

§15.3 Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \underline{\hspace{2cm}}$$

Example 89. Find the area of the region R bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$.

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \underline{\hspace{2cm}}$$

Example 90. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

Example 91. *You try it!* Find the average value of the function $f(x, y) = x^2 + y^2$ on the region $R = [0, 2] \times [0, 2]$.

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Example 92. Find the average value of the function $f(x, y) = \sin(x + y)$ on (a) the region $R_1 = [0, \pi] \times [0, \pi]$, and (b) the region $R_2 = [0, \pi] \times [0, \pi/2]$.

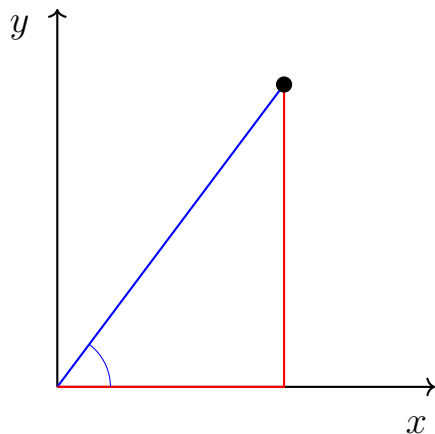
Hint: choose your order of integration carefully!

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.

Example 93. *You try it!* Which value is larger for the function $f(x, y) = xy$: the average value of f over the square $R_1 = [0, 1] \times [0, 1]$, or the average value of f over R_2 the quarter circle $x^2 + y^2 \leq 1$ in Quadrant I? Verify your guess with calculations.

§15.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates



Cartesian coordinates: Give the distances in _____ and _____ directions from _____

Polar coordinates:

- r = distance from _____ to _____
- θ = angle between the ray _____ and the positive _____

We can use trigonometry to go back and forth.

Polar to Cartesian:

$$x = r \cos(\theta) \quad y = r \sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \quad \tan(\theta) = \frac{y}{x}$$

Example 94. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy -plane.**

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

d) *You try it!* Write a Cartesian equation describing the points that satisfy $r = 2 \sin(\theta)$.

Goal: Given a region R in the xy -plane described in polar coordinates and a function $f(r, \theta)$ on R , compute $\iint_R f(r, \theta) \, dA$.

Example 95. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Remember: In polar coordinates, the area form $dA =$ _____

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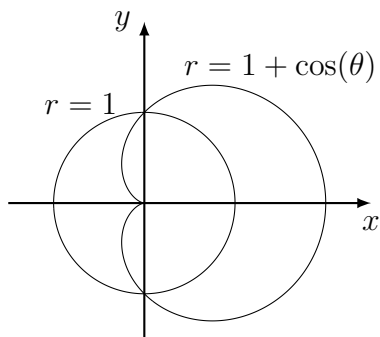
Cont.

Remember: In polar coordinates, the area form $dA =$ _____

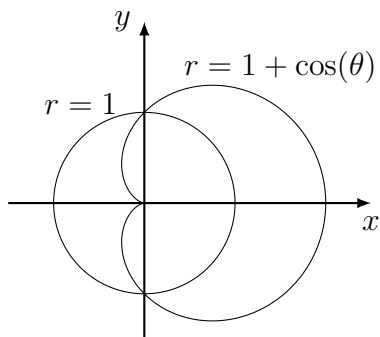
Example 96. Compute $\iint_D e^{-(x^2+y^2)} dA$ on the washer-shaped region $1 \leq x^2 + y^2 \leq 4$.

Example 97. Compute the area of the smaller region bounded by the circle $x^2 + (y - 1)^2 = 1$ and the line $y = x$.

Example 98. *You try it!* Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



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Example 100. Convert the integral in polar coordinates to an equivalent integral in cartesian coordinates, but do not evaluate. Then, evaluate the original integral to find the value of $\iint_R f(x, y) \, dA$.

$$\int_{\pi/6}^{\pi/2} \int_1^{\csc \theta} r^2 \cos \theta \, dr \, d\theta$$

Tips and tricks

For horizontal lines such as $x = 2$:

For vertical lines such as $y = 1$ (e.g., Example [100](#)):

For off-set circles such as $x^2 + (y - 1)^2 = 1$ (e.g., Example [98](#)):

Example 101. *You try it!* Find the area of the region R which is the smaller part bounded between the circle $x^2 + y^2 = 4$ and the line $x = 1$.

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Math 2551 Worksheet: Exam 2 Review

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0) ? Give reasons for your answers.
 - (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
 - (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
 - (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f .
 - (d) At (x_0, y_0) , the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.
2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi)$, $x = e^t$, and $y = \ln(t + 1)$.
3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.
5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy -plane.
6. Evaluate the integral $\int_0^1 \int_{2y}^2 4 \cos(x^2) \, dx \, dy$. Describe why you made any choices you did in the course of evaluating this integral.
7. If $f(x, y) \geq 2$ for all (x, y) , is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\frac{2}{\pi}$?
8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.