Quiz 4

Be sure to follow the quiz instructions in order to avoid a deduction in points. Submissions are due in Gradescope by 11:59pm on Friday; no late work is accepted.

Name:

Question #1: Find and classify the critical points of the function $z = f(x, y)$.

$$f(x, y) = x^3 - 3xy^2 + 4x^2 + 2y^2$$

Step 1 Find crit pts. solve $\nabla f = \vec{0}$.

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y^2 + 8x \\ -6xy + 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $3x^2 - 3y^2 + 8x = 0$ ①

and $-6xy + 4y = 0$ ②

From ② $y(4 - 6x) = 0$

$\Rightarrow y = 0$ or $x = 4/6 = 2/3$

Now sub into ①

If $y = 0$ then $3x^2 + 8x = 0 \Rightarrow x(3x + 8) = 0$

$\Rightarrow x = 0$ or $x = -8/3$ get $(0,0)$ & $(-8/3, 0)$

If $x = 2/3$ then $3(\frac{2}{3})^2 - 3y^2 + 8(\frac{2}{3}) = 0 \Rightarrow \frac{4}{3} - 3y^2 + \frac{16}{3} = 0$

$\Rightarrow 3y^2 = \frac{20}{3} \Rightarrow y^2 = \frac{20}{9}$

$\Rightarrow y = \pm\sqrt{20/9} = \pm\frac{2\sqrt{5}}{3}$ get $(\frac{2}{3}, \frac{2\sqrt{5}}{3})$ and $(\frac{2}{3}, -\frac{2\sqrt{5}}{3})$

Step 2 Classify crit points w/ Hessian $Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$f_{xx} = 6x + 8$ $f_{xy} = -6y$

$f_{yx} = -6y$ $f_{yy} = -6x + 4$

@ $(0,0)$ $Hf(0,0) = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}$, $\det Hf(0,0) = 32 > 0$ and $f_{xx}(0,0) > 0$ So

@ $(-8/3, 0)$ $Hf(-8/3, 0) = \begin{bmatrix} -8 & 0 \\ 0 & 20 \end{bmatrix}$, $\det Hf = -160 < 0$ So

@ $(2/3, \frac{2\sqrt{5}}{3})$ $Hf(2/3, \frac{2\sqrt{5}}{3}) = \begin{bmatrix} 12 & -4\sqrt{5} \\ -4\sqrt{5} & 0 \end{bmatrix}$, $\det Hf = -80 < 0$ So

@ $(2/3, -\frac{2\sqrt{5}}{3})$ $Hf(2/3, -\frac{2\sqrt{5}}{3}) = \begin{bmatrix} 12 & 4\sqrt{5} \\ 4\sqrt{5} & 0 \end{bmatrix}$, $\det Hf = -80 < 0$ So

MIN @ $(0,0)$

SADDLE @ $(-8/3, 0)$

SADDLE @ $(2/3, 2\sqrt{5}/3)$

SADDLE @ $(2/3, -2\sqrt{5}/3)$

Name: Quiz 1

Question #2: Consider the surface of $\mathbb{R}^3$ containing the point P and defined by

$$2x^2 + xy - 3y^2 + z^2 = 1 \ , \ P(1,2,3).$$

Identify the function $F(x,y,z)$ such that the surface is a level set of F. Then, find ∇F and an equation for the plane tangent to the surface at P. Finally, find a parametric equation for the line normal to the surface at P.

① $F(x,y,z) = 2x^2+xy-3y^2+z^2 = 1$ $(k=1)$ $F(1,2,3) = 2+2-12+9 = 1$ ✓

② $\nabla F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 4x+y \\ x-6y \\ 2z \end{bmatrix}$ @ $(1,2,3)$ $\boxed{\nabla F(1,2,3) = \begin{bmatrix} 6 \\ -11 \\ 6 \end{bmatrix}}$

So plane equation w/ normal $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ & point $P(x_0,y_0,z_0)$ is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

becomes $\boxed{\begin{array}{l} 6(x-1)-11(y-2)+6(z-3)=0 \\ \text{or} \quad 6x-11y+6z=2 \end{array}}$ $D = 6-22+18 = 2$

