§15.1 Double Integrals, Iterated Integrals, Change of Order

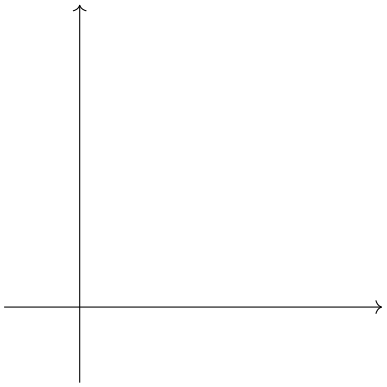
Recall: Riemann sum and the definite integral from single-variable calculus.

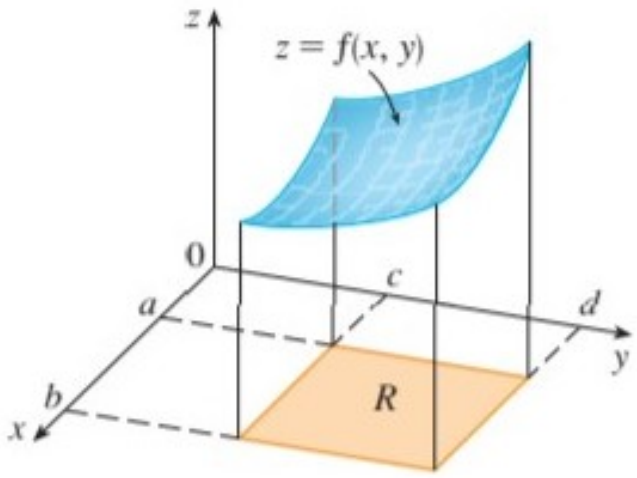
Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

Definition 79. The _____ of $f(x, y)$ over a rectangle R is

$$\iint_R f(x, y)\ dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

if this limit exists.

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Question: How can we compute a double integral?

Answer:

Let $f(x, y) = 2xy$ and lets integrate over the rectangle $R = [1, 3] \times [0, 4]$.

We want to compute $\int_1^3 \int_0^4 f(x, y)\ dy\ dx$, but lets consider the slice at $x = 2$.

What does $\int_0^4 f(2, y)\ dy$ represent here?

In general, if $f(x,y)$ is integrable over $R = [a,b]\times[c,d]$, then $\int_c^d f(2,y)\,dy$ represents:

What about $\int_c^d f(x,y)dy$?

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$= \int_a^b A(x)dx =$$

This is called an _____.

Example 80. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\ dy\ dx$.

Theorem 81 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 82. *You try it!* Integrate:

a) $\int_0^2 \int_{-1}^1 x - y \, dy \, dx$ **easy**

b) $\int_0^1 \int_0^1 \frac{y}{1+xy} \, dx \, dy$ **medium**

c) $\int_1^4 \int_1^e \frac{\ln x}{xy} \, dx \, dy$ HARD!

Example 83. Compute $\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.

Hint: Fubini's Theorem.

§15.2 Double Integrals on General Regions

Question: What if the region R we wish to integrate over is not a rectangle?

Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 84. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

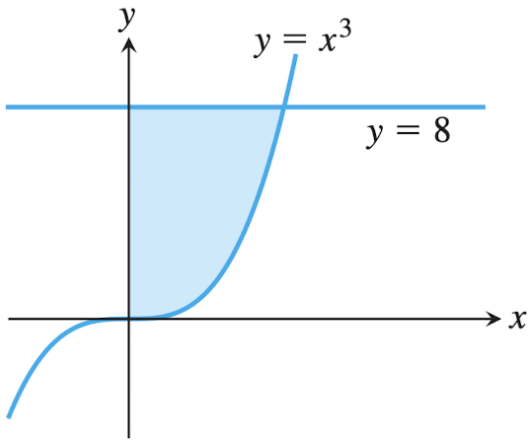
Vertically simple:

Horizontally simple:

Example 85. Write the two iterated integrals for $\iint_R 1 \, dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Example 86. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2 y \, dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

Example 87. *You try it!* Write the two iterated integrals for $\iint_R 1 \, dA$ for the region R which is bounded by $x = 0, y = 8$, and $y = x^3$.



Example 88. Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x,y)\ dy\ dx.$$

Then write an equivalent iterated integral in the order $dx\ dy$.