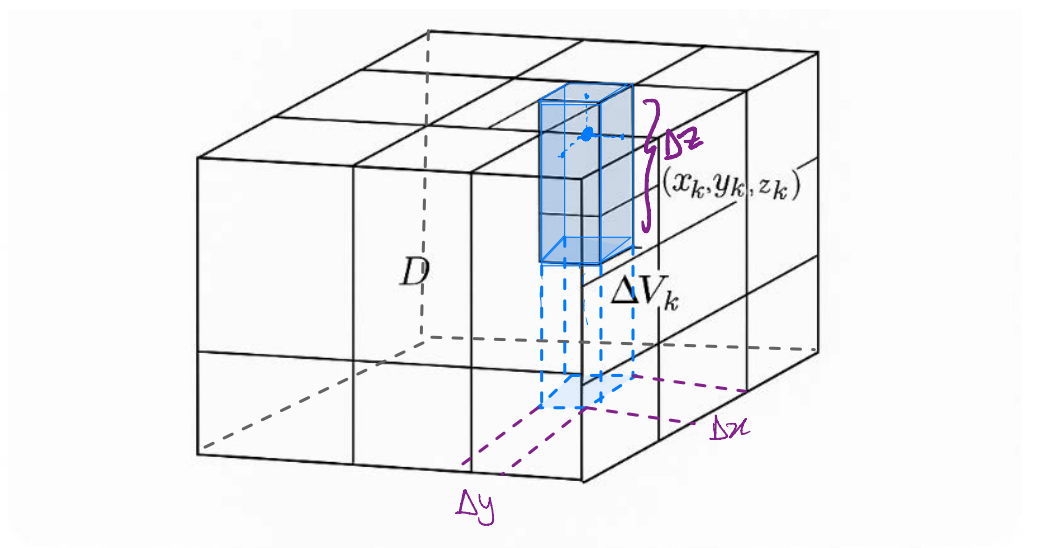


§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D , e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^n f(x_k, y_k, z_k) \Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k .



$$\Delta V_k = \Delta x \Delta y \Delta z$$

Taking the limit gives a

$$\text{_____} : \iiint_D f(x, y, z) \, dV$$

Important special case:

$$\iiint_D 1 \, dV = \text{_____}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

$$\int_a^b \int_c^d \int_e^f f(x, y, z) \, dz \, dy \, dx = \int_x^* \int_x^* \int_x^* f(x, y, z) \, d\cancel{x} \, d\cancel{y} \, d\cancel{z}$$

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\delta = \delta(x, y, z)$ is the density at (x, y, z) .

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \delta \, dV, \quad M_{xz} = \iiint_D y \delta \, dV, \quad M_{xy} = \iiint_D z \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\delta = \delta(x, y)$ is the density at (x, y) .

First moments: $M_y = \iint_R x \delta \, dA, \quad M_x = \iint_R y \delta \, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

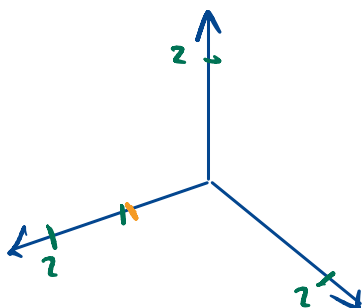
doesn't seem to appear in any sample exams?



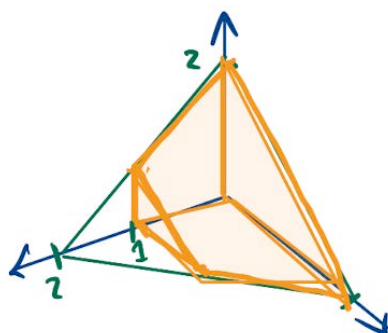
Example 102. 1. **How to do the computation:**

Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz \, dy \, dx.$

2. **What does it mean:** What shape is this the volume of?



3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy \, dz \, dx.$



Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a)
$$\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} \, dx \, dy \, dz$$

(b)
$$\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z \, dx \, dy \, dz$$

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} dx dy dz$

$$= \int_1^e \int_1^{e^2} \frac{1}{yz} \ln(x) \Big|_1^{e^3} dy dz = \int_1^e \int_1^{e^2} \frac{1}{yz} * 3 dy dz$$

$$= \int_1^e \frac{3}{z} \ln y \Big|_1^{e^2} dz = \int_1^e \frac{3}{z} (2-0) dz = 6 \ln z \Big|_1^e = 6-0 = 6$$

(b) $\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z dx dy dz$

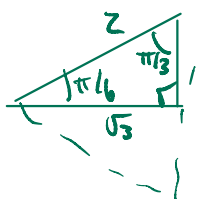
$$= \int_0^{\pi/3} \int_0^1 y \sin z \cdot x \Big|_{-2}^3 dy dz = \int_0^{\pi/3} \int_0^1 5y \sin z dy dz$$

$$= \int_0^{\pi/3} \frac{5}{2} \sin z \cdot y^2 \Big|_0^1 dz = \int_0^{\pi/3} \frac{5}{2} \sin z dz$$

$$= -\frac{5}{2} \cos z \Big|_0^{\pi/3} = -\frac{5}{2} [\cos(\pi/3) - \cos(0)]$$

$$= -\frac{5}{2} \left(\frac{1}{2} - 1 \right)$$

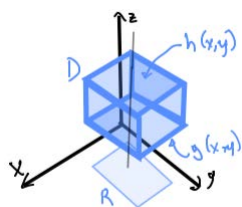
$$= \frac{5}{4}$$



We will think about converting triple integrals to iterated integrals in terms of the _____ of D on one of the coordinate planes.

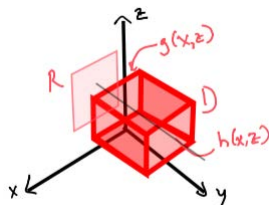
Case 1: **z -simple**) region. If R is the projection of D on the xy -plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x, y, z) \, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z) \, dz \right) dy \, dx$$



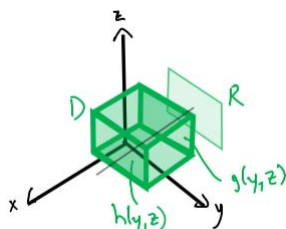
Case 2: **y -simple**) region. If R is the projection of D on the xz -plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x, y, z) \, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x, y, z) \, dy \right) dz \, dx$$



Case 3: **x -simple**) region. If R is the projection of D on the yz -plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x, y, z) \, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x, y, z) \, dx \right) dz \, dy$$



$z \in [3-x^2-y^2, 2y]$ & $z=0$ & $x \in [0, \sqrt{3}]$ & $y \in [0, 1]$

$$\text{mass} = \iiint_D \delta(x, y, z) dV$$

Example 104. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z -simple and b) x -simple. Is the solid also y -simple?

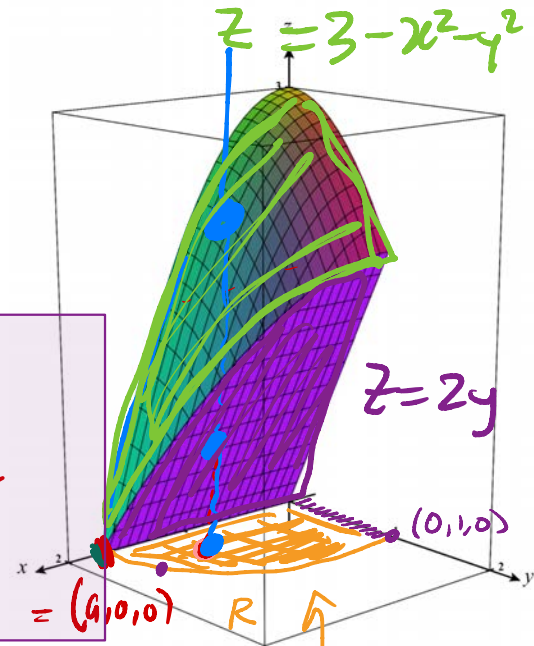
Case 1:

z -simple

$$\iint_R \int_{g(x,y)}^{h(x,y)} \delta(x, y, z) dz dy dx = M$$

top cap
bottom cap

$$\text{mass} = \int_0^{\sqrt{3}} \int_0^{\sqrt{4-x^2}-1} \int_{2y}^{3-x^2-y^2} (x^2y + 0.1) dz dy dx$$



Find what x - y pairs (w/ $x \in [0, \sqrt{3}]$) have a corresponding z -value in the range $2y \leq z \leq 3 - x^2 - y^2$

top $z = 3 - x^2 - y^2$ bot $z = 2y$

set equal & find relationship between x & y .

