Definition 35. A ____________________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a _________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 36. Describe the largest possible domain of the function

$$f(x, y, z) = \frac{1}{4 - x^2 - y^2 - z^2}.$$

Example 37. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

§14.2 Limits & Continuity

Definition 38. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ ______________________________ regardless of how we approach the point (x_0, y_0).

Definition 39. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. ______________________________
2. ______________________________
3. ______________________________

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 40. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Example 41. *You try it!* Evaluate $\displaystyle\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$, if it exists.

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 42. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

This idea is called the **two-path test:**

If we can find ______________________ to (x_0, y_0) along which ______________________ takes on two different values, then ______________________.

Example 43. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Example 44. *You try it!* Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ is DNE by using the two-path test.

Example 45. **[Challenge:]** Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^2}$$

does exist using the Squeeze Theorem.

Theorem 46 (Squeeze Theorem)**.** *If* $f(x,y) = g(x,y)h(x,y)$, *where* $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ *and* $|h(x,y)| \leq C$ *for some constant* C *near* (a,b), *then* $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.