$$

2. **What does it mean:** What shape is this the volume of?



$$\frac{7}{6} = \int_0^1\int_0^{2-x}\int_0^{2-x-y} 1\, dz\, dy\, dx$$

Start w/ $x \in [0,1)$

Then $y \in [0, 2-x]$

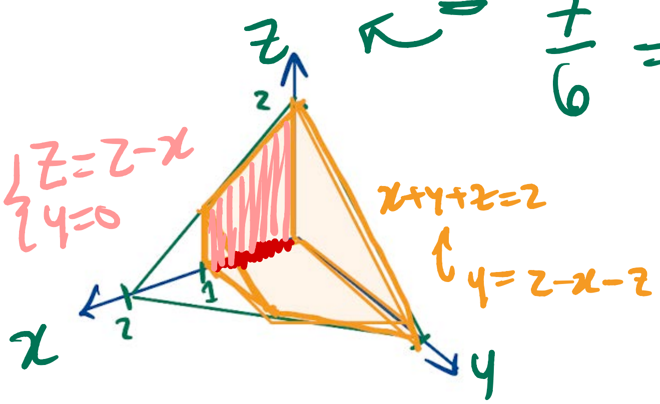
draw $y = 2-x$ in xy-plane (w/ $z=0$)

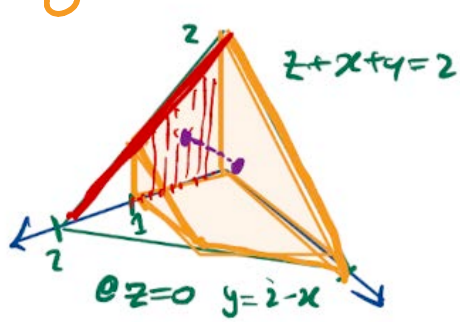
next $z \in [0, 2-x-y]$

$z = 2-x-y$

3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\frac{7}{6} = \int_0^1\int_0^{2-x}\int_0^{2-x-z} 1\; dy\, dz\, dx$$





Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\, dx\, dy\, dz$

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

$$\frac{1}{yz} * \frac{1}{x} = \frac{1}{xyz} \checkmark$$

$$\ln(e^3) = 3$$
$$\ln(1) = 0$$

(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \boxed{\frac{1}{xyz}}\, dx\; dy\; dz$

$$= \int_1^e \int_1^{e^2} \boxed{\frac{1}{yz}\ln(x)}\Big|_1^{e^3} dy\, dz = \int_1^e \int_1^{e^2} \boxed{\frac{1}{yz}} * 3 \; dy\, dz$$

$$= \int_1^e \frac{3}{z} \boxed{\ln y}\Big|_1^{e^2} dz = \int_1^e \frac{3}{z}(2-0)\, dz = 6 \ln z\Big|_1^e = 6-0$$

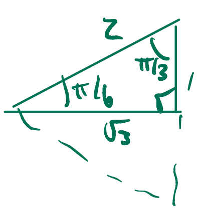
$$= \boxed{6}$$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z \; dx\; dy\; dz$

$$= \int_0^{\pi/3} \int_0^1 y \sin z \cdot x\Big|_{-2}^3 dy\, dz = \int_0^{\pi/3} \int_0^1 5y \sin z \, dy\, dz$$

$$= \int_0^{\pi/3} \frac{5}{2} \sin z \cdot y^2\Big|_0^1 dz = \int_0^{\pi/3} \frac{5}{2} \sin z \, dz$$

$$= -\frac{5}{2}\cos z\Big|_0^{\pi/3} = -\frac{5}{2}\left[\cos(\pi/3) - \cos(0)\right]$$



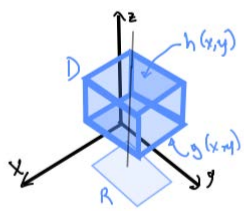
$$= -\frac{5}{2}\left(\frac{1}{2} - 1\right)$$

$$= \boxed{5/4}$$

We will think about converting triple integrals to iterated integrals in terms of the ________________ of D on one of the coordinate planes.

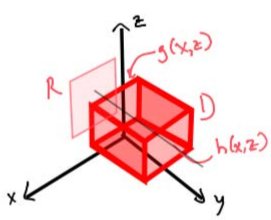
Case 1: z**-simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$



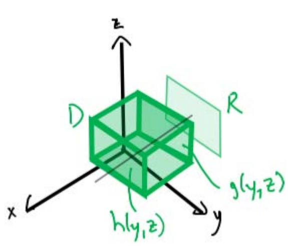
Case 2: y**-simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\ dy \right)\ dz\ dx$$



Case 3: x**-simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



$\boxed{2y \le 3 - x^2 - y^2}$ & $z=0$ & $x \ge 0, y \ge 0$

set $z=0$.

