Example 53. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2+t \\ 2s-t \\ st \end{bmatrix}$ have? Compute them.

So, we computed partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 54. *You try it!* Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

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In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 55. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 56. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) its graph has a unique tangent plane at $(a,b,f(a,b))$.

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.

§14.4 The Chain Rule

Recall the Chain Rule from single variable calculus:

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Example 58. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Example 59. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$

$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$

$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$

$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

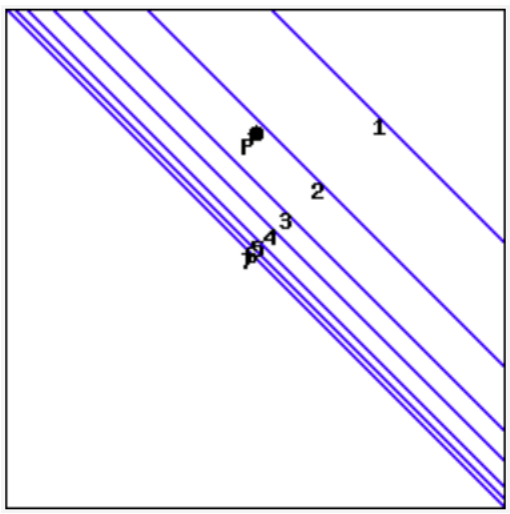
Find $W_s(1,0)$ and $W_t(1,0)$.

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Example 60. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

§14.5 Directional Derivatives & Gradient Vectors

Example 61. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?



Let's go back to our hill example again, $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2, 1)$ if we move in the direction $\langle -1, 1\rangle$?

Definition 62. The ______________________________ of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

if this limit exists.

E.g. for our hill example above we have:

Note that $D_{\mathbf{i}} f =$ $D_{\mathbf{j}} f =$ $D_{\mathbf{k}} f =$

Definition 63. If $f : \mathbb{R}^n \to \mathbb{R}$, then the ________________ of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ____ (or ________) defined by

$$\nabla f(\mathbf{p}) =$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

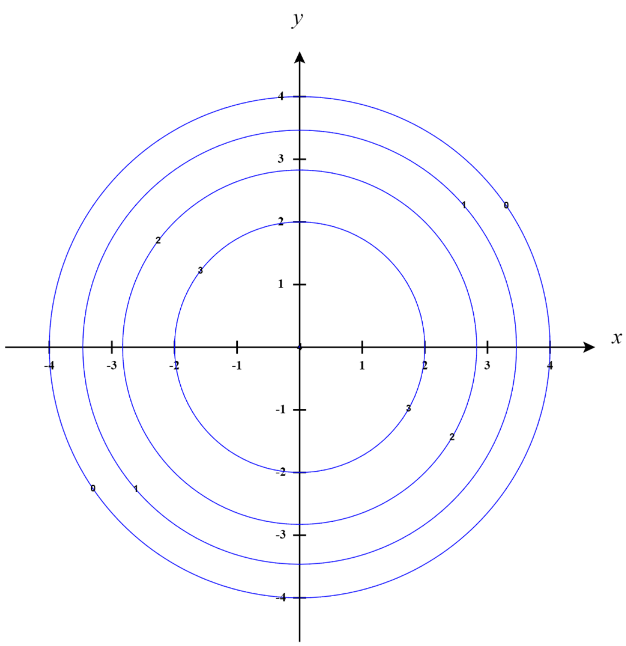
$$D_{\mathbf{u}} f(\mathbf{p}) =$$

Example 64. *You try it!* Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x, y) = \ln(x^2 + y^2), \mathbf{p} = (-1, 1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

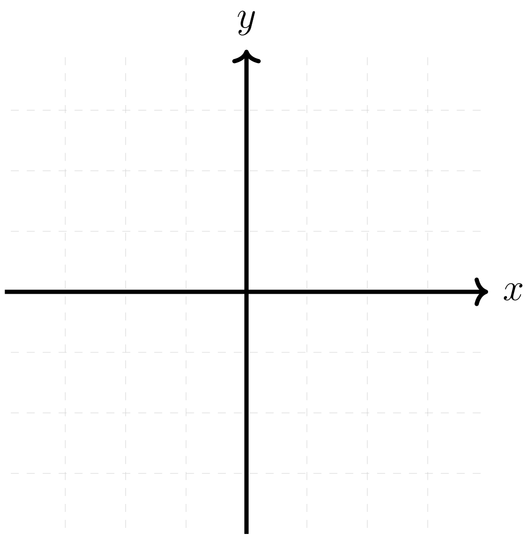
Example 65. If $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2, 0), (0, 4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2, 0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



Note that the gradient vector ∇f is ________________ to the level curves of the function _____________.

Similarly, for $f(x, y, z)$, $\nabla f(a, b, c)$ is ____________________________________

Example 66. *You try it!* Sketch the curve $x^2 + y^2 = 4$ together with (a) the vector $\nabla f\,|_P$ and (b) the tangent line at $P(\sqrt{2}, \sqrt{2})$. Be sure to label the tangent line with the equation which defines it.



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what's the latex symbol \nabla

The LaTeX symbol `\nabla` renders as:

$$\nabla$$

It is called **"nabla"** or the **del operator**, and it is used primarily in vector calculus. It represents the vector differential operator:

$$\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right]$$

Common Uses:

- **Gradient** of a scalar function f:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \right]$$

- **Divergence** of a vector field $\vec{F}$:

$$\nabla \cdot \vec{F}$$

- **Curl** of a vector field $\vec{F}$:

$$\nabla \times \vec{F}$$

- **Laplacian** of a scalar field f:

$$\nabla^2 f = \nabla \cdot \nabla f$$

So in summary, `\nabla` is a compact and powerful symbol in multivariable calculus, especially when working with fields and differential operators.

\+ Tools