Solve for y

$$3 - x^2 - y^2 = 2y \Rightarrow 3 = x^2 + y^2 + 2y$$

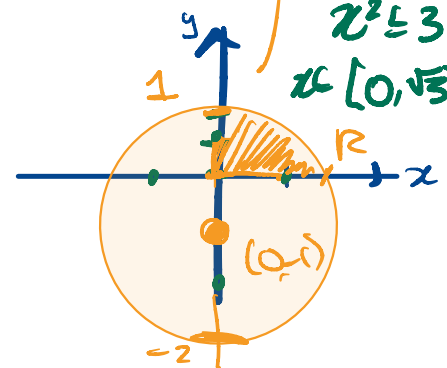
$$\Rightarrow 4 = x^2 + y^2 + 2y + 1$$

$$\Rightarrow 4 = x^2 + (y+1)^2$$

$$\Rightarrow 4 - x^2 = (y+1)^2$$

$$\Rightarrow \sqrt{4 - x^2} = y + 1 \Rightarrow y = \sqrt{4 - x^2} - 1$$

offset circle of radius $r=2$ center @ $(0, -1)$



Example 104 (cont.)

$$D: 2y \leq z \leq 3 - x^2 - y^2$$

Case 3:

(b) x -simple

$$M = \iint_R \int_{g(x,y)}^{h(x,y)} \delta(x,y,z) dz dy$$

pink region R defined by $y \in [0, a]$ intersection of $z = 2y$ and $z = 3 - y^2$ is @ $y = a$

$$\Rightarrow (\text{set } z=z) \quad 2y = 3 - y^2$$

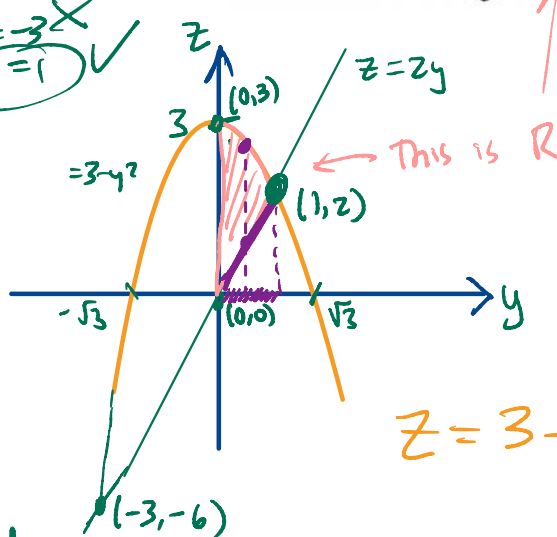
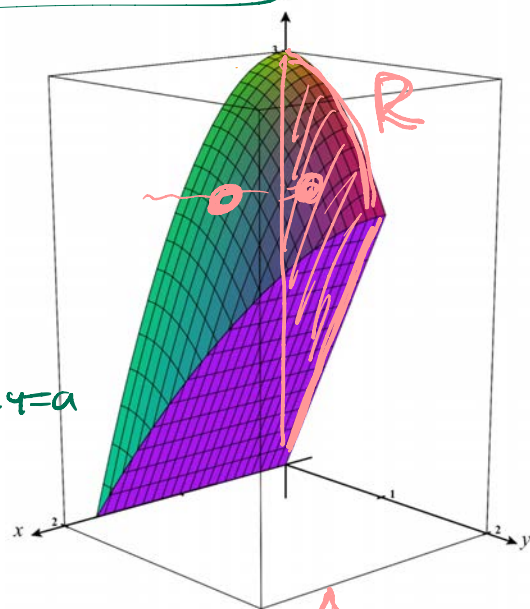
$$\Rightarrow y^2 + 2y - 3 = 0$$

$$\Rightarrow (y+3)(y-1) = 0 \rightarrow y = -3 \text{ or } y = 1$$

$$y \in [0, 1]$$

$$z \in [2y, 3 - y^2]$$

$$\text{mass} = \int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-z-y^2}} x^2 + 0.1 dx dz dy$$

Set @ $x=0$
but cap
 $z = 2y$ Set @ $x=0$
(@ $x=0$)
 $z = 3 - y^2$ on top cap.top cap in terms
of x is

$$z = 3 - x^2 - y^2$$

$$x^2 = 3 - z - y^2$$

$$\Rightarrow x = \sqrt{3 - z - y^2}$$

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

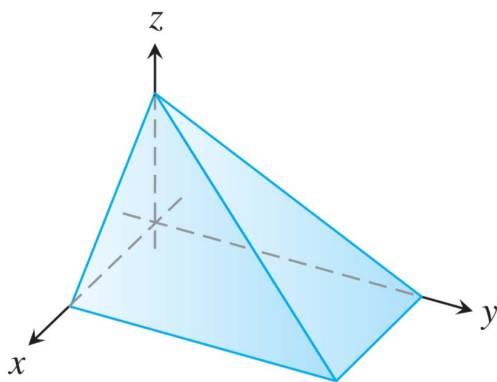
Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

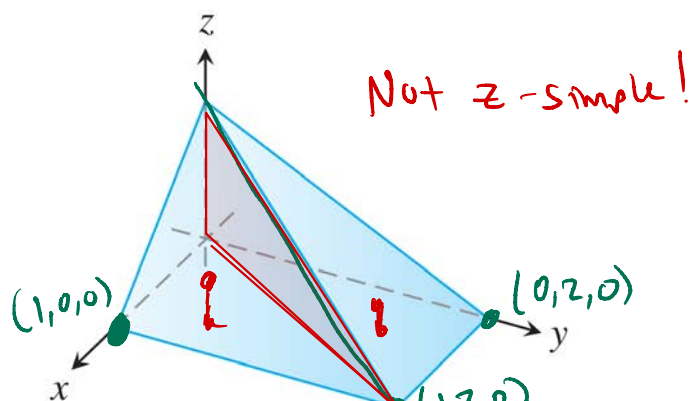
Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

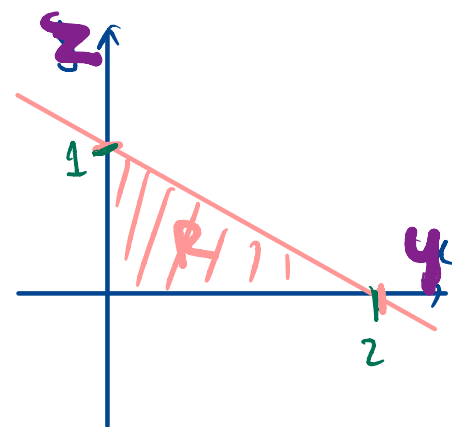
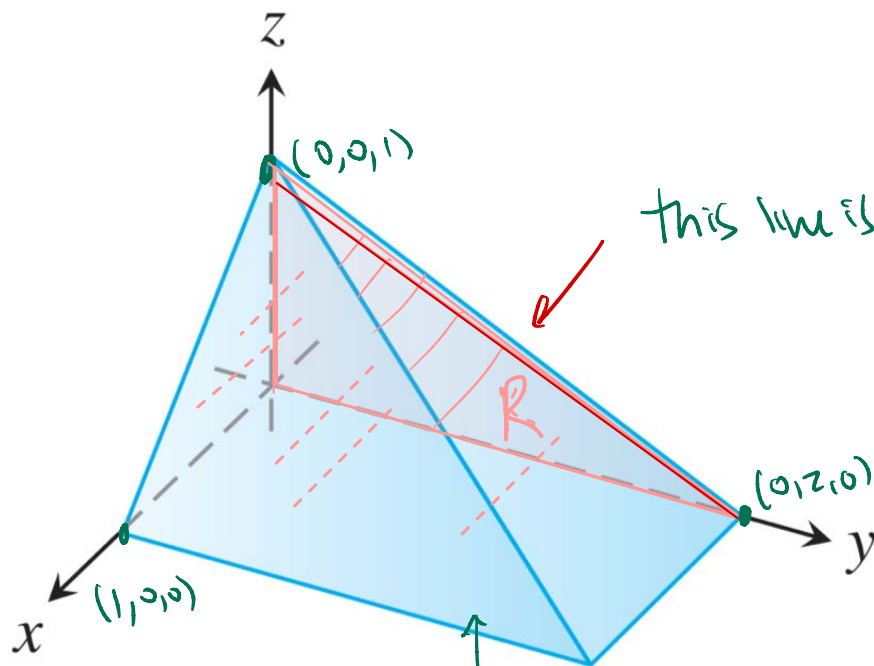
Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.