mass $= \iiint_D \delta(x,y,z)\, dV$

Example 104. Write an integral for the mass of the solid D in the first octant with $\boxed{2y \le z \le 3 - x^2 - y^2}$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

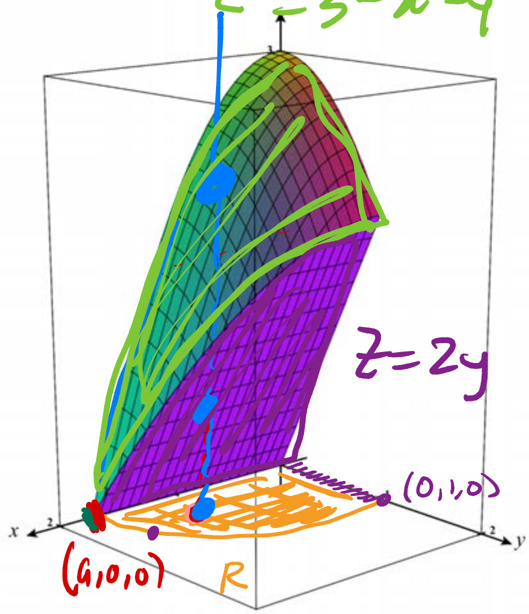
Case 1:

z-Simple.

$$\iint_R \int_{\boxed{g(x,y)}}^{\boxed{h(x,y)}} \delta(x,y,z)\, dz\, dy\, dx = M$$

top cap

bot cap



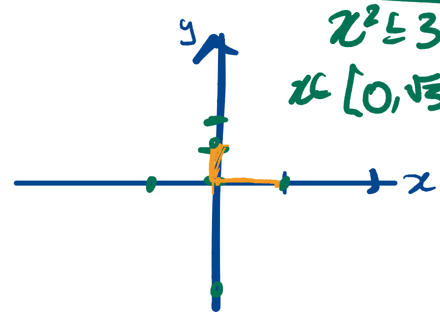
$$\text{mass} = \int_0^{\sqrt{3}} \int \int_{2y}^{3-x^2-y^2} x^2y + 0.1 \; dz\, dy\, \cancel{dx}$$

$z=0,\ y=0$

$\boxed{2(0) \le 0 \le 3 - x^2 - 0}$

$x^2 \le 3$.

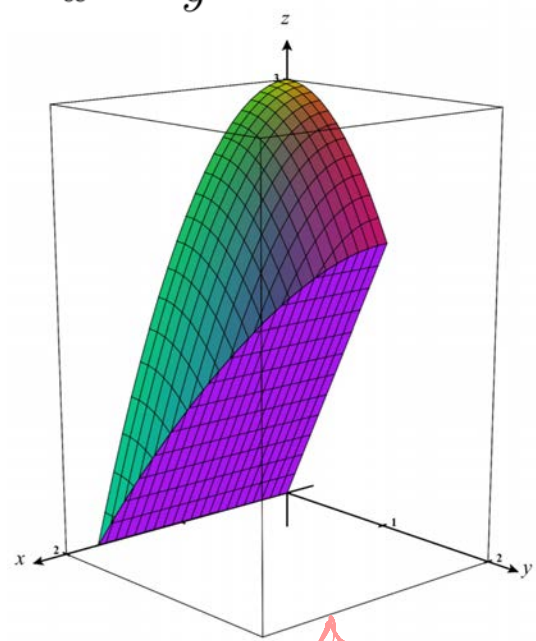
$x \in [0, \sqrt{3}]$

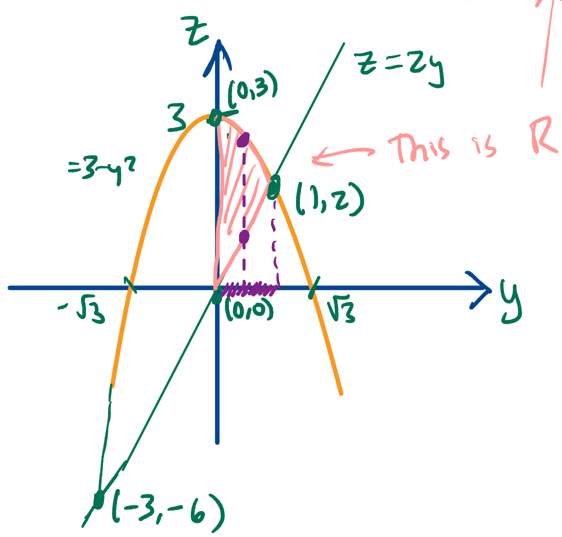


Example 104 (cont.) D: $2y \leq z \leq 3 - x^2 - y^2$

Case 3:
(b) x-simple

$$M = \iint_R \int_{g(x,y)}^{h(x,y)} \delta(x,y,z)\, dx\, dz\, dy$$





-z

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

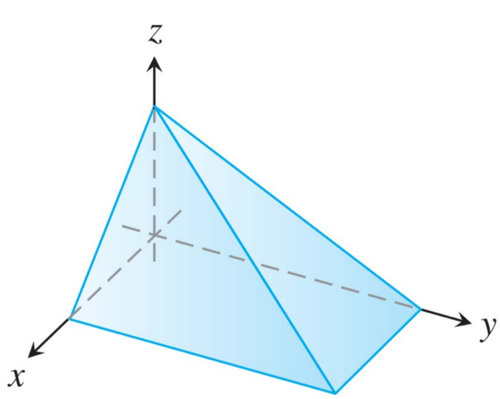
Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

