**Example 71.** *You try it!* Determine which of the functions below have a critical point at $(0, 0)$ .

a) $f(x, y) = 3x + y^3 + 2y^2$

b) $g(x, y) = \cos(x) + \sin(x)$

c) $h(x, y) = \dfrac{4}{x^2 + y^2}$

d) $k(x, y) = x^2 + y^2$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at $(a, b)$ is

$$Hf(a, b) =$$

**Theorem 72** (2nd Derivative Test)**.** *Suppose $(a, b)$ is a critical point of $f(x, y)$ and $f$ has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, $f(a, b)$ *is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, $f(a, b)$ *is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, $f$ *has a saddle point at* $(a, b)$
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, $f$ is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, $f$ is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, $f$ is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, $f$ may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, $f$ may have either a local maximum or neither at $\mathbf{p}$.

**Example 73.** Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 70.

**Two Local Maxima, No Local Minimum:** The function $g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** $(x,y)$ in the domain of $f$. A global minimum is defined similarly.

**Theorem 74.** *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*

**Closed:**

**Bounded:**

**Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain $R$**

1. Find all critical points of $f$ inside $R$.
2. Find all critical points of $f$ on the boundary of $R$
3. Evaluate $f$ at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

**Example 75.** Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region $R$ bounded by $y = x^2$ and $y = 4$.

**Example 76.** Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region $R$ bounded by $y = x^2$ and $y = 4$.

*(Cont.)*

# §14.8 Constrained Optimization, Lagrange Multipliers

**Goal:** Maximize or minimize $f(x, y)$ or $f(x, y, z)$ subject to a *constraint*, $g(x, y) = c$.

**Example 77.** A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the $xy$-plane. What is the highest point on the hill on this path?

**Objective function:**

**Constraint equation:**

**Example 77.** A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the $xy$-plane. What is the highest point on the hill on this path?

*(Cont.)*

**Method of Lagrange Multipliers:** To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of $f$ at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

**Example 78.** Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

**Example 78.** Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

*(Cont.)*

**Example 79.** *You try it!* Find the points on the curve $x^2 + xy + y^2 = 1$ in the $xy$-plane that are nearest to and farthest from the origin.