24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$ quadrant



this line $\begin{cases} x+z=1 \\ y+2z=2 \end{cases}$ $\begin{matrix} @ z=0 & @ x=y=0 \\ x=1 & z=1 \\ y=2 \end{matrix}$

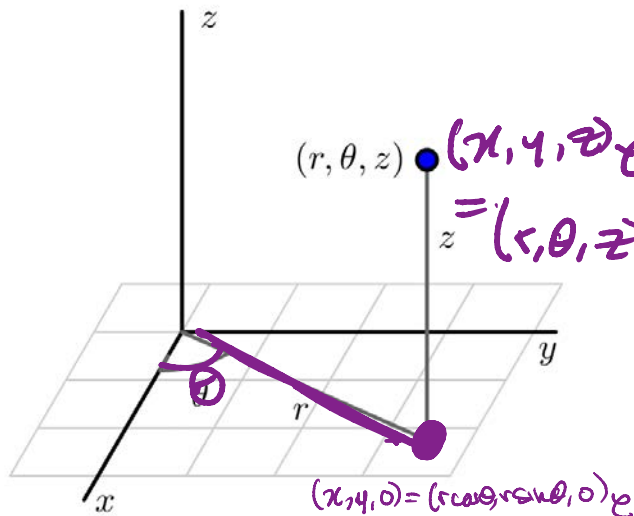


So

$$V = \int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{1-z} 1 \, dx \, dz \, dy$$

§15.7 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinate System

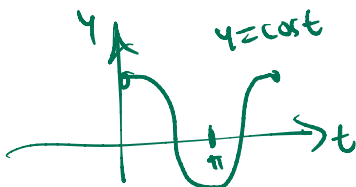


Cylindrical to Cartesian: $(r, \theta, z) \rightarrow (x, y, z)$

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

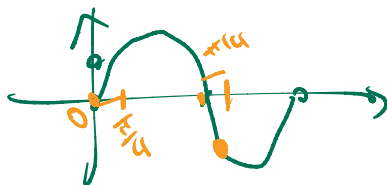
$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$



$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$



$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$



Conventions: As before for double-integrals: $r \geq 0$ and

$$\theta \in [0, 2\pi] \text{ or } \theta \in [-\pi, \pi]$$

Example 108. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$. $= (r, \theta, z)$

want $(r, \theta, z) = ? = (2, \frac{2\pi}{3}, 3)$



$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \pi - \pi/3 = 2\pi/3$$

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

$$x = 2 * \cos\left(\frac{5\pi}{4}\right) = 2 * -\frac{\sqrt{2}}{2} = -\sqrt{2}$$

$$y = 2 * \sin\left(\frac{5\pi}{4}\right) = 2 * -\frac{\sqrt{2}}{2} = -\sqrt{2}$$

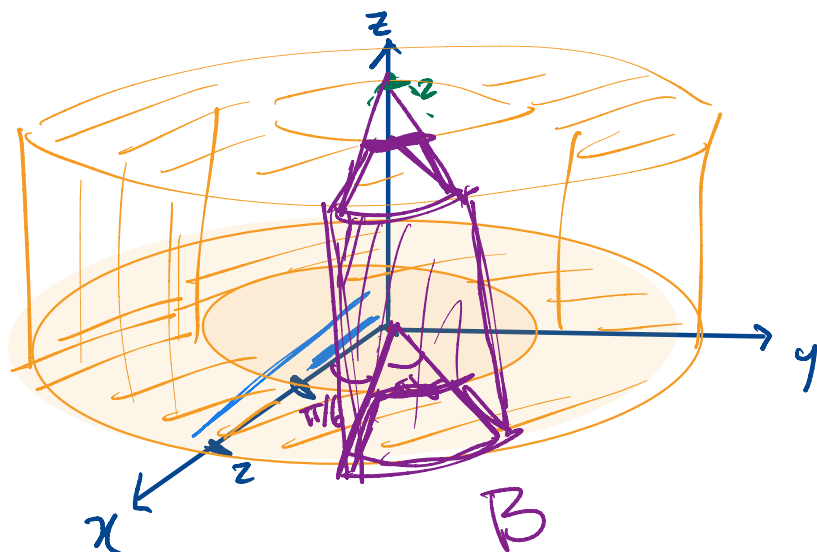
$$(-\sqrt{2}, -\sqrt{2}, 1)$$

Ex. Find the mass of region

occupying B w/ $\delta(x, y, z) = xy$

Example 109. In xyz -space sketch the cylindrical box

$$B = \{(r, \theta, z) \mid 1 \leq r \leq 2, \pi/6 \leq \theta \leq \pi/3, 0 \leq z \leq 2\}.$$



$$\int_0^2 \int_{\pi/6}^{\pi/3} \int_1^2 (r \cos \theta)(r \sin \theta) r \, dz \, d\theta \, dr$$

$\delta(x, y, z)$
change
into
cylindrical
 dV ?

use
cylindrical
volume
measure.

Triple Integrals in Cylindrical Coordinates

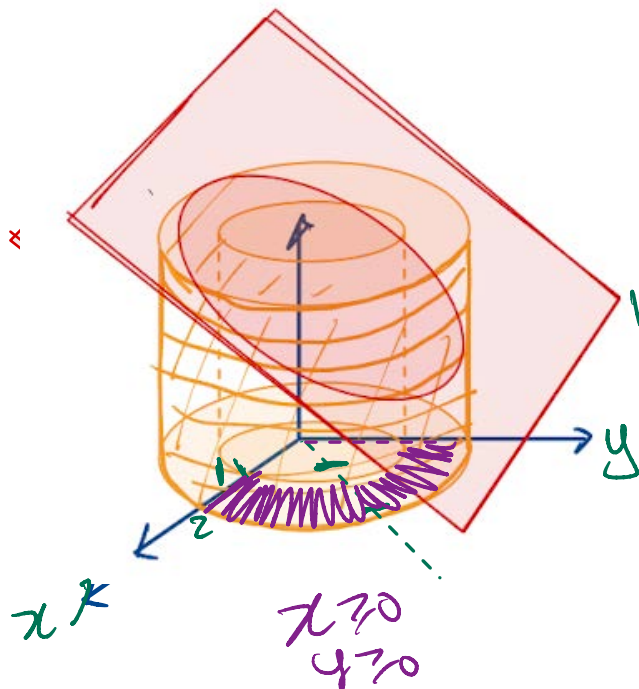
We have $dV = r \, dz \, d\theta \, dr$

Example 110. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = \sqrt{x^2 + y^2} + 2$, above the xy -plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$r \in [1, 2]$$

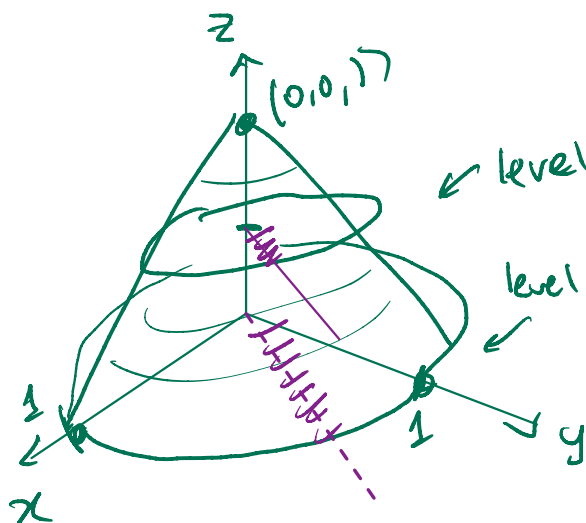
$$\theta \in [0, 2\pi]$$

$$z \in [0, \sqrt{x^2 + y^2} + 2] \quad ??$$



$$\text{Vol} = \int_1^2 \int_0^{2\pi} \int_0^{\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} + 2} 1 \cdot r \, dz \, d\theta \, dr$$

Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.



$$\theta \in [0, 2\pi]$$

$$r \in [0, 1 - z]$$

$$z \in [0, 1]$$

$$\text{mass} = \int_0^{2\pi} \int_0^{1-z} \int_0^1 z \, r \, dz \, dr \, d\theta$$

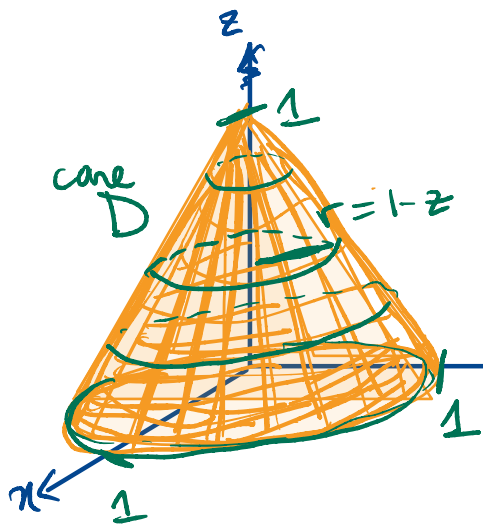
???

$$\text{mass} = \int_0^{2\pi} \int_0^1 \int_0^{1-z} z \, r \, dr \, dz \, d\theta$$

???

$$\sqrt{x^2 + y^2} = 1 - z$$

Example 111. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.



$$\theta \in [0, 2\pi]$$

$$z \in [0, 1]$$

$$r \in [0, 1 - z]$$

$$\text{so } M = \iiint_D \delta(r, \theta, z) r \, dr \, d\theta \, dz$$

$$M = \int_0^{2\pi} \int_0^1 \int_0^{1-z} z r \, dr \, d\theta \, dz$$

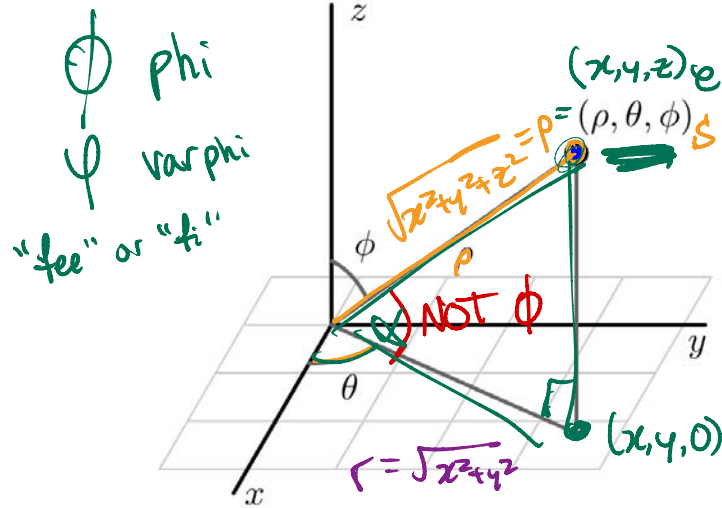
Other options also work

$$M = \int_0^{2\pi} \int_0^1 \int_0^{1-r} z r \, dz \, dr \, d\theta$$

or even

$$M = \int_0^1 \int_0^{1-r} \int_0^{2\pi} z r \, d\theta \, dz \, dr \quad \text{also fine!}$$

Spherical Coordinate System



Conventions:

$$\rho \geq 0$$

$$\theta \in [0, 2\pi) \text{ or } (-\pi, \pi]$$

$$\phi \in [0, \pi]$$

Example 112. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

$$\rho = \sqrt{(-2)^2 + 2^2 + (\sqrt{8})^2} = \sqrt{4 + 4 + 8} = \sqrt{16} = 4 \checkmark$$

$$\theta = 3\pi/4$$

$$\phi = \pi/4.$$

Spherical to Cartesian:

$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\phi) = \frac{\sqrt{x^2 + y^2}}{z}$$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

$$\uparrow \rho \quad \uparrow \theta \quad \uparrow \phi$$

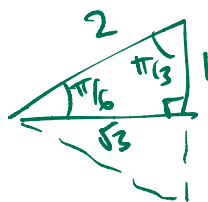
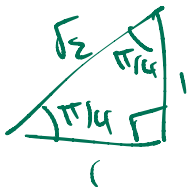
$$(2, \pi/2, \pi/3) \rightarrow (0, 1)$$

$$x = 2 \sin(\pi/3) \cos(\pi/2) = 2 * \frac{\sqrt{3}}{2} * 0$$

$$y = 2 \sin(\pi/3) \sin(\pi/2) = 2 * \frac{\sqrt{3}}{2} * 1$$

$$z = 2 \cos(\pi/3) = 2 * \frac{1}{2} = 1$$

$$(2, \pi/2, \pi/3) = (0, \sqrt{3}, 1)$$



🌀 The two common lowercase phi symbols:

ϕ (curly phi) — sometimes called “script phi” or “open phi”

Looks like a curly or loopy “C” with a vertical line

Unicode: U+03D5

Often used in physics and engineering (e.g., magnetic flux)

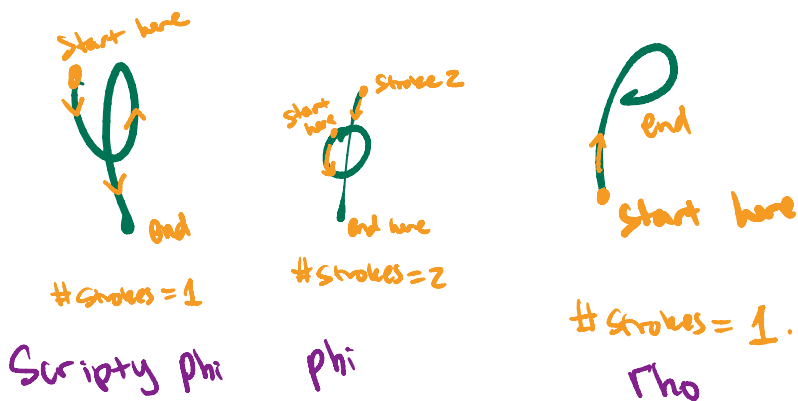
φ (straight phi) — often just “phi”

Looks like a circle with a vertical line through it

Unicode: U+03C6

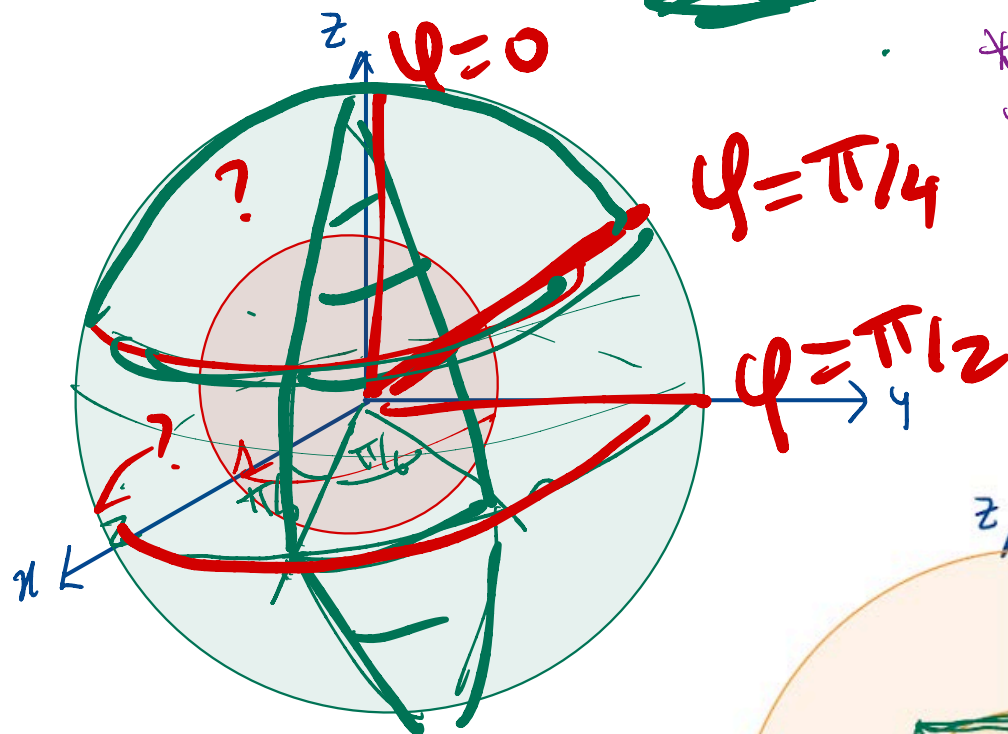
Often used in math, philosophy, and logic