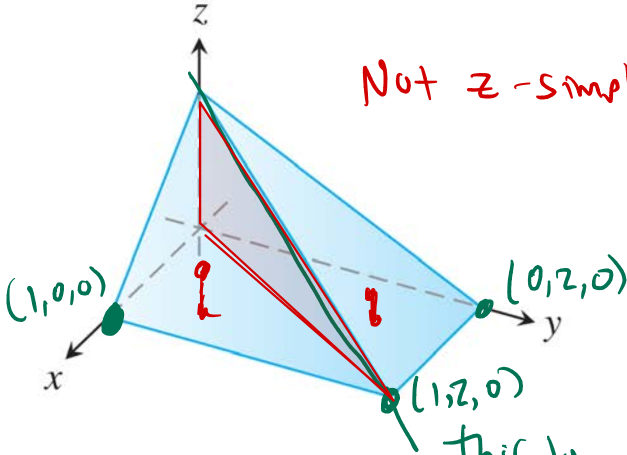
Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.

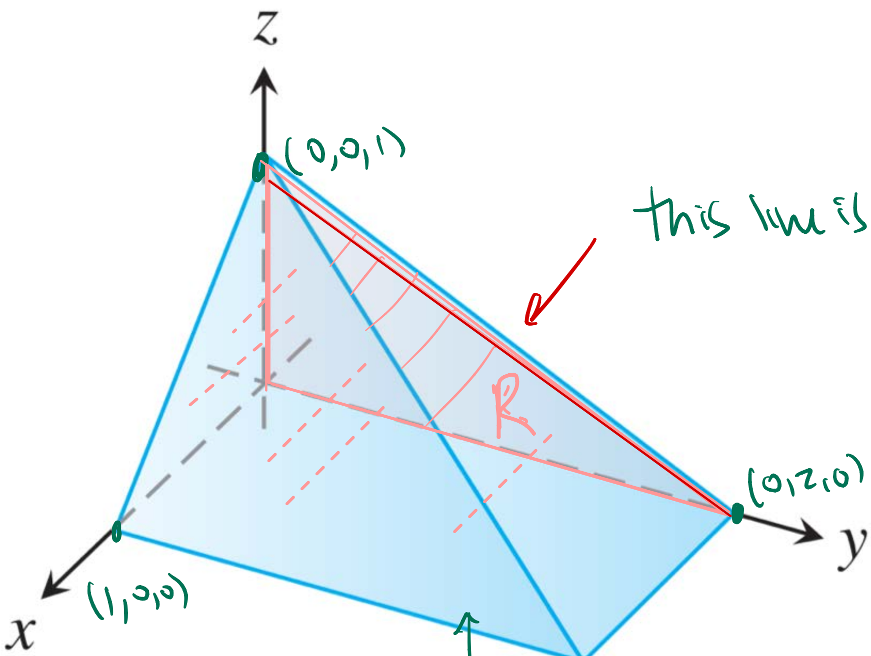


24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$ quadrant

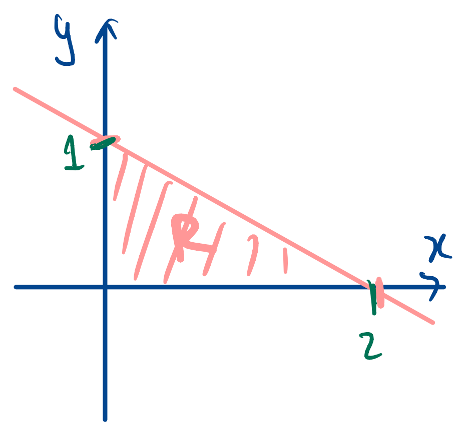
Not z-simple!



This line $\begin{cases} x+z=1 \\ y+2z=2 \end{cases}$ @ $z=0$: $x=1$, $y=2$ @ $x=y=0$: $z=1$



this line is $\begin{cases} x=0 \\ y+2z=2 \end{cases} \Rightarrow \begin{cases} x=0 \\ z=1-\frac{1}{2}y \end{cases}$



this plane is $x+z=1$
$\Leftrightarrow x=1-z$

So

$$\boxed{\text{Vol} = \int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{1-z} 1 \, dx \, dz \, dy}$$