§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = A$

To motivate: maybe try $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

Eg. $dA = dx\,dy = r\,dr\,d\theta$ so

converting double integral into polar coord.

r "integration factor"

$$A = \int_*^* \frac{1}{\sqrt{4-(2\sin\theta)^2}} * 2\cos\theta\, d\theta$$

$$= \int_*^* \frac{2\cos\theta}{\sqrt{\underbrace{4-4\sin^2\theta}_{4\cos^2\theta}}}\, d\theta$$

$$= \int_*^* \frac{2\cos\theta}{2\cos\theta}\, d\theta$$

$$= \int_*^* 1\, d\theta$$

$$= \theta \Big|_*^*$$

$$= \sin^{-1}\left(\frac{x}{2}\right)\Big|_1^{\sqrt{3}}$$

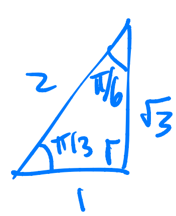
$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(1/2)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\pi/6}$$

$dV = dx\,dy\,dz$

$= r\,dr\,d\theta\,dz$

$= \rho^2 \sin\varphi\, d\varphi\, d\rho\, d\theta$

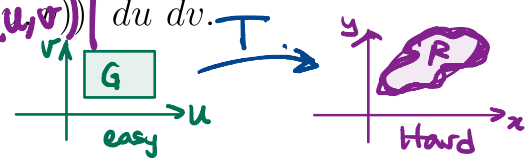


When you convert triple integrals into cylindrical or spherical coords.

Theorem 116 (Substitution Theorem). *Suppose* $\mathbf{T}(u,v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

integration factor.

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))\,\left|\det(D\mathbf{T}(u,v))\right|\,du\,dv.$$



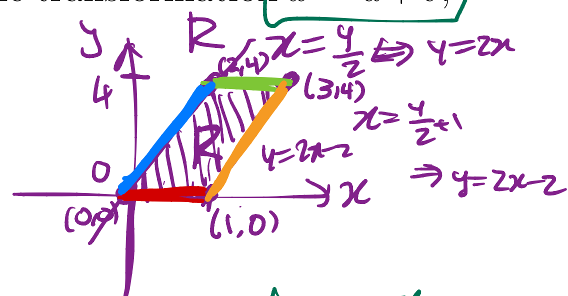
$$\underbrace{\int_0^4\int_{y/2}^{y/2+1}}_{R} \frac{2x-y}{2}\,dx\,dy = \underbrace{\int_*^*\int_*^*}_{G} \boxed{?}\; 2\,du\,dv$$

The "Jacobian determinant" measures how the $\Delta x \Delta y$ needs to get converted to $\Delta u \Delta v$

e.g. if $|\det DT| = 2$ then $2 * \text{Area } G = \text{Area } R$.

Example 117. Evaluate $\displaystyle\int_0^4\int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $x = u+v$, $y = 2v$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$



1. **Find T:** $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$

$$DT = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|\det DT| = 2.$$

What about the integration limits

@ $y=0 \Rightarrow 0 = y = 2v$ so $v = 0$

@ $y=4 \Rightarrow 4 = y = 2v$ so $v = 2$

So if $y \in [0,4)$ then $v \in [0,2)$.

@ $x = \frac{y}{2} \Rightarrow \frac{y}{2} = x = u+v \Rightarrow u = \frac{y}{2} - v$

$\overset{②}{\Rightarrow} u = \frac{2v}{2} - v$

@ $x = \frac{y}{2}+1 \Rightarrow \frac{y}{2} + 1 = u + v \Rightarrow u = 0$

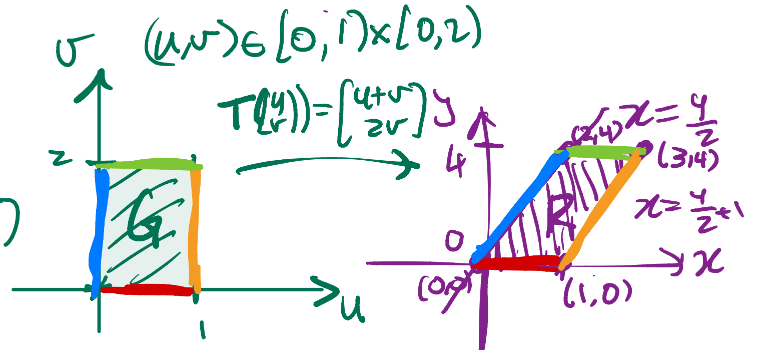
$\overset{②}{\Rightarrow} \frac{2v}{2} + 1 = u + v \Rightarrow u = 1$