How to draw them:



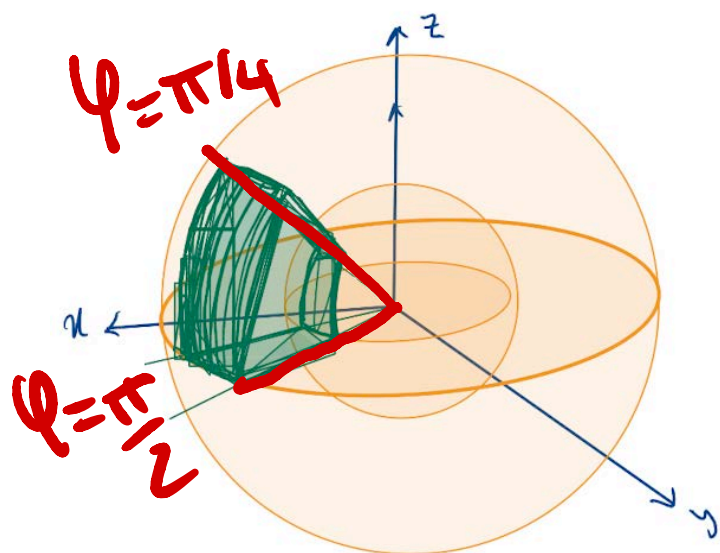
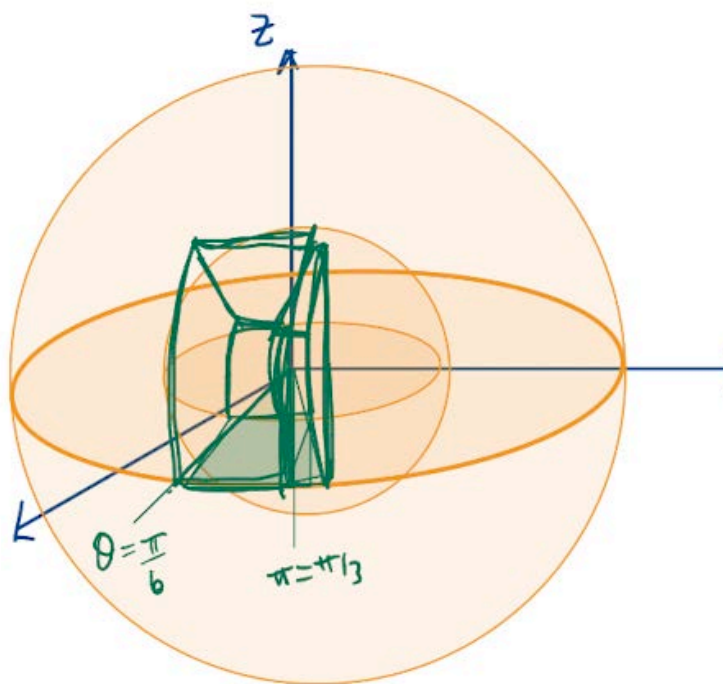
Example 113. In xyz -space sketch the spherical box

$$B = \{(\rho, \varphi, \theta) \mid 1 \leq \rho \leq 2, \pi/4 \leq \varphi \leq \pi/2, \pi/6 \leq \theta \leq \pi/3\}.$$



* According to Dr. H.

"Spheres, curves, and planes through the z -axis are fair game."

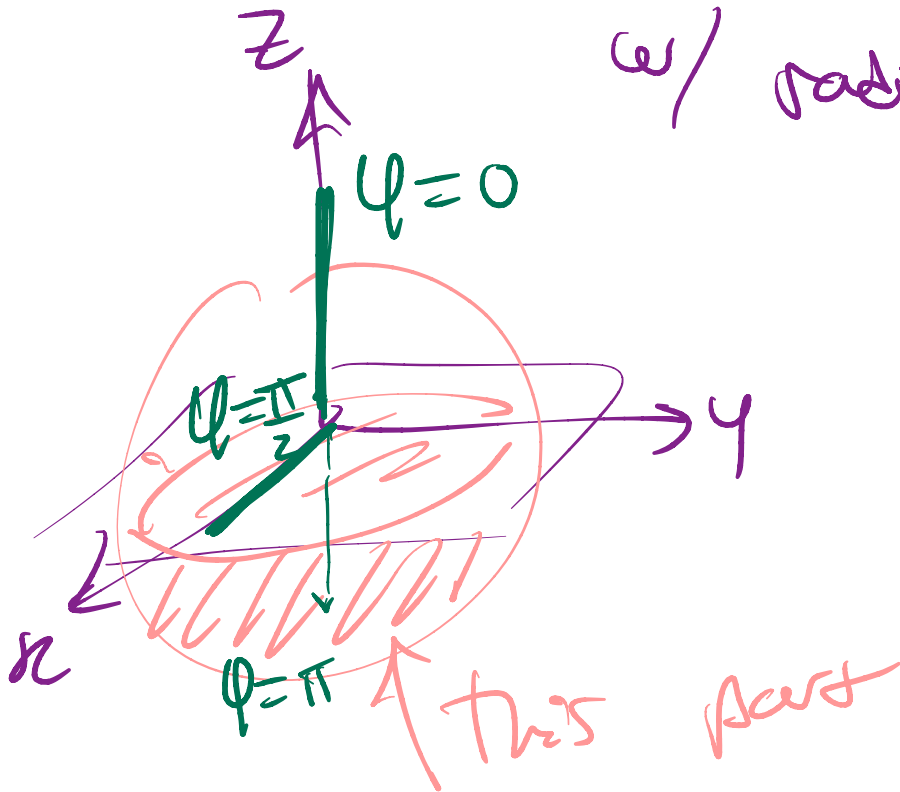


Set up triple for ^{mass} ~~volume~~ of lower half sphere

w/ radius $\rho = 3$.

a) density function

$$\delta = -z.$$



$$z = \rho \cos \varphi$$

$$\varphi \in [\pi/2, \pi]$$

$$\theta \in [0, 2\pi]$$

$$\rho \in [0, 3]$$

~~$$\text{mass} = \int_0^3 \int_0^{2\pi} \int_{\pi/2}^{\pi} \rho \cos \varphi \, d\varphi d\theta d\rho$$~~

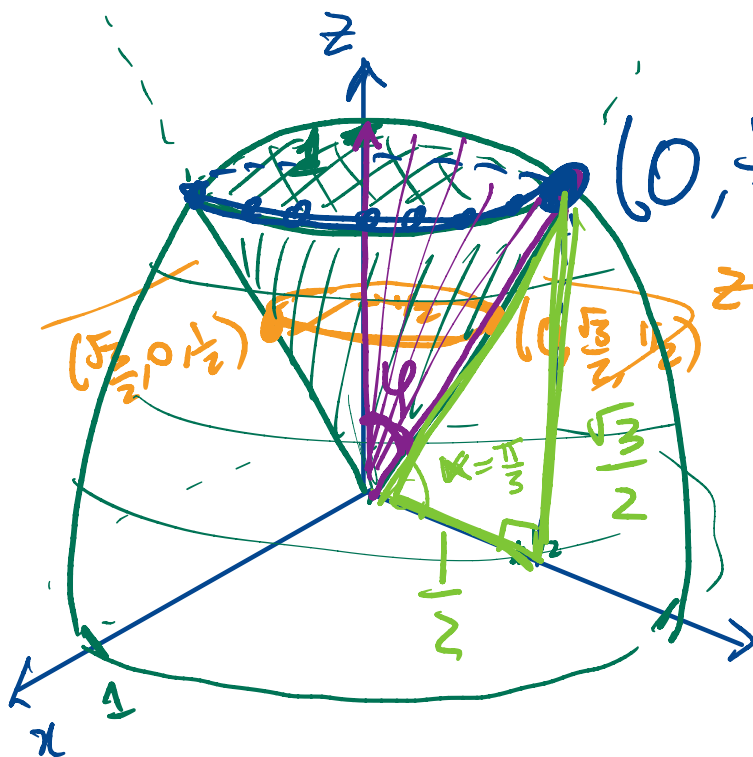
$$\rho = \sin \varphi$$

$$\text{mass} = \int_0^3 \int_0^{2\pi} \int_{\pi/2}^{\pi} \rho^3 \cos \varphi \sin \varphi \, d\varphi d\theta d\rho$$

