§15.5-15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit gives a

$$\underline{\hspace{6cm}} : \iiint_D f(x,y,z)\, dV$$

Important special case:

$$\iiint_D 1\, dV = \underline{\hspace{5cm}}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta\, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x\,\delta\, dV, \qquad M_{xz} = \iiint\limits_D y\,\delta\, dV, \qquad M_{xy} = \iiint\limits_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta\, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x\,\delta\, dA, \qquad M_x = \iint\limits_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Example 102. 1. **How to do the computation:**

Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} \, dz \, dy \, dx.$

2. **What does it mean:** What shape is this the volume of?

3. **How to reorder the differentials:** Write an equivalent iterated integral in the order $dy \, dz \, dx$.

Example 103. *You try it!* Evaluate the triple integrals. What is the shape of the region of integration D in each case?

(a) $\displaystyle \int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

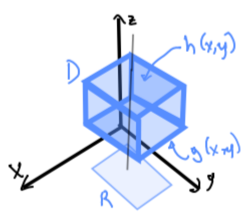
(b) $\displaystyle \int_0^{\pi/3} \int_0^1 \int_{-2}^3 y \sin z\, dx\, dy\, dz$

We will think about converting triple integrals to iterated integrals in terms of the

________________ of D on one of the coordinate planes.

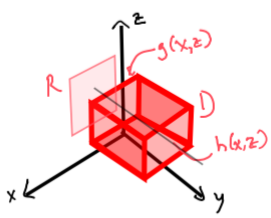
Case 1: z**-simple**) region. If R is the projection of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x,y)$ and $z = g(x,y)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz \right)\ dy\ dx$$



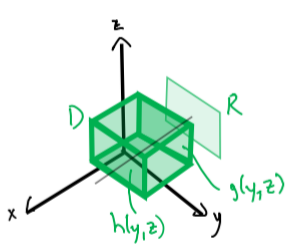
Case 2: y**-simple**) region. If R is the projection of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x,z)$ and $y = g(x,z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\ dy \right)\ dz\ dx$$



Case 3: x**-simple**) region. If R is the projection of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y,z)$ and $x = g(y,z)$, then

$$\iiint_D f(x,y,z)\ dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\ dx \right)\ dz\ dy$$



Example 104. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

Example 104 (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. *You try it!* Find the volume of the region in the first quadrant bounded by the coordinate planes and the planes $x + z = 1,\ y + 2z = 2$.

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