§13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \leq t \leq 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?

b) How far does the fly travel between $t = 0$ and $t = \pi$?

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

d) Compute the integral $\displaystyle\int_0^{\pi} \|\mathbf{v}(t)\|\, dt$. What do you notice?

Definition 17. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from ____________ to ____________ that is traced out exactly once is

$$L = \underline{\hspace{6cm}}$$

Example 18. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Example 19. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = \langle 6\sin(2t), 6\cos(2t), 5t\rangle,\ 0 \leq t \leq 2\pi$.

Check your intuition

Example 20. *You try it!* Find the length of the portion of the curve in $\mathbb{R}^3$ given by the parametrization $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \leq t \leq 8$.

Check your intuition

Arc length parametrization

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \underline{\hspace{6cm}}$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

Example 21. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

Example 22. *You try it!* Find (a) an arc length parameterization $s(t)$ of the curve $\mathcal{C}$, the portion of the helix of radius 4 in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle, 0 \le t \le \pi/2$, and (b) use $s(t)$ to find L the length of $\mathcal{C}$

§13.3 & 13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the <u>curvature</u> of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: ________________
- In terms of any parameter t: ________________

This lets us define the **curvature**, $\kappa(s) =$ ________________

Example 23. In Example 21 we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$________________

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ ________________
- $\mathbf{N}(t) =$ ________________
- $\kappa(t) =$ ________________ or ________________________

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$, $t \in \mathbb{R}$.

Example 25. *You try it!* Find $\mathbf{T}, \mathbf{N}, \kappa$ for the curve parametrized by

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ t \in \mathbb{R}.$$