NOTE: often you are given $T^{-1}(x,y)$

e.g. $T^{-1}(x,y) = \begin{bmatrix} (2x-y)/2 \\ y/2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

When that happens set up system & solve for x, y to find $T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix}$

2. **Find G and sketch:**



$$\left.\begin{array}{l} @\ y=0 \Rightarrow v=0 \\ @\ y=4 \Rightarrow v=2 \end{array}\right\}\ \text{so } y\in[0,4) \text{ becomes } v\in[0,2)$$

$$\left.\begin{array}{l} @\ x=\frac{y}{2} \Rightarrow u=0 \\ @\ x=\frac{y}{2}+1 \Rightarrow u=1 \end{array}\right\}\ \text{So } x\in\left[\tfrac{y}{2},\tfrac{y}{2}+1\right) \text{ becomes } u\in[0,1)$$

"$(x,y)\in\left[\frac{y}{2},\frac{y}{2}+1\right)\times[0,4)$"

$$\underbrace{\int_0^4\int_{y/2}^{\frac{y}{2}+1}}_{R} \frac{2x-y}{2}\,dx\,dy = \underbrace{\int_0^2\int_0^1}_{G} \ ?\ 2\,du\,dv$$

to convert $f(x,y)$ use rules $x=u+v,\ y=2v$

the Jacobian determinant

3. **Find Jacobian:**

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}u+v\\2v\end{bmatrix} \qquad DT=\begin{bmatrix}1&1\\0&2\end{bmatrix} \qquad |\det DT| = 2$$

4. **Convert and use theorem:**

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|\,du\,dv.$$

$$\int_0^4\int_{y/2}^{y/2+1}\frac{2x-y}{2}\,dx\,dy = \int_0^2\int_0^1 \frac{2(u+v)-2v}{2}\,2\,du\,dv$$

$$= \int_0^2\int_0^1 u\,2\,du\,dv = \int_0^2 \frac{2}{2}u^2\Big|_0^1 dv$$

$$= \int_0^2 1\,dv = 1v\Big|_0^2 = \boxed{2}$$

Example 118. a) *You try it!* Find the Jacobian of the transformation

Step 1: write T

$$x = u + (1/2)v, \;\; y = v.$$

The Jacobian Matrix

Step 2: compute DT total derivative of T.

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix}$$

$$\text{So } DT = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \quad |\det DT| = 1.$$

the Jacobian determinant.

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dx\, dy?$$

i) $u = x, v = y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii) $\boxed{u = 2x - y}, \boxed{v = y^3}$

iv) $u = y$, $v = 2x - y$

v) $u = 2x - y$, $v = y$

vi) $u = e^{(2x-y)^2}$, $v = y^3$

(iii) $\displaystyle\iint_G v u e^{u^2} \overset{??}{\boxed{|DT|}}\, du\, dv$?? set up w-substitution

$w = u^2$

$dw = 2u\, du$

$\frac{1}{2} dw = u\, du$

$y = v^{1/3}$

$x = \frac{u+y}{2} = \frac{u + v^{1/3}}{2}$

(0) need $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$??

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} (u+v^{1/3})/2 \\ v^{1/3} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad DT = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

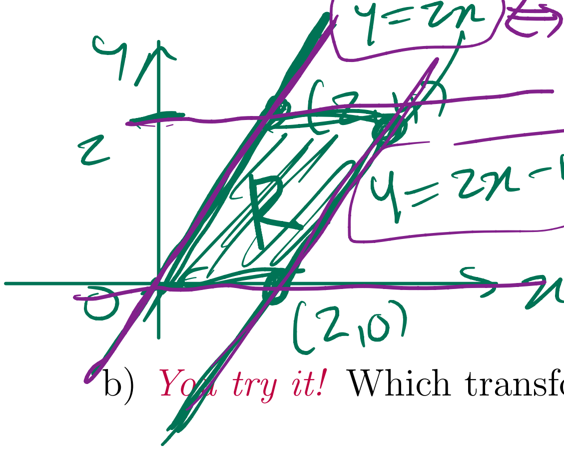
(1) the total derivative DT Matrix.