Triple Integrals in Spherical Coordinates

We have $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

Example 114. Write an iterated integral for the volume of the “ice cream cone” D bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$.



$$(0, \frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$z = 1/2$$

$$\rho = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$$\frac{1}{2} = \sqrt{3} \sqrt{x^2 + y^2}$$

$$\frac{\sqrt{3}}{2} = \sqrt{x^2 + y^2}$$

$$\Rightarrow \frac{3}{4} = x^2 + y^2$$

circle of radius $\sqrt{3}/2 = r$

$$\rho \in [0, 1]$$

$$\theta \in [0, 2\pi]$$

$$\phi \in [0, \pi/6]$$

$$\phi + \alpha = \frac{\pi}{2} \text{ and } \alpha = \frac{\pi}{3} \Rightarrow \phi = \frac{\pi}{6}$$

The intersection curve is all triples (x, y, z) s.t.

$$x^2 + y^2 + z^2 = 1 \text{ and } z = \sqrt{3} \sqrt{x^2 + y^2}$$

just substitute

$$x^2 + y^2 + (\sqrt{3} \sqrt{x^2 + y^2})^2 = 1 \Rightarrow x^2 + y^2 + 3(x^2 + y^2) = 1$$

$$\Rightarrow 4x^2 + 4y^2 = 1$$

$$\text{So } z = \sqrt{3} \times \frac{1}{2}$$

$$\Rightarrow x^2 + y^2 = 1/4 \text{ (circle of radius } \sqrt{1/4} = 1/2)$$

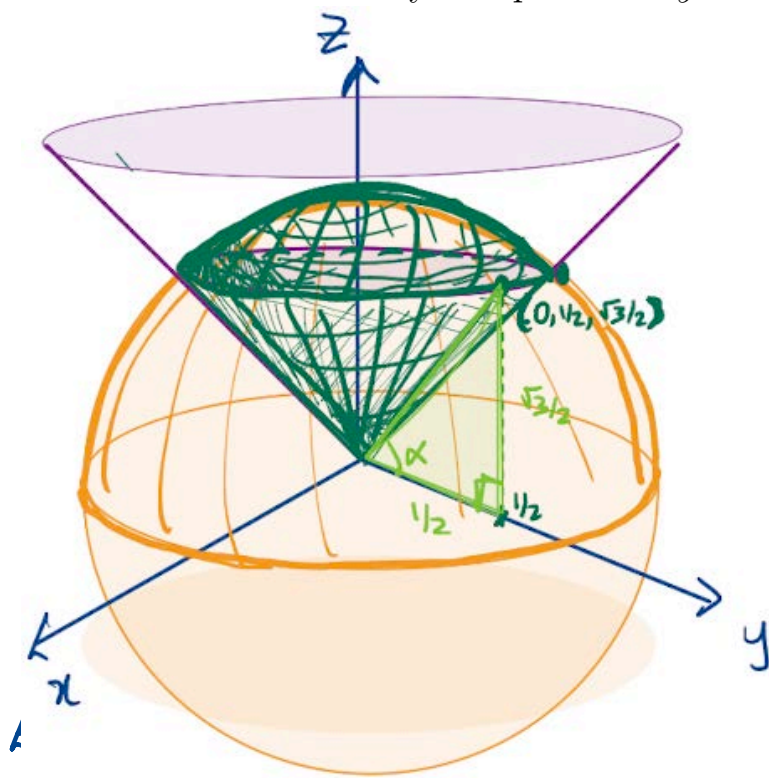
Triple Integrals in Spherical Coordinates

We have $dV =$

$$\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

don't
for set

Example 114. Write an iterated integral for the volume of the “ice cream cone” D bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$.



$$\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{2}$$

$$\rho \in [0, 1]$$

$$\theta \in (0, 2\pi)$$

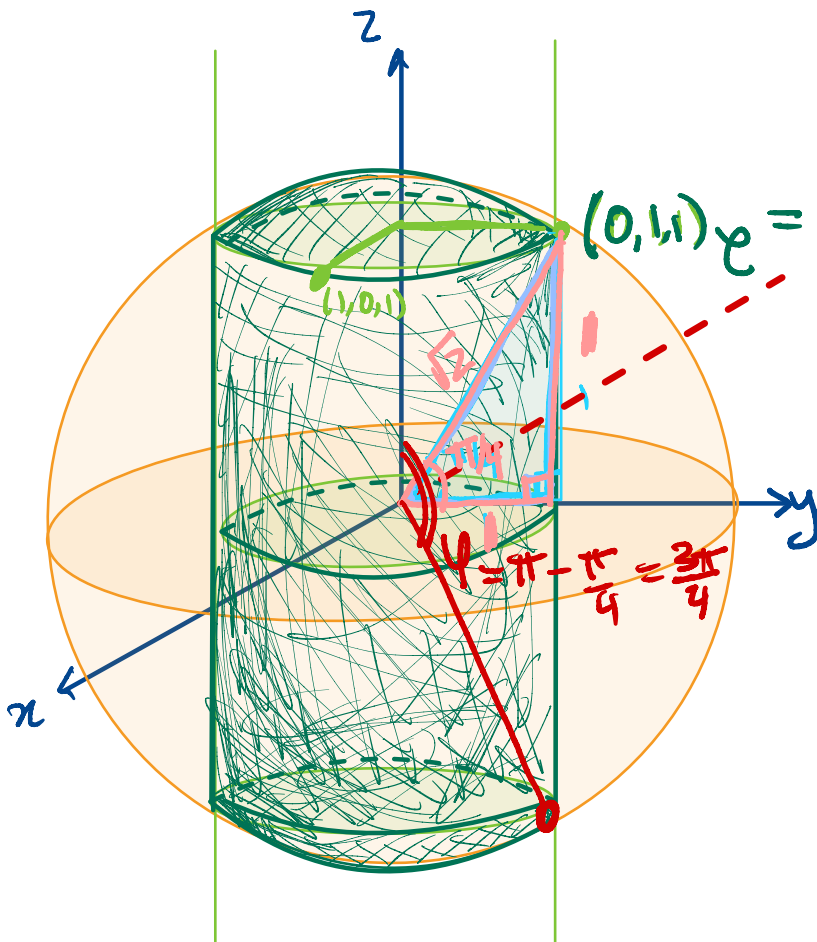
$$\phi \in [0, \pi/6]$$

So the triple integral is

???

$$V = \iiint_D 1 \, dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



$$\begin{array}{ccc} \rho & \theta & \psi \\ \downarrow & \downarrow & \downarrow \\ (0, 1, 1)_C = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)_S \end{array}$$

$$\rho = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$\theta = \pi/2$$

$$\psi = \pi/4$$

$$\alpha = \pi/4 \text{ so}$$

$$\psi = \pi/2 - \pi/4 = \pi/4$$

for solid D orange shell

$$\left\{ \begin{array}{l} \theta \in [0, 2\pi] \\ \psi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \\ \rho \in [\csc \psi, \sqrt{2}] \end{array} \right.$$

$$\psi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$\rho \in [\csc \psi, \sqrt{2}]$$

how to change x, y, z 's into ρ, ψ, θ 's

$$\begin{cases} x = \rho \sin \psi \cos \theta \\ y = \rho \sin \psi \sin \theta \\ z = \rho \cos \psi \end{cases}$$

plug in into cylinder eqn.

$$x^2 + y^2 = 1$$

$$\Rightarrow (\rho \sin \psi \cos \theta)^2 + (\rho \sin \psi \sin \theta)^2 = 1$$

$$\Rightarrow \rho^2 \sin^2 \psi \cos^2 \theta + \rho^2 \sin^2 \psi \sin^2 \theta = 1$$