③ parametric vector equation for line is

$\ell(t) = \vec{OP} + t\vec{v}$, $t \in \mathbb{R}$ so

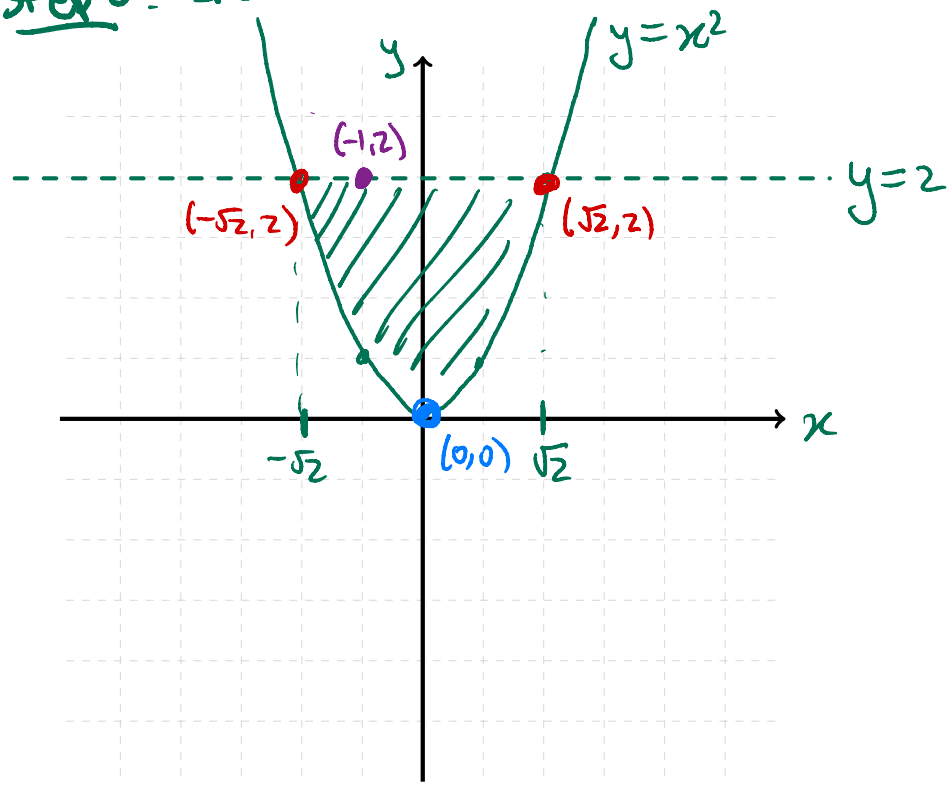
$$\boxed{\ell(t) = \langle 1,2,3 \rangle + t\langle 6,-11,6 \rangle, \ t \in \mathbb{R}}$$

Name: Quiz 1

Question #3: Sketch the domain R on the axes provided, and find the absolute maxima and minima of the function $f(x, y) = x^2 + xy + y^2$ on R.

Step 0: sketch

R is bounded by $y = x^2$ and $y = 2$.



Step 1: interior of R

$f(x,y) = x^2+xy+y^2$

So

$$\nabla f = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x+y=0 \\ x+2y=0 \end{cases}$$

$$\Rightarrow \begin{cases} y=-2x \\ y=-\frac{1}{2}x \end{cases}$$

$\Rightarrow -2x+\frac{1}{2}x=0 \Rightarrow x=0$

So get $(0,0)$ only.
(no crit pts in interior)

Step 2: bdry curves

@ $y=2$ $f(x,2) = x^2+2x+4$, so $f'(x) = 2x+2=0 \Rightarrow x=-1$
get $(-1,2)$

@ $y=x^2$ $f(x,x^2) = x^2+x^3+x^4$, so $f'(x) = 4x^3+3x^2+2x=0$

$\Rightarrow x(4x^2+3x+2)=0$ $\Rightarrow$ ✓ $x=0$ or ✗ $x = \frac{-3\pm\sqrt{9-32}}{8}$ non-real roots

get $(0,0)$

Step 3: Evaluate

(x,y)	$f(x,y) = x^2+xy+y^2$
$(0,0)$	0 MIN @ $(0,0)$
$(-1,2)$	$1-2+4=3$
$(\sqrt{2},2)$	$2+2\sqrt{2}+4 = 6+2\sqrt{2}$ MAX @ $(\sqrt{2},2)$
$(-\sqrt{2},2)$	$2-2\sqrt{2}+4 = 6-2\sqrt{2} \approx 3.17$