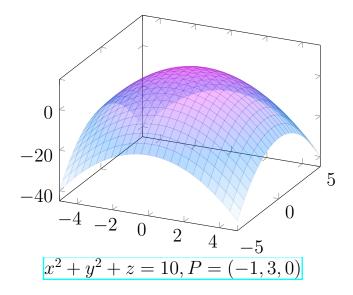
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§14.6 Tangent Planes to Level Surfaces

Suppose S is a surface with equation F(x, y, z) = k. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



Example 67. Find the equation of the tangent plane at the point (-2, 1, -1) to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have z = f(x, y) and a point (a, b, f(a, b)), the equation of the tangent plane is

This should look familiar: it's _____

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Example 68. You try it! Consider the surface in \mathbb{R}^3 containing the point P and defined by

$$x^{2} + 2xy - y^{2} + z^{2} = 7$$
, $P(1, -1, 3)$.

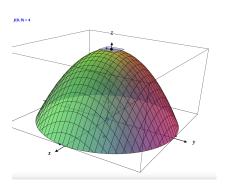
Identity the function F(x, y, z) such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

§14.7 Optimization: Local & Global

Gradient: If f(x,y) is a function of two variables, we said $\nabla f(a,b)$ points in the direction of greatest change of f.

Back to the hill
$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$
.

What should we expect to get if we compute $\nabla h(0,0)$? Why? What does the tangent plane to z = h(x,y) at (0,0,4) look like?



Definition 68. Let f(x,y) be defined on a region containing the point (a,b). We say

- f(a,b) is a ______ value of f if f(a,b) _____ f(x,y) for all domain points (x,y) in a disk centered at (a,b)
- f(a,b) is a ______ value of f if f(a,b) _____ f(x,y) for all domain points (x,y) in a disk centered at (a,b)

In \mathbb{R}^3 , another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near (0,0).

This is called a _____

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 69. If f(x, y) is a function of two variables, a point (a, b) in the domain of f with Df(a, b) = or where Df(a, b) is called a ______ of f.

Example 70. Find the critical points of the function

$$f(x,y) = x^3 + y^3 - 3xy.$$

Example 71. You try it! Determine which of the functions below have a critical point at (0,0).

a)
$$f(x,y) = 3x + y^3 + 2y^2$$

$$b)g(x,y) = \cos(x) + \sin(x)$$

c)
$$h(x,y) = \frac{4}{x^2 + y^2}$$

$$d)k(x,y) = x^2 + y^2$$

To classify critical points, we turn to the **second derivative test** and the **Hessian** matrix. The **Hessian matrix** of f(x,y) at (a,b) is

$$Hf(a,b) =$$

Theorem 72 (2nd Derivative Test). Suppose (a,b) is a critical point of f(x,y) and f has continuous second partial derivatives. Then we have:

- If det(Hf(a,b)) > 0 and $f_{xx}(a,b) > 0$, f(a,b) is a local minimum
- If det(Hf(a,b)) > 0 and $f_{xx}(a,b) < 0$, f(a,b) is a local maximum
- If det(Hf(a,b)) < 0, f has a saddle point at (a,b)
- If det(Hf(a,b)) = 0, the test is inconclusive.

More generally, if $f: \mathbb{R}^n \to \mathbb{R}$ has a critical point at **p** then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from \mathbf{p} and so has a local minimum at \mathbf{p} .
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from \mathbf{p} and so has a local maximum at \mathbf{p} .
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from \mathbf{p} and concave down in others, so has neither a local minimum or maximum at \mathbf{p} .
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at \mathbf{p} .
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at \mathbf{p} .

Example 73. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 70.

Two Local Maxima, No Local Minimum: The function $g(x,y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at (-1,0) and (1,2). Both are local maxima, and the function never has a local minimum!

A global maximum of f(x, y) is like a local maximum, except we must have $f(a, b) \ge f(x, y)$ for all (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 74. On a closed \mathcal{E} bounded domain, any continuous function f(x,y) attains a global minimum \mathcal{E} maximum.

Closed:

Bounded:

Strategy for finding global min/max of f(x,y) on a closed & bounded domain R

- 1. Find all critical points of f inside R.
- 2. Find all critical points of f on the boundary of R
- 3. Evaluate f at each critical point as well as at any endpoints on the boundary.
- 4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 75. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and y = 4.

Example 76. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and y = 4. (Cont.)