§14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 47. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.

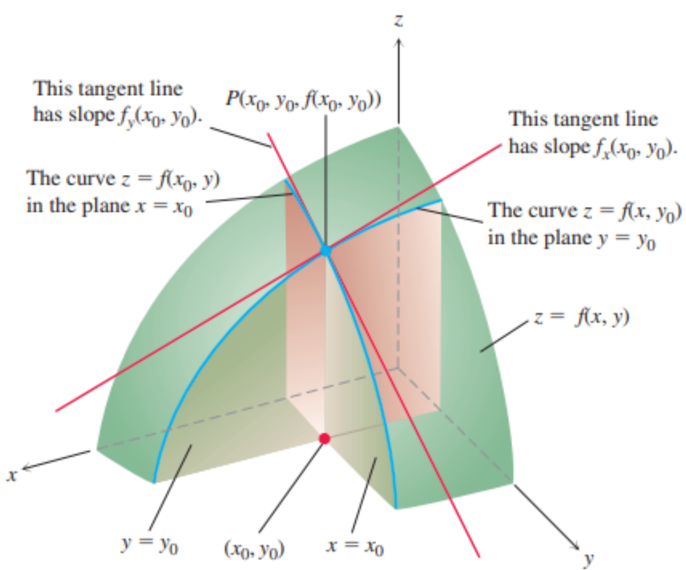
Definition 48. If f is a function of two variables x and y, its ____________________

are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \underline{\hspace{4cm}} \qquad f_y(x, y) = \lim_{h \to 0} \underline{\hspace{4cm}}$$

Notations:

Interpretations:



Example 49. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x,y) = \sqrt{5x-y}$

b) $f(x,y) = \tan(xy)$

Question: How would you define the second partial derivatives?

Example 50. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

$f(x, y) = \sqrt{5x - y}$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 51 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f, f_x, f_y, f_{xy}, f_{yx} are all continuous on D, then*

Example 52. *You try it!* What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $L(x) = f(a) + f'(a)(x - a) \approx f(x)$, near $x = a$.

Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 56. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.