MATH 2550 G/J w/ Dr. Sal Barone

- Dr. Barone, Prof. Sal, or just Sal, as you prefer

Daily Announcements & Reminders:

Goals for Today: Sections 12.1, 12.4, 12.5

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
-
-

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

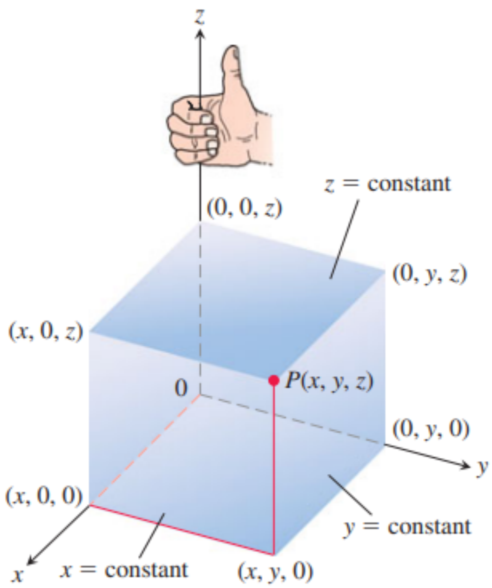
Differential Calculus

Integral Calculus

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

§12.1: Three-Dimensional Coordinate Systems



Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?

§12.3, 12.4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{8cm}}$$

This product tells us about ____.

In particular, two vectors are **orthogonal** if and only if their dot product is ________.

Example 2. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1.

2.

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \underline{\hspace{8cm}}$$

Example 4. Find $\langle 1, 2, 0 \rangle \times \langle 3, -1, 0 \rangle$.

Example 5. *You try it!* Find $\langle 2, 1, 0\rangle \times \langle 1, 2, 1\rangle$.

Some common [AJN] things to look out for.

[A] Accuracy

- simplify answer
- box answer

[J] Justification

- minus sign on $\mathbf{j}$ component
- show intermediate steps

[N] Notation

- use = sign for expressions that are equal
- vector notation vs. point notation

A Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$

The cross product $\mathbf{u} \times \mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}$$

where $\mathbf{n}$ is a unit vector which is normal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

Example 6. Find the area of the parallelogram determined by the points P, Q, and R.

$$P(1, 1, 1),\ Q(2, 1, 3),\ R(3, -1, 1)$$

§12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

Example 8. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.

§12.6 Quadric Surfaces

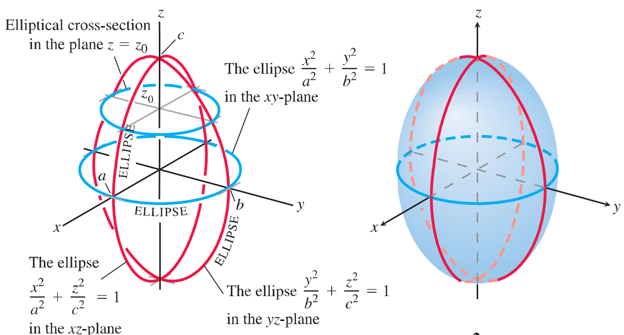
Definition 9. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

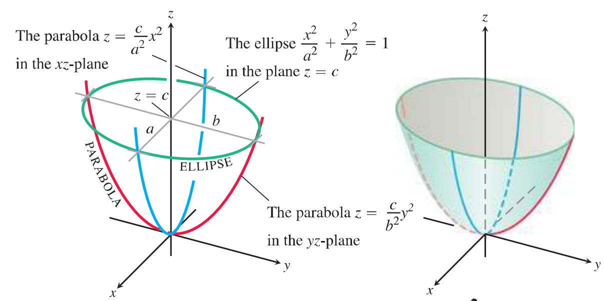
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 10. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

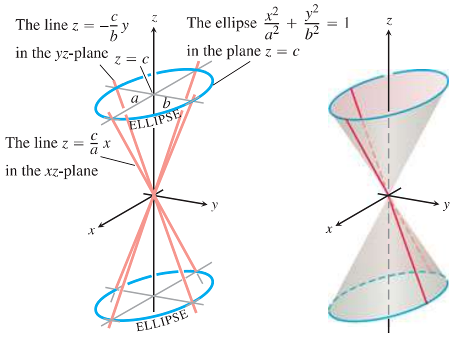
TABLE 12.1 Graphs of Quadric Surfaces



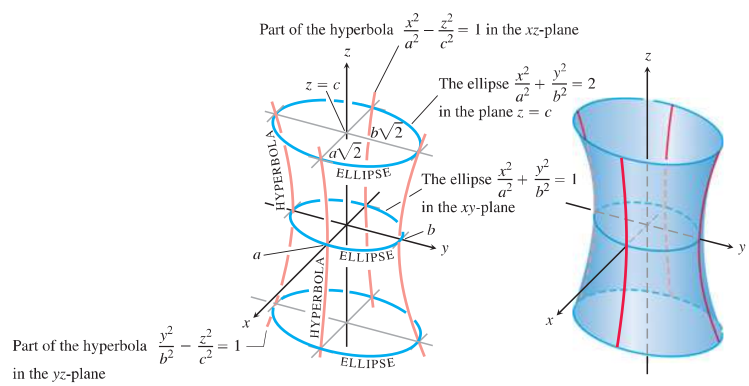
ELLIPSOID $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



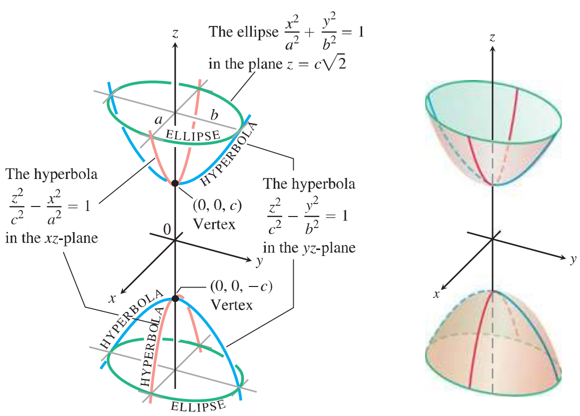
ELLIPTICAL PARABOLOID $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$



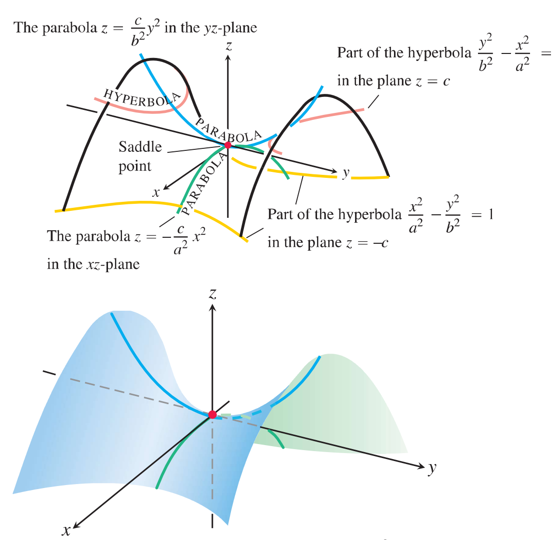
ELLIPTICAL CONE $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$



HYPERBOLOID OF ONE SHEET $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



HYPERBOLOID OF TWO SHEETS $$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



HYPERBOLIC PARABOLOID $$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$$

§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ________________ the curve C, and $\mathbf{r}$ is called a __________________ of C.

Example 11. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 12. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$.

And with continuity:

Example 13. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2 - 4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

And with derivatives:

Example 14. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________
- $|\mathbf{r}'(t)|$ gives ______________________
- $\mathbf{r}''(t)$ gives ______________________

Let's see this graphically

Example 15. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

And with integrals:

Example 16. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t) \rangle \; dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 17. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.

§13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?

b) How far does the fly travel between $t = 0$ and $t = \pi$?

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

d) Compute the integral $\displaystyle\int_0^{\pi} \|\mathbf{v}(t)\| \, dt$. What do you notice?

Definition 18. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from ____________ to ____________ that is traced out exactly once is

$$L = \underline{\hspace{6cm}}$$

Example 19. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \leq t \leq 2\pi$.

Check your intuition

Example 21. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \leq t \leq 8.$

Check your intuition

Arc length parametrization

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

Example 22. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \leq t \leq 2\pi$.

Example 23. *You try it!* Find (a) an arc length parameterization $s(t)$ of the curve $\mathcal{C}$, the portion of the helix of radius 4 in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t \rangle, 0 \leq t \leq \pi/2$, and (b) use $s(t)$ to find L the length of $\mathcal{C}$

§13.3 & 13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: ____________________
- In terms of any parameter t: ____________________

This lets us define the **curvature**, $\kappa(s) =$ ____________________

Example 24. In Example **??** we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ ________________

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ ________________
- $\mathbf{N}(t) =$ ________________
- $\kappa(t) =$ ________________ or ________________________

Example 25. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$, $t \in \mathbb{R}$.

Example 26. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ \ t \in \mathbb{R}.$$

§14.1 Functions of Multiple Variables

Definition 27. A ______________________________ is a rule that assigns to each __________ of real numbers (x, y) in a set D a ________________ denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

Example 28. Three examples are

$$f(x,y) = x^2 + y^2, \qquad g(x,y) = \ln(x+y), \qquad h(x,y) = \frac{1}{\sqrt{x+y}}.$$

Example 29. Find the largest possible domains of f, g, and h.

Definition 30. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 31. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

In 3D, it looks like this.

Definition 32. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

Example 33. Create a contour diagram of $f(x, y) = x^2 - y^2$

Definition 34. The ______________ of a surface are the curves of ______________________ of the surface with planes parallel to the __________________________.

Example 35. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 36. A ________________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a ________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 37. Describe the largest possible domain of the function

$$f(x, y, z) = \frac{1}{4 - x^2 - y^2 - z^2}.$$

Example 38. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

§14.2 Limits & Continuity

Definition 39. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ ____________________________ regardless of how we approach the point (x_0, y_0).

Definition 40. A function $f(x,y)$ is **continuous** at (x_0, y_0) if

1. ____________________________
2. ____________________________
3. ____________________________

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 41. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Example 42. *You try it!* Evaluate $\displaystyle\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 43. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

This idea is called the **two-path test:**

If we can find ____________________ to (x_0, y_0) along which ________________ takes on two different values, then ____________________.

Example 44. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Example 45. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test.

Example 46. **[Challenge:]** Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^2}$$

does exist using the Squeeze Theorem.

Theorem 47 (Squeeze Theorem)**.** *If* $f(x,y) = g(x,y)h(x,y)$, *where* $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ *and* $|h(x,y)| \leq C$ *for some constant* C *near* (a,b), *then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.

§14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 48. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

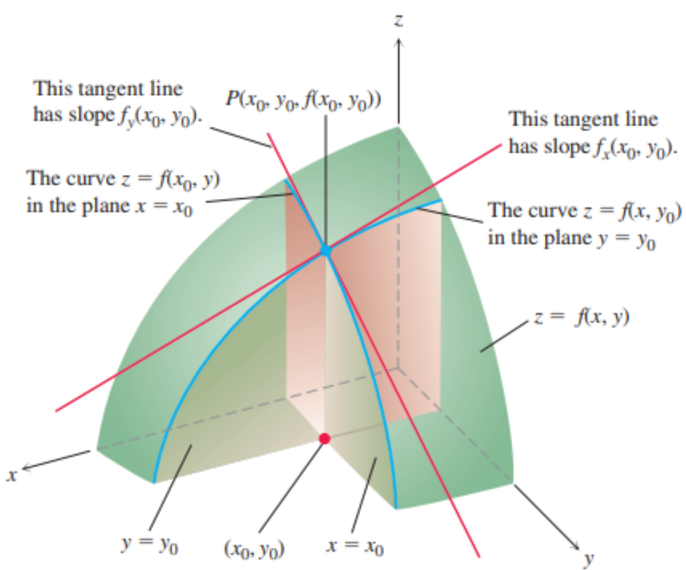
Let's investigate graphically.

Definition 49. If f is a function of two variables x and y, its ____________________ are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \_\_\_\_\_\_\_\_\_\_\_\_ \qquad f_y(x, y) = \lim_{h \to 0} \_\_\_\_\_\_\_\_\_\_\_\_$$

Notations:

Interpretations:



Example 50. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x,y) = \sqrt{5x - y}$

b) $f(x,y) = \tan(xy)$

Question: How would you define the second partial derivatives?

Example 51. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

$f(x, y) = \sqrt{5x - y}$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 52 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f, f_x, f_y, f_{xy}, f_{yx} are all continuous on D, then*

Example 53. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

Example 54. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 55. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2 y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $L(x) = f(a) + f'(a)(x - a) \approx f(x)$, near $x = a$.

Definition 56. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 57. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 58. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.

§14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Example 59. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Example 60. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$
$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$
$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$
$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

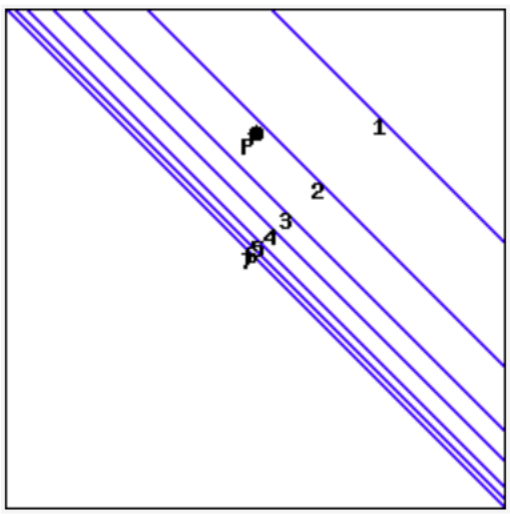
Find $W_s(1,0)$ and $W_t(1,0)$.

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Example 61. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

§14.5 Directional Derivatives & Gradient Vectors

Example 62. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?



Let's go back to our hill example again, $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2,1)$ if we move in the direction $\langle -1, 1 \rangle$?

Definition 63. The ______________________________ of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

if this limit exists.

E.g. for our hill example above we have:

Note that $D_{\mathbf{i}}f =$ $D_{\mathbf{j}}f =$ $D_{\mathbf{k}}f =$

Definition 64. If $f : \mathbb{R}^n \to \mathbb{R}$, then the _______________ of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ____ (or ________) defined by

$$\nabla f(\mathbf{p}) =$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

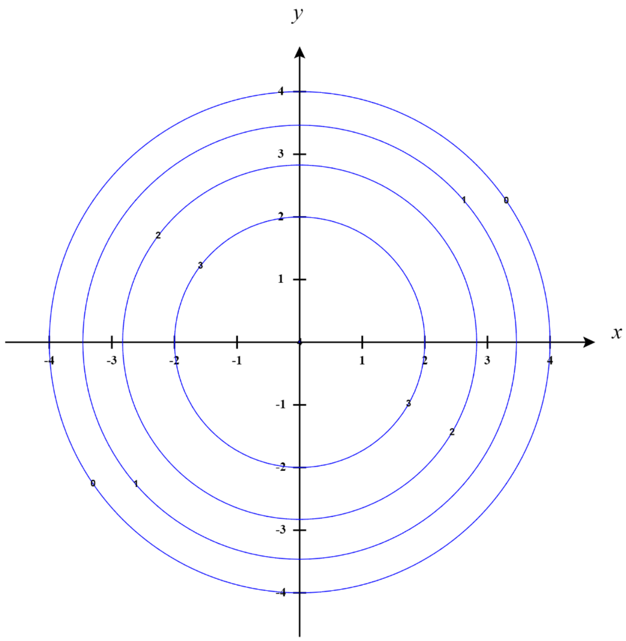
$$D_{\mathbf{u}}f(\mathbf{p}) =$$

Example 65. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x, y) = \ln(x^2 + y^2), \mathbf{p} = (-1, 1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

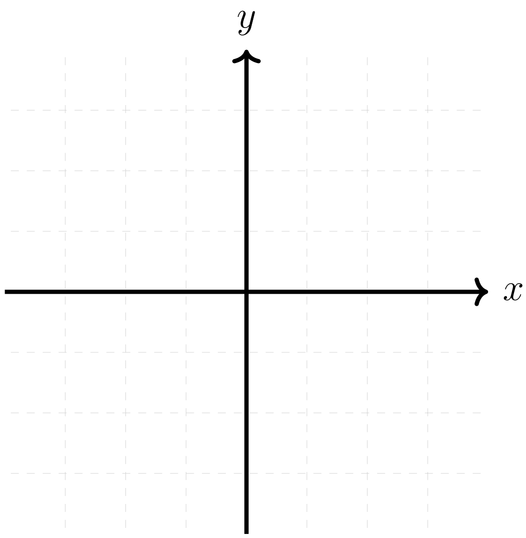
Example 66. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



Note that the gradient vector ∇f is ________________ to the level curves of the function _____________.

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is ____

Example 67. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f \,|_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.



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what's the latex symbol \nabla

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

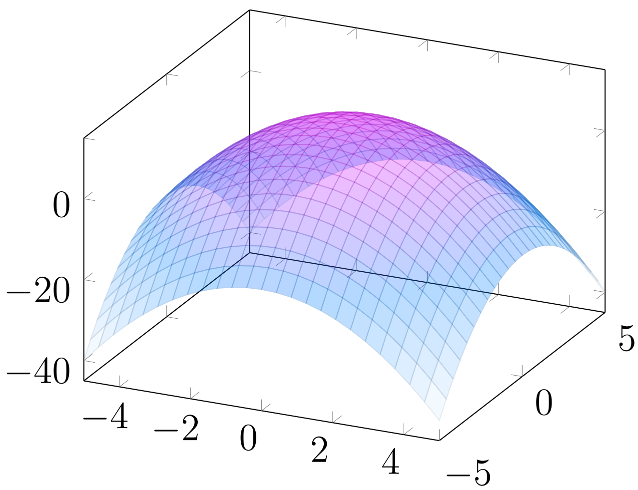
$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

+ Tools

§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

Example 68. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

This should look familiar: it's ______________________

Example 69. *You try it!* Consider the surface in $\mathbb{R}^3$ containing the point P and defined by

$$x^2 + 2xy - y^2 + z^2 = 7, \ P(1, -1, 3).$$

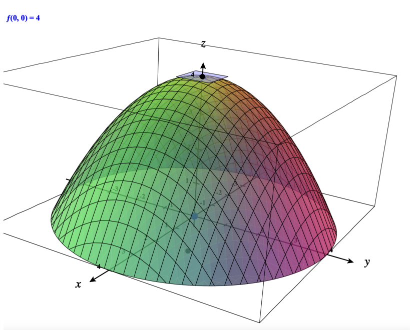
Identity the function $F(x, y, z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

§14.7 Optimization: Local & Global

Gradient: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$.

What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?



Definition 70. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ _____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a ____________________ value of f if $f(a, b)$ _____ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a ____________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 71. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$______________ or where $Df(a, b)$ ____________________ is called a ____________________ of f.

Example 72. Find the critical points of the function

$$f(x,y) = x^3 + y^3 - 3xy.$$

Example 73. *You try it!* Determine which of the functions below have a critical point at $(0, 0)$.

a) $f(x, y) = 3x + y^3 + 2y^2$

b) $g(x, y) = \cos(x) + \sin(x)$

c) $h(x, y) = \dfrac{4}{x^2 + y^2}$

d) $k(x, y) = x^2 + y^2$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a, b) =$$

Theorem 74 (2nd Derivative Test)**.** *Suppose* (a, b) *is a critical point of* $f(x, y)$ *and* f *has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$*,* $f(a, b)$ *is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$*,* $f(a, b)$ *is a local maximum*
- *If* $\det(Hf(a, b)) < 0$*,* f *has a saddle point at* (a, b)
- *If* $\det(Hf(a, b)) = 0$*, the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 75. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example **??**.

Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 76. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 77. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Example 77. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

(Cont.)

§14.8 Constrained Optimization, Lagrange Multipliers

Goal: Maximize or minimize $f(x,y)$ or $f(x,y,z)$ subject to a *constraint*, $g(x,y)=c$.

Example 78. A new hiking trail has been constructed on the hill with height $h(x,y)=4-\frac{1}{4}x^2-\frac{1}{4}y^2$, above the points $y=-0.5x^2+3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function:

Constraint equation:

Example 78. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

(Cont.)

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 79. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Example 79. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

(Cont.)

§15.1 Double Integrals, Iterated Integrals, Change of Order

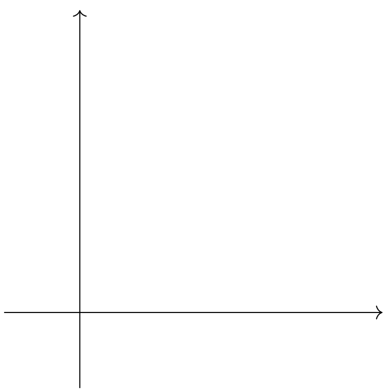
Recall: Riemann sum and the definite integral from single-variable calculus.

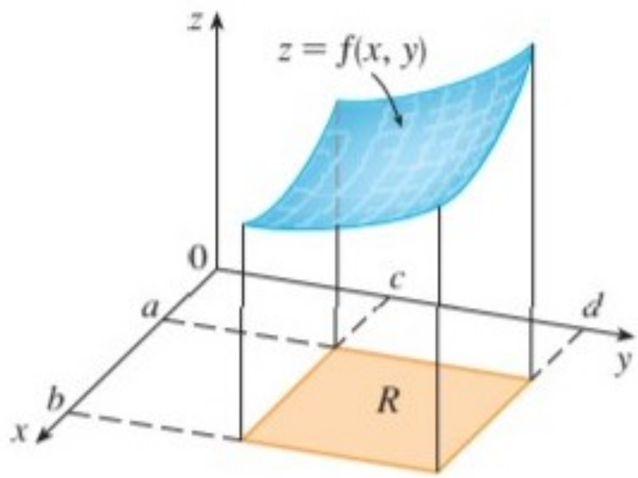
Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

Definition 80. The ____________________ of $f(x, y)$ over a rectangle R is

$$\iint_R f(x, y)\; dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

if this limit exists.

-
-

Question: How can we compute a double integral?

Answer:

Let $f(x, y) = 2xy$ and lets integrate over the rectangle $R = [1, 3] \times [0, 4]$.

We want to compute $\int_1^3 \int_0^4 f(x, y)\ dy\ dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2, y)\ dy$ represent here?

In general, if $f(x,y)$ is integrable over $R = [a,b] \times [c,d]$, then $\int_c^d f(2,y)\, dy$ represents:

What about $\int_c^d f(x,y)dy$?

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$= \int_a^b A(x)dx =$$

This is called an ____________________.

Example 81. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\ dy\ dx.$

Theorem 82 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 83. *You try it!* Integrate:

a) $\displaystyle\int_0^2 \int_{-1}^1 x - y \; dy \; dx$ **easy**

b) $\displaystyle\int_0^1 \int_0^1 \frac{y}{1 + xy} \; dx \; dy$ **medium**

c) $\displaystyle\int_1^4 \int_1^e \frac{\ln x}{xy} \; dx \; dy$ HARD!

Example 84. Compute $\iint_R xe^{e^{e^y}}\, dA$, where R is the rectangle $[-1, 1] \times [0, 4]$.

Hint: Fubini's Theorem.

§15.2 Double Integrals on General Regions

Question: What if the region R we wish to integrate over is not a rectangle?

Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 85. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

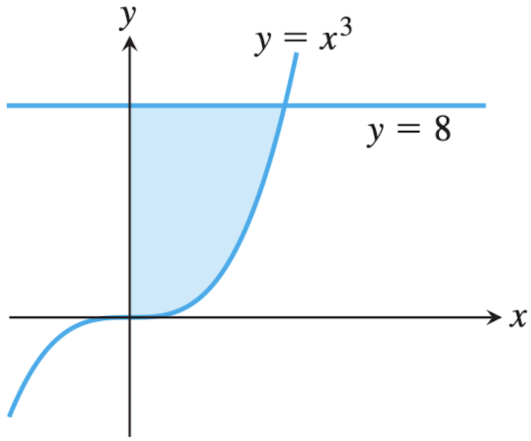
Vertically simple:

Horizontally simple:

Example 86. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Example 87. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\ dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

Example 88. *You try it!* Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



Example 89. Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x, y)\ dy\ dx.$$

Then write an equivalent iterated integral in the order $dx\ dy$.

§15.3 Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \underline{\hspace{4cm}}$$

Example 90. Find the area of the region R bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \underline{\hspace{5cm}}$$

Example 91. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

Example 92. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

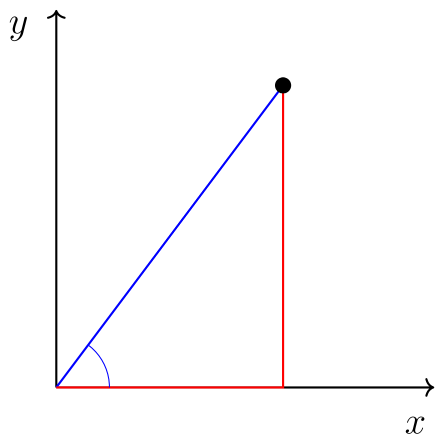
Example 93. Find the average value of the function $f(x, y) = \sin(x + y)$ on (a) the region $R_1 = [0, \pi] \times [0, \pi]$, and (b) the region $R_2 = [0, \pi] \times [0, \pi/2]$.

Hint: choose your order of integration carefully!

Example 94. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.

§15.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates



Cartesian coordinates: Give the distances in ________ and ________ directions from ________

Polar coordinates:

- r = distance from ________ to ________
- θ = angle between the ray ________ and the positive ____________

We can use trigonometry to go back and forth.

Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

Example 95. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 96. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Remember: In polar coordinates, the area form $dA =$________________

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\ dA$.

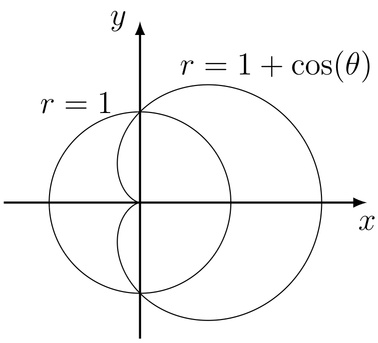
Example 97. Compute the area of the disk of radius 5 centered at $(0, 0)$.
Cont.

Remember: In polar coordinates, the area form $dA =$__________________

Example 98. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \leq x^2+y^2 \leq 4$.

Example 99. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

Example 100. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Example 101. Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x,y)\ dA$.

$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta\ dr\ d\theta$$

Tips and tricks

For horizontal lines such as $x = 2$:

For vertical lines such as $y = 1$ (e.g., Example **??**):

For off-set circles such as $x^2 + (y - 1)^2 = 1$ (e.g., Example **??**):

Example 102. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit gives a

$$\underline{\hspace{6cm}} : \iiint_D f(x,y,z)\, dV$$

Important special case:

$$\iiint_D 1\ dV = \underline{\hspace{5cm}}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta\, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta\, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\, dA, \qquad M_x = \iint_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Example 103. 1. **How to do the computation:**

Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} \, dz \; dy \; dx$.

2. **What does it mean:** What shape is this the volume of?

3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy\ dz\ dx$.

Example 104. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

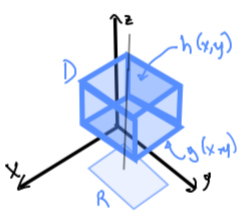
(a) $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

(b) $\displaystyle\int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\, dx\, dy\, dz$

We will think about converting triple integrals to iterated integrals in terms of the ________________ of D on one of the coordinate planes.

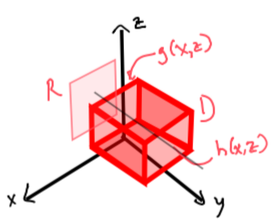
Case 1: z**-simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$



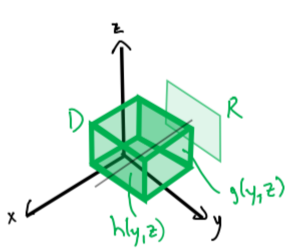
Case 2: y**-simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\ dy \right)\ dz\ dx$$



Case 3: x**-simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



Example 105. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2 y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

Example ?? (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 106. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1,\ y + 2z = 2$.

Example 106. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$.

Example 107. Set up an integral for the volume of the region D defined by

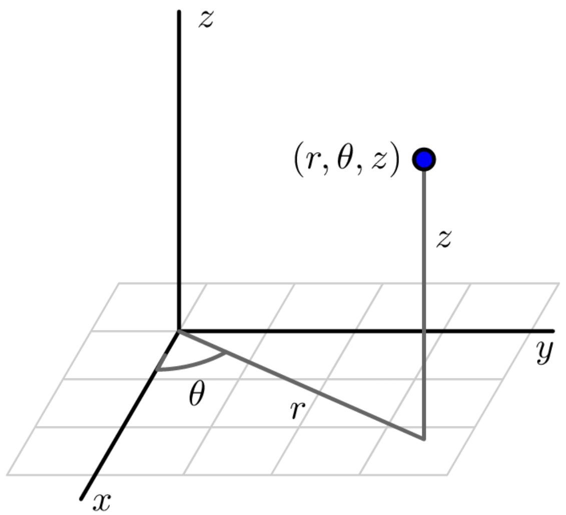
$$x + y^2 \leq 8, \quad y^2 + 2z^2 \leq x, \quad y \geq 0$$

Example 108. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3 y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad z = 0, \quad x + y + z = 2.$$

§15.7 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

Conventions:

Example 109. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

Example 110. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \leq r \leq 2,\ \pi/6 \leq \theta \leq \pi/3,\ 0 \leq z \leq 2\}.$$

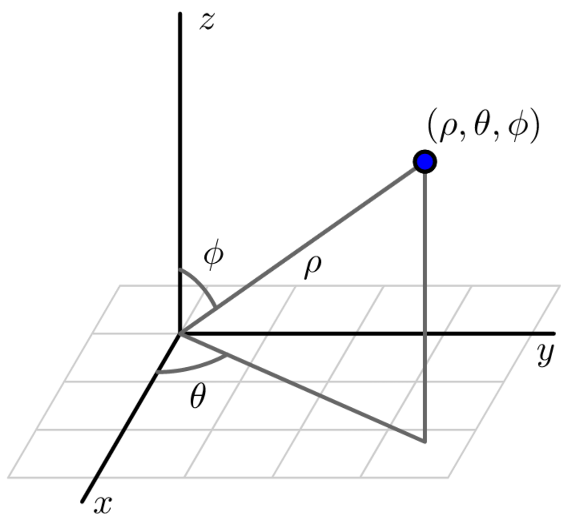
Triple Integrals in Cylindrical Coordinates

We have $dV =$ ______________________

Example 111. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x+2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 112. *You try it!* Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

Spherical Coordinate System



Conventions:

Example 113. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

Spherical to Cartesian:

$$x = \rho \sin(\varphi) \cos(\theta)$$
$$y = \rho \sin(\varphi) \sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 114. In xyz-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq \pi/4,\ \pi/6 \leq \theta \leq \pi/3\}.$$

Triple Integrals in Spherical Coordinates

We have $dV =$ ______________________________

Example 115. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.

Example 116. *You try it!* Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

§15.8 Change of Variables in Multiple Integrals

Thinking about single variable calculus: Compute $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Theorem 117 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u,v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x,y)\;dx\;dy = \iint_G f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|\;du\;dv.$$

Example 118. Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\;dx\;dy$ via the transformation $x = u+v$, $y = 2v$.

1. **Find T:**

2. **Find G and sketch:**

3. **Find Jacobian:**

4. **Convert and use theorem:**

Example 119. a) *You try it!* Find the Jacobian of the transformation

$$x = u + (1/2)v, \;\; y = v.$$

b) *You try it!* Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$

iv) $u = y, \; v = 2x - y$

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

v) $u = 2x - y, \; v = y$

iii) $u = 2x - y, \; v = y^3$

vi) $u = e^{(2x-y)^2}, \; v = y^3$

Theorem 120 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the* uv*-plane to a region* R *in the* xy*-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

Example 121. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

1. **Find the rectangle G in the uv plane that is mapped to R**

2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

3. **Use the Substitution Theorem to compute the integral.**

§16.1 Line Integrals of Scalar Functions

Chapter 16: Vector Calculus

Goals:

- Extend ____________ integrals to ____________ objects living in higher-dimensional space
- Extend the ________________________________ in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 122. Suppose we build a wall whose base is the straight line from $(0, 0)$ to $(1, 1)$ in the xy-plane and whose height at each point is given by $h(x, y) = 2x + y^2$ meters. What is the area of this wall?

Definition 123. The **line integral** of a scalar function $f(x, y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x, y)\ ds =$$

What things can we compute with this?

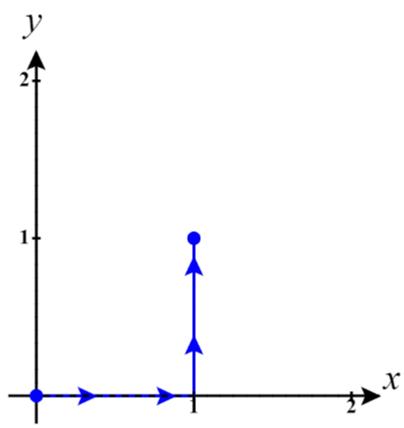
- If $f = 1$:
- If $f = \delta$ is a density function:
- If f is a height:

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x, y, z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 124. *You try it!* Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

Example 125. Compute $\int_C 2x + y^2\ ds$ along the curve C pictured below.



Example 126. *You try it!* Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Example 127. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2 + 1$ from $(2, 5)$ to $(-1, 2)$ and use it to write the integral $\int_C x^2 + y^2 \, ds$ as an integral with respect to your parameter.

§16.2 Vector Fields & Vector Line Integrals

Vector Fields:

Definition 128. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

Examples:

-
-
-
-
-

Graphically: For each point (a, b) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b)$ with its base at (a, b).

Tools: CalcPlot3d
Field Play

Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The ________________________ by a field **F** on an object moving along a curve C is given by

Example 129. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y\rangle$ N. Find the work done by **F** on a moving object from $(0, 3)$ to $(3, 0)$ in a straight line, where x, y are measured in meters.

1. The ____________________ along a curve C of a velocity field $\mathbf{F}$ for a fluid in motion is given by

When C is __________, this is called _________________. C is called _____________ if it does not intersect itself.

Example 130. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.

Example 131. *You try it!* What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $\mathbf{r}'(t)$.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$
4. Integrate

Idea: ____________ across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).

We compute this with the integral

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds.$$

The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of ________ or in the opposite direction.

We can choose $\mathbf{n}$ to be either of

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \pm \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt$ (sign based on choice of normal)
4. Integrate

Example 132. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.

§16.3 Conservative Vector Fields & Fundamental Theorem

Definition 133. A vector field $\mathbf{F}$ is **path independent** on an open region D if

______________________________ for all paths C in the region that have the same endpoints.

When $\mathbf{F}$ is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F} \cdot d\mathbf{r}.$$

Example 134. If C is any closed path and $\mathbf{F}$ is path independent on a region containing C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} =$$

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

For example, is $\mathbf{F}(x, y) = \langle x, y \rangle$ a path independent vector field on its domain?

Example 135. *You try it!* Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x \rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

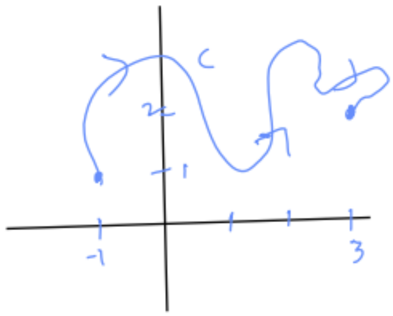
A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a ____________________ for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

Is $\mathbf{F}(x, y) = \langle x, y \rangle$ conservative?

Theorem 136 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

Example 137. Compute $\int_C \langle x, y \rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



It follows that **every conservative field is path independent.**

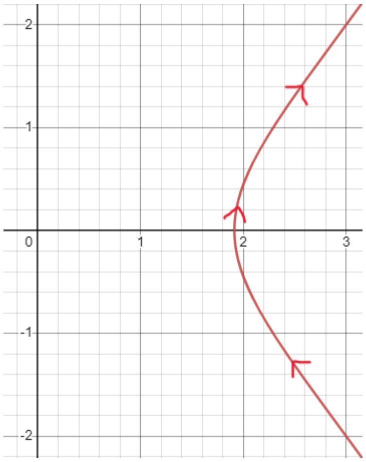
In fact, by carefully constructing a potential function, we can show the converse is also true: ________________________________

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0, 0, 0\rangle$, then $\mathbf{F}$ is conservative.