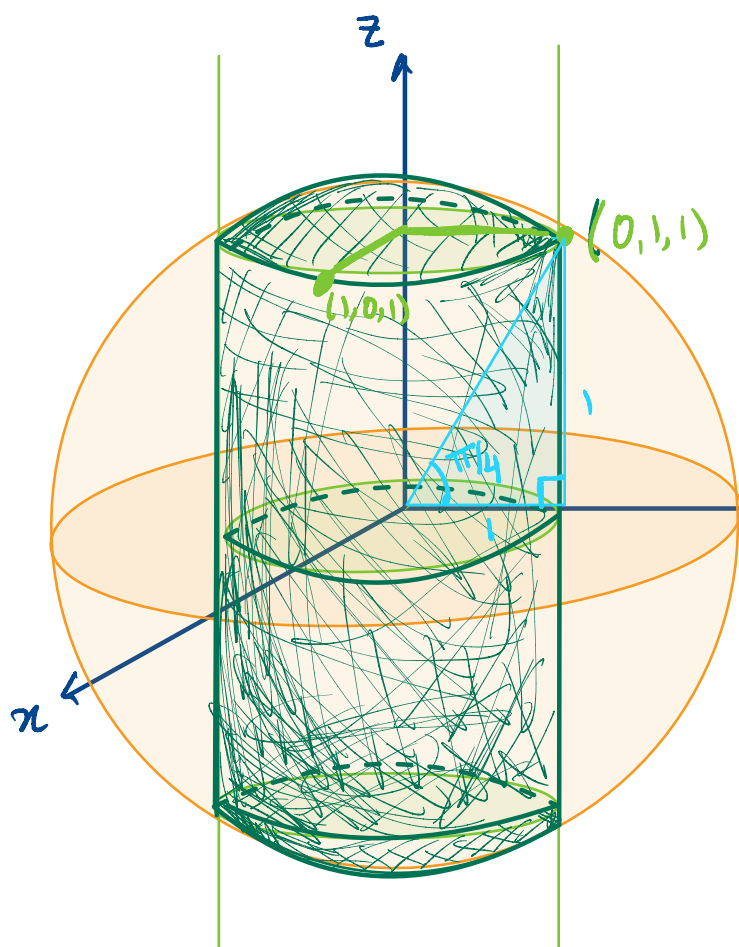
$$\Rightarrow \rho^2 \sin^2 \psi (\cos^2 \theta + \sin^2 \theta) = 1$$

$$\Rightarrow \rho^2 \sin^2 \psi = 1 \Rightarrow \rho^2 = \frac{1}{\sin^2 \psi} = \csc^2 \psi$$

$$\Rightarrow \rho = \csc \psi \quad (\rho \geq 0)$$

$$\uparrow \rho(\psi, \theta) = ?$$

Example 115. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.



Sphere:
 $x^2 + y^2 + z^2 = 2 \Leftrightarrow \rho = \sqrt{2}$

intersection
 $x^2 + y^2 + z^2 = 2 \quad \& \quad x^2 + y^2 = 1$

$$\Rightarrow 1 + z^2 = 2$$

$$\Rightarrow z^2 = 1 \Rightarrow z = \pm 1$$

One point in intersection
 is $(0, 1, 1)_{\mathcal{C}} = (1, \frac{\pi}{2}, \frac{\pi}{4})_{\mathcal{S}}$

Same idea as last example

$$x^2 + y^2 = \rho^2 \sin^2 \varphi \text{ so}$$

$$\text{cylinder is } \rho^2 \sin^2 \varphi = 1$$

$$\Rightarrow \rho \sin \varphi = 1$$

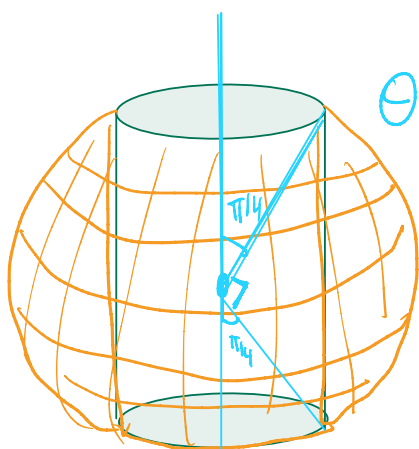
$$\Rightarrow \rho = \csc \varphi$$

we want outer part outside
 cylinder & inside sphere so

$$\varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4} \right]$$

$$\rho \in [\csc \varphi, \sqrt{2}]$$

$$\text{and } \theta \in [0, 2\pi)$$



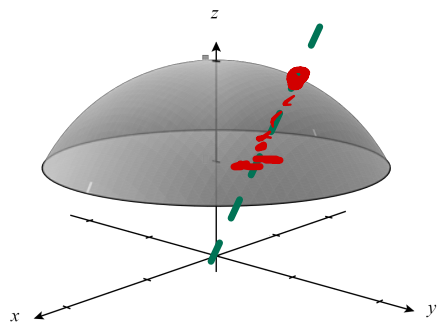
$$\theta = \left[\frac{\pi}{4}, \frac{3\pi}{4} \right]$$

$$Vol = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc \varphi}^{\sqrt{2}} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

5. (6 points) Let D be the smaller cap cut from a solid ball of radius 1 units by the plane $z = 1/2$. The point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$ is on D at the intersection of the ball and the plane.

(a) Find the ~~cylindrical~~ ^{spherical} coordinates (ρ, φ, θ) of the point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$. [AN]

(b) Express the volume of D as an iterated triple integral in spherical coordinates. *Do not evaluate!* [AN]



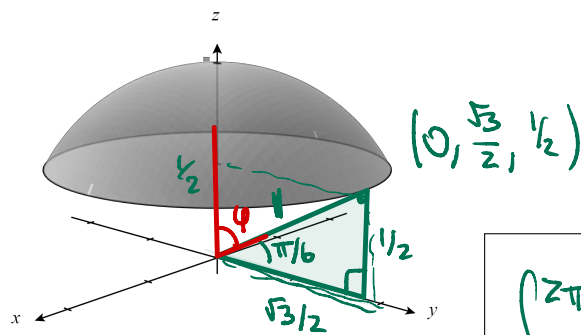
Volume=

$$(0, \frac{\sqrt{3}}{2}, \frac{1}{2})_{cyl} =$$

5. (6 points) Let D be the smaller cap cut from a solid ball of radius 1 units by the plane $z = 1/2$. The point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$ is on D at the intersection of the ball and the plane.

(a) Find the ~~cylindrical~~ ^{spherical} coordinates (ρ, φ, θ) of the point $P(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$. [AN]

(b) Express the volume of D as an iterated triple integral in spherical coordinates. *Do not evaluate!* [AN]



Volume =

$$(0, \frac{\sqrt{3}}{2}, \frac{1}{2}) \rightarrow (1, \pi/3, \pi/2)$$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{2}\sec\varphi}^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\varphi = \pi/2 - \pi/6 = \pi/3$$

$$\rho = \sqrt{0^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$$

$$\theta = \pi/2 \text{ (on y-axis)}$$

$$z = 1/2 = \rho \cos \varphi \Rightarrow \rho = \frac{1}{2 \cos \varphi} = \frac{1}{2} \sec \varphi$$

$$\text{so } D: \theta \in [0, 2\pi], \varphi \in [0, \pi/3], \rho \in [\frac{1}{2} \sec \varphi, 1]$$

§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx = A$

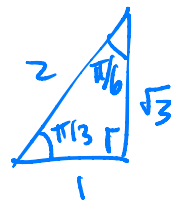
To motivate: maybe try $x = 2 \sin \theta$
 $dx = 2 \cos \theta d\theta$

Eg. $dA = dx dy = r dr d\theta$ so $A = \int_{\pi/4}^{\pi/3} \frac{1}{\sqrt{4-(2\sin\theta)^2}} \cdot 2 \cos \theta d\theta$
 converting double integral
 into polar coord.
 r "integration factor"

$$dV = dx dy dz$$

$$= r dr d\theta dz$$

$$= \rho^2 \sin \phi d\phi d\rho d\theta$$



When you convert
 triple integrals into
 cylindrical or spherical coords.

$$= \int_{\pi/4}^{\pi/3} \frac{2 \cos \theta}{\sqrt{4-4 \sin^2 \theta}} d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{2 \cos \theta}{2 \cos \theta} d\theta$$

$$= \int_{\pi/4}^{\pi/3} 1 d\theta$$

$$= \theta \Big|_{\pi/4}^{\pi/3}$$

$$= \sin^{-1}\left(\frac{x}{2}\right) \Big|_1^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\pi/6}$$

Theorem 116 (Substitution Theorem). Suppose $\mathbf{T}(u, v)$ is a one-to-one, differentiable transformation that maps the region G in the uv -plane to the region R in the xy -plane. Then

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(\mathbf{T}(u, v)) |\det(D\mathbf{T}(u, v))| \, du \, dv.$$

integration factor.

