§12.1

Exercises 12.1

Geