§12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

Example 8. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.

Example 9. *You try it!* Find the plane containing the lines parameterized by

$$\begin{aligned} \ell_1(t) &= \langle 1,1,1\rangle + t\langle 2,1,0\rangle, \qquad -\infty < t < \infty \\ \ell_2(s) &= \langle 0,-1,0\rangle + s\langle 1,2,1\rangle, \qquad -\infty < s < \infty \end{aligned}$$

Give your answer in the form $Ax+By+Cz=D$ or $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

§13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ______________ the curve C, and $\mathbf{r}$ is called a ________________ of C.

Example 10. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

§13.2: Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with <u>limits:</u>

Example 11. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

And with <u>continuity</u>:

Example 12. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

And with <u>derivatives</u>:

Example 13. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

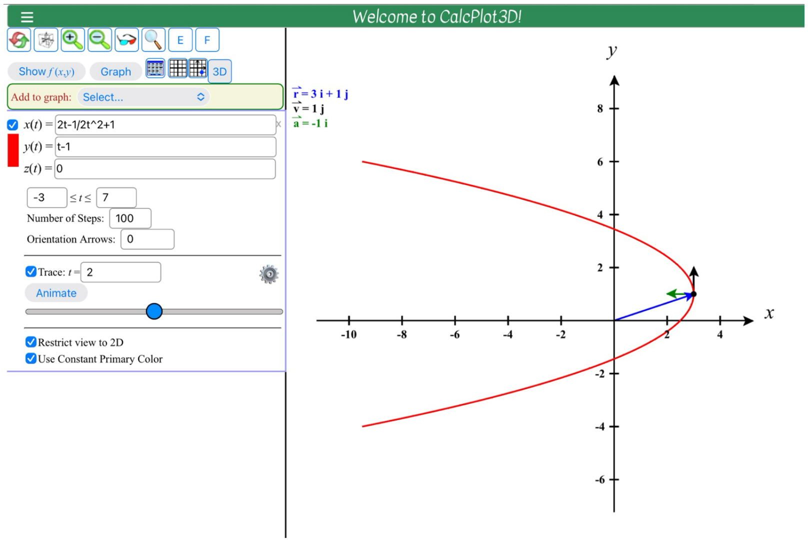
Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________________
- $|\mathbf{r}'(t)|$ gives ______________________________
- $\mathbf{r}''(t)$ gives ______________________________

Let's see this graphically

Example 14. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

Example 14. (cont.) Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.



☑ $x(t) =$ 2t-1/2t^2+1

$y(t) =$ t-1

$z(t) =$ 0

-3 $\leq t \leq$ 7

Number of Steps: 100

Orientation Arrows: 0

☑ Trace: $t =$ 2

☑ Restrict view to 2D

☑ Use Constant Primary Color

$\vec{\mathbf{r}} = 3\,\mathbf{i} + 1\,\mathbf{j}$

$\vec{\mathbf{v}} = 1\,\mathbf{j}$

$\vec{\mathbf{a}} = -1\,\mathbf{i}$

And with integrals:

Example 15. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \; dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 16. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.