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$
- This is also called the **mixed-partials test**, because

Example 138. Evaluate $\int_C (10x^4 - 2xy^3)\ dx - 3x^2y^2\ dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.



§16.4 Divergence, Curl, Green's Theorem

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} =$

- $\nabla \times \mathbf{F} =$

How do we measure the change of a vector field?

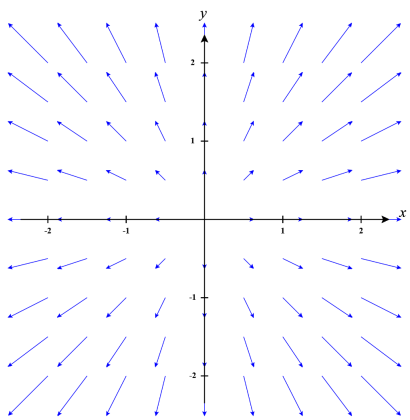
1. **Curl** (in $\mathbb{R}^3$)

   - Tells us ____________________
   - Measures ____
   - Is a ____________
   - Direction gives ____________________
   - Magnitude gives ____________________
   - $\operatorname{curl}\mathbf{F} =$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle$

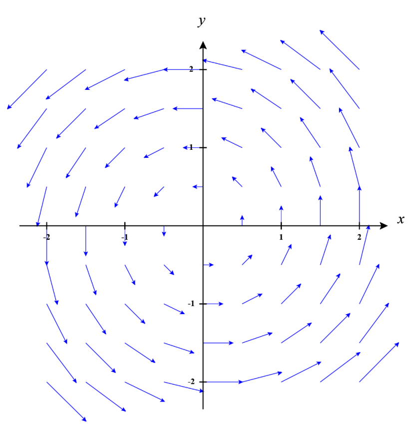
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us ____________________
   - Measures ____
   - Is a ____________
   - $\operatorname{div}\mathbf{F} =$

Example 139. Let $\mathbf{F}(x,y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Example 140. *You try it!* Let $\mathbf{F}(x,y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Question: How is this useful?

Answer: We can relate __________________________________ inside a region to the behavior of the vector field on the boundary of the region.

Theorem 141 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If $\mathbf{F} = \langle P, Q\rangle$ has continuous partial derivatives around R, then*

a) Circulation form:

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C P\ dx + Q\ dy = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA = \iint_R Q_x - P_y\ dA$$

b) Flux form:

$$\int_C \mathbf{F}\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \iint_R (\nabla\bullet\mathbf{F})\ dA = \iint_R P_x + Q_y\ dA$$

Example 142. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.

Example 143. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first octant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

Example 144. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

§16.5, 16.6 Surfaces & Surface Integrals

Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$	$z = f(x, y)$
Implicit:	$F(x, y) = 0$	$F(x, y, z) = 0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t)\rangle$	

Example 145. Give parameteric representations for the surfaces below.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

c) $x^2 + y^2 + z^2 = 9$

d) $x^2 + y^2 = 25$

What can we do with this?

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is ______________________________
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a rectangle of size ______________________ on the surface in $\mathbb{R}^3$.
- Thus, $\text{Area}(S) =$

Example 146. *You try it!* Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$.

Example 147. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

§16.6, 16.7 Flux Surface Integrals, Stokes' Theorem

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of ____________________

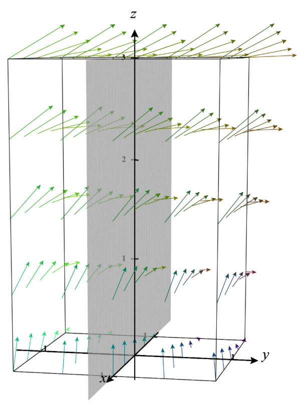
If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Example 148. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the upper hemisphere of $x^2 + y^2 + z^2 = 4$, oriented away from the origin.

Example 149. *You try it!* Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 150. *You try it!* Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?



Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Example 151. *You try it!* Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

Theorem 152 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\ ds.$$

- If S is a region R in the xy-plane, then we get:

- An **oriented surface** is one where ______________________________

- S and C are oriented compatibly if:

§16.7 Stokes' Theorem

Theorem 153 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\; ds.$$

Example 154 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2=2$ above $z=1$, oriented away from the origin.

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

Example 155. Suppose $\operatorname{curl} \mathbf{F} = \langle y^{y^y} \sin(z^2), (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}} \rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.

§16.8 Divergence Theorem

Theorem 156 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ d\sigma = \iiint_D \nabla \cdot \mathbf{F}\ dV.$$

Example 157. Let $\mathbf{F} = \langle y^{1234} e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.

Final Exam Review

Questions/Topics?

Example 158. Evaluate the integral $\int_C y^2\,dx + x^2\,dy$ where C is the circle $x^2 + y^2 = 4$.

Example 159. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i}+2yz\mathbf{j}+2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

Example 160. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ on an object moving along the plane curve $\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t) \rangle$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$.

Example 161. Find the flux of the field $\mathbf{F} = \langle 2xy + x, xy - y \rangle$ outward across the boundary of the square bounded by $x = 0, x = 1, y = 0, x = 1$.

Example 162. Find the flux of $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the upper cap cut from the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$, oriented away from the xy-plane.