$$= \begin{bmatrix} 1/2 & \frac{1}{6}v^{-2/3} \\ 0 & \frac{1}{3}v^{-2/3} \end{bmatrix}$$

(2) compute $\det DT = |DT|$

$$|DT| = \det\begin{bmatrix} 1/2 & \frac{1}{6}v^{-2/3} \\ 0 & \frac{1}{3}v^{-2/3} \end{bmatrix} = \boxed{\frac{1}{6}v^{-2/3}}$$

Example 118. a) *You try it!* Find the Jacobian of the transformation



$y=2x \Leftrightarrow \boxed{2x-y=0}$ $u=0$

$$x = u + (1/2)v, \ \ y = v.$$

$y=2x-4 \Leftrightarrow \boxed{2x-y=4}$ $u=4$

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} (u+v)/2 \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

b) *You try it!* Which transformation(s) seem suitable for the integral

linear substitution
$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy?$$

$$\begin{cases} x = \dfrac{u+y}{2} = \dfrac{u+v}{2} \\ y = v \end{cases}$$

i) $u = x, v = y$ iv) $u = y$, $v = 2x - y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$ v) $u = 2x - y$, $v = y$

iii) $u = 2x - y$, $v = y^3$ vi) $u = e^{(2x-y)^2}$, $v = y^3$

try (v)

$$\boxed{\int_0^2\int_0^4 v^3\, u\, e^{u^2}\, |DT|\, du\, dv}$$

1/2

|J| and how

Solve for x, y

$u = 2x - y$
$v = y$

(a) $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}u + \frac{1}{2}v \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ y \end{bmatrix}$

has standard matrix

$$\left|\,|DT|\,\right| = \left|\det\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}\right| = \boxed{1/2}$$

abs value of this.

(J) $M = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & +1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

For R: $y \in [0,2]$ $x \in \left[y/2, \frac{y+4}{2}\right]$

$\Rightarrow v \in [0,2]$ and $u \in [0,4]$

Example 118. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \;\; y = v.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix} = T(u,v) \quad \text{So} \quad DT = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

The Jacobian MATRIX

and $|DT| = 1$

The Jacobian determinant

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dx\, dy?$$

i) $u = x, v = y$ does nothing

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$?

G?? iii) $u = 2x - y,\; v = y^3$

iv) $u = y,\; v = 2x - y$ ✓

v) $u = 2x - y,\; v = y$ ✓

?? vi) $u = e^{(2x-y)^2},\; v = y^3$

Try (iii) $\iint_G v\,\underbrace{u\, e^{u^2}}_{\text{Can do } w\text{-sub}}\, * |DT|\, du\, dv$ ✓ Seems helpful.

Try (iv) $\iint_G u^3\,\underbrace{v\, e^{v^2}}_{\text{Can do } w\text{-sub}}\, * |DT|\, du\, dv$ ✓ Seems helpful.

try (v) $\iint_G v^3\, u\, e^{u^2} * |DT|\, du\, dv$ " — " ✓

Might be easier to figure out bounds for these two?

try (vi) $\iint_G v\,(?!?)\,u\,|DT|\,du\,dv$ how to isolate $2x-y$??

 Prev example $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

Theorem 119 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

To get $|DT|$
$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

1 General fact about invertible matrices

$$\det A = \frac{1}{\det A^{-1}} \checkmark.$$

2 solve for $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

Warning!! DON'T write $|DT^{-1}|$ for the answer!