The "Jacobian determinant" measures how the $\Delta x \Delta y$ needs to get converted to $\Delta u \Delta v$.

e.g. if $|\det DT| = 2$ then $2 \times \text{Area } G = \text{Area } R$.

$$\underbrace{\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2} \, dx \, dy}_R = \underbrace{\int_0^2 \int_0^1 \boxed{?} \, du \, dv}_G$$

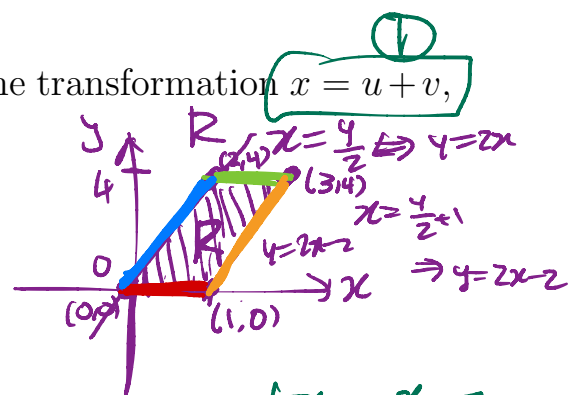
Example 117. Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2} \, dx \, dy$ via the transformation $x = u + v$,

$y = 2v$

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

1. Find T :

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$$



$$DT = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|\det DT| = 2.$$

What about the integration limits

@ $y=0 \Rightarrow 0=y=2v$ so $v=0$

@ $y=4 \Rightarrow 4=y=2v$ so $v=2$

So if $y \in [0, 4]$ then $v \in [0, 2]$

@ $x = \frac{y}{2} \Rightarrow \frac{y}{2} = x = u + v \Rightarrow u = \frac{y}{2} - v$

@ $x = \frac{y}{2} + 1 \Rightarrow \frac{y}{2} + 1 = u + v \Rightarrow u = \frac{y}{2} - v + 1$

NOTE: often you are given $T^{-1}(x, y)$

e.g. $T^{-1}(x, y) = \begin{bmatrix} (2x-y)/2 \\ y/2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

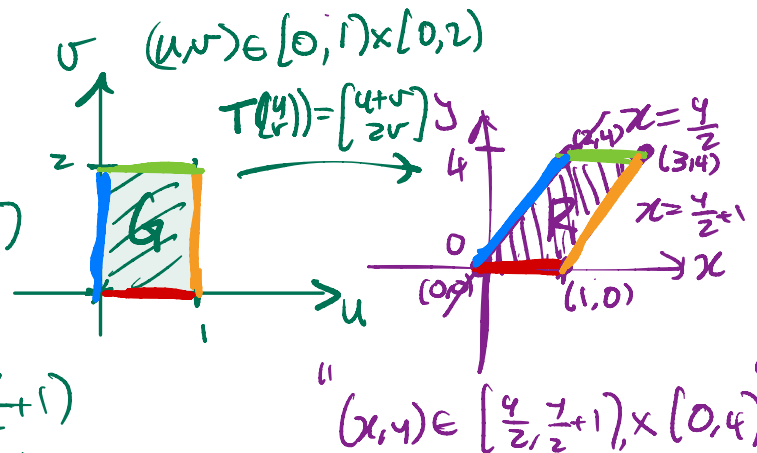
When that happens set up system & solve for x, y to find $T(u, v) = \begin{bmatrix} x \\ y \end{bmatrix}$

@ $x = \frac{y}{2} + 1 \Rightarrow \frac{y}{2} + 1 = u + v \Rightarrow u = 1$

2. Find G and sketch:

$$\begin{aligned} @ y=0 &\Rightarrow v=0 \\ @ y=4 &\Rightarrow v=2 \end{aligned} \left. \vphantom{\begin{aligned} @ y=0 \\ @ y=4 \end{aligned}} \right\} \begin{aligned} &\text{so } y \in (0, 4) \\ &\text{becomes } v \in (0, 2) \end{aligned}$$

$$\begin{aligned} @ x=\frac{y}{2} &\Rightarrow u=0 \\ @ x=\frac{y}{2}+1 &\Rightarrow u=1 \end{aligned} \left. \vphantom{\begin{aligned} @ x=\frac{y}{2} \\ @ x=\frac{y}{2}+1 \end{aligned}} \right\} \begin{aligned} &\text{so } x \in \left[\frac{y}{2}, \frac{y}{2}+1\right) \\ &\text{becomes } u \in (0, 1) \end{aligned}$$



$$\underbrace{\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2} dx dy}_R = \underbrace{\int_0^2 \int_0^1}_{G} ? \mathbf{2} du dv$$

to convert for (x,y)
use rules
 $x = u+v, y = 2v$

3. Find Jacobian:

$$T\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} u+v \\ 2v \end{pmatrix} \quad DT = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad |\det DT| = \mathbf{2}$$

the Jacobian determinant

4. Convert and use theorem:

$$\iint_R f(x,y) dx dy = \iint_G f(T(u,v)) |\det(DT(u,v))| du dv.$$

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2} dx dy = \int_0^2 \int_0^1 \frac{2(u+v)-2v}{2} \mathbf{2} du dv$$

$$= \int_0^2 \int_0^1 u \mathbf{2} du dv = \int_0^2 \left. \frac{2}{2} u^2 \right|_0^1 dv$$

$$= \int_0^2 \mathbf{1} dv = \left. \frac{1}{2} v \right|_0^2 = \mathbf{2}$$

Example 118. a) *You try it!* Find the Jacobian of the transformation

Step 1: write T

$$x = u + (1/2)v, \quad y = v.$$

Step 2: compute DT total derivative of T .

$$T\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} u + \frac{1}{2}v \\ v \end{pmatrix}$$

$$\text{So } DT = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

The Jacobian Matrix

$$|\det DT| = 1.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} dx dy?$$

The Jacobian determinant.

i) $u = x, v = y$

iv) $u = y, v = 2x - y$

ii) $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$

v) $u = 2x - y, v = y$

iii) $u = 2x - y, v = y^3$

vi) $u = e^{(2x-y)^2}, v = y^3$

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \quad y = v.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix} = T(u, v)$$

The Jacobian MATRIX

So $DT = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

and $|DT| = 1$

↪ The Jacobian determinant.

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} dx dy$$

i) $u = x, v = y$ does nothing ☹

iv) $u = y, v = 2x - y$

ii) $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$?

v) $u = 2x - y, v = y$

iii) $u = 2x - y, v = y^3$

vi) $u = e^{(2x-y)^2}, v = y^3$

Try (iii) $\iint_G v \underbrace{u e^{u^2}}_{\text{Can do } w\text{-sub } \checkmark} * |DT| du dv$

Seems helpful.

Try (iv) $\iint_G u^3 \underbrace{v e^{v^2}}_{\text{Can do } w\text{-sub } \checkmark} * |DT| du dv$

Seems helpful.

try (v) $\iint_G v^3 u e^{u^2} * |DT| du dv$

↔↔↔ ✓

] Might be easier to figure out bounds for these two?

try (vi) $\iint_G v (?) u du dv$ how to isolate $2x-y$??

Theorem 119 (Derivative of Inverse Coordinate Transformation). *If $\mathbf{T}(u, v)$ is a one-to-one differentiable transformation that maps a region G in the uv -plane to a region R in the xy -plane and $T(u_0, v_0) = (x_0, y_0)$, then we have*

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

General fact about invertible matrices

$$\det A = \frac{1}{\det A^{-1}} \quad \checkmark.$$

Example 120. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy -plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

- 1. Find the rectangle G in the uv plane that is mapped to R

=

1,3

- 2. Evaluate $f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

3. Use the Substitution Theorem to compute the integral.