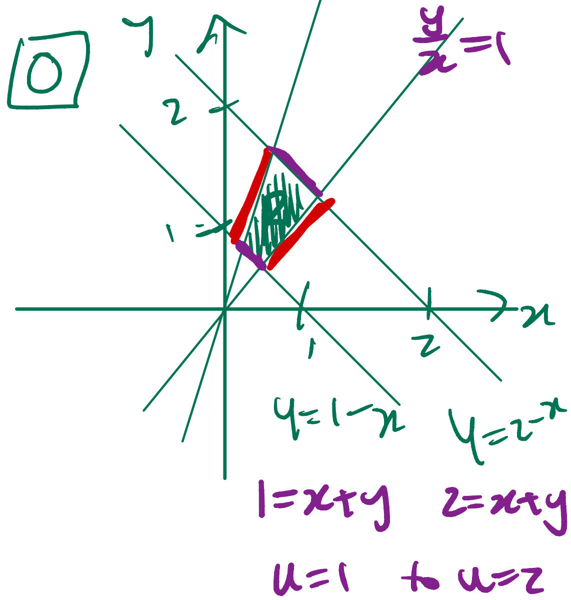
Example 120. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

0 Draw R.

Find $x+y=u$
$y/x = v$

$\frac{y}{x}=3$
$y=3x$

1. **Find the rectangle G in the uv plane that is mapped to R**

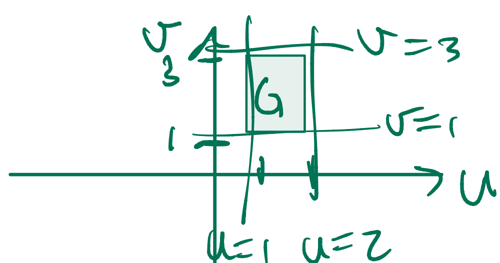


replace u=x+y in the eqns for boundaries

bot $y=1-x \Rightarrow u=x+y=1$ so $u=1$
top $y=2-x \Rightarrow u=x+y=2$ to $u=2$
left $y=3x \Rightarrow \frac{y}{x}=v=3$ so
right $y=x \Rightarrow \frac{y}{x}=v=1$

$v=1$ to 3.

1



$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

2. **Evaluate $f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|$ in terms of u and v without directly solving for T using the theorem above**

$\begin{cases} u=x+y \Rightarrow y=u-x \\ v=\frac{y}{x} \Rightarrow xv=y=u-x \end{cases}$

$\Rightarrow u = xv + x$

$\Rightarrow u = x(v+1)$

$\Rightarrow x = \frac{u}{v+1}$

Start w/ $f(T(u,v))$

Then $|DT| = \frac{1}{|DT^{-1}|}$

$$\boxed{\frac{y(x+y)}{x^3} = \frac{y}{x}(x+y)\cdot\frac{1}{x^2} = v * u * \frac{1}{x^2}}$$

$$= \frac{uv}{\left(u/(v+1)\right)^2} = \left(\frac{v+1}{u}\right)^2 uv = \frac{(v+1)^2}{u} * v$$

Example 120. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

Find $|DT|$ R.

1. ~~**Find the rectangle** G **in the** uv **plane that is mapped to** R~~

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y/x \end{bmatrix} \qquad DT^{-1} = \begin{bmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

$$|DT^{-1}| = \frac{1}{x} - \frac{-y}{x^2} = \frac{x+y}{x^2}$$

So $|DT| = \frac{1}{(x+y/x^2)} = \boxed{\frac{x^2}{x+y}}$

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

2. **Evaluate** $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ **in terms of** u **and** v **without directly solving for T using the theorem above**

Start w/ $f(T(u,v))$

Then $|DT| = \frac{1}{|DT^{-1}|}$

$$\frac{y(x+y)}{x^3}|DT| = \frac{y}{x}(x+y) \cdot \frac{1}{x^2}|DT| = v * u * \frac{1}{x^2}|DT|$$

$$= v * u * \frac{1}{\cancel{x^2}} \frac{\cancel{x^2}}{x+y}$$

$$= v * u * \frac{1}{u} = \boxed{v}$$

3. **Use the Substitution Theorem to compute the integral.**

$$\iint_R \frac{y(x+y)}{x^3}\,dA = \int_1^2 \int_1^3 v \; dv\, du$$

$$= \int_1^2 \frac{1}{2}v^2 \Big|_1^3 du$$

$$= \int_1^2 \frac{9}{2} - \frac{1}{2}\, du = \int_1^2 4\, du$$

$$= 4u\Big|_1^2 = 8 - 4 = \boxed{4}$$

4. (8 points) In this problem you will compute the integral

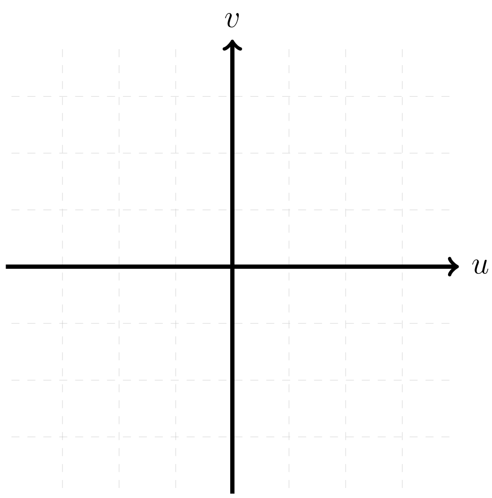
$$M = \iint_R (x+y)e^{x^2-y^2}\,dx\,dy$$

for the region R in the first quadrant bounded by the lines $x-y=0$, $x-y=3$, $x+y=1$, $x+y=3$ using the transformation $u=x-y$, $v=x+y$.

(a) On the axes provided, sketch the new region of integration G in the uv-plane. [A]

(b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e., find $\mathbf{T}(u,v)$ and $|\det D\mathbf{T}(u,v)|$. [AJN]

(c) Evaluate the double integral using a change of variables and your results from parts (a) and (b) to find the value of M. *Hint: integrate with the order $du\,dv$.* [AJN]



$\mathbf{T}(u,v)=$

$|\det \mathbf{T}(u,v)|=$

4. (8 points) In this problem you will compute the integral

$$M = \iint_R (x+y)e^{x^2-y^2}\,dx\,dy$$

$u=0$ $u=3$ $v=1$

for the region R in the first quadrant bounded by the lines $x-y=0$, $x-y=3$, $x+y=1$, $x+y=3$ using the transformation $u=x-y$, $v=x+y$.

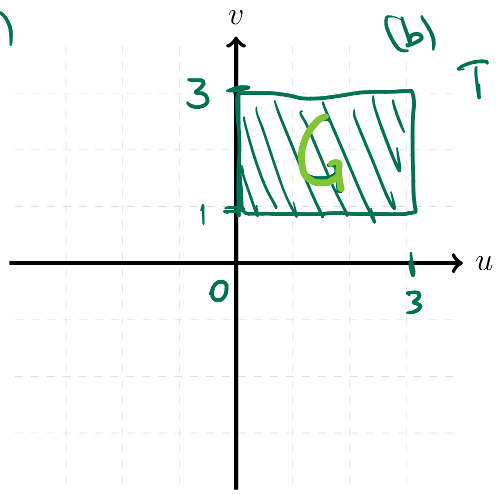
$v=3$

(a) On the axes provided, sketch the new region of integration G in the uv-plane. [A]

(b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e., find $\mathbf{T}(u,v)$ and $|\det D\mathbf{T}(u,v)|$. [AJN]

(c) Evaluate the double integral using a change of variables and your results from parts (a) and (b) to find the value of M. *Hint: integrate with the order $du\,dv$.* [AJN]

(a)



(b)

$$T^{-1}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x-y\\x+y\end{bmatrix}=\begin{bmatrix}u\\v\end{bmatrix}$$

$$A=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}1/2&1/2\\-1/2&1/2\end{bmatrix}$$

$\det A^{-1} = 1/2$

$\mathbf{T}(u,v)=$ $\boxed{\begin{bmatrix}1/2\,u+1/2\,v\\-1/2\,u+1/2\,v\end{bmatrix}}$

$|\det \mathbf{T}(u,v)|=$ $\boxed{1/2}$

(c)

$$M=\iint_R (x+y)e^{x^2-y^2}\tfrac{1}{2}\,dx\,dy = \frac{1}{2}\int_1^3\int_0^3 v e^{uv}\,du\,dv$$

$$=\frac{1}{2}\int_1^3 v * \frac{1}{v}e^{uv}\Big|_0^3\,dv = \frac{1}{2}\int_1^3 e^{3v}-1\,dv$$

$$=\frac{1}{2}\left(\frac{1}{3}e^{3v}-v\Big|_1^3\right)=\frac{1}{2}\left[\left(\frac{1}{3}e^9-3\right)-\left(\frac{1}{3}e^3-1\right)\right]$$

$$=\boxed{\frac{1}{6}\left(e^9-e^3\right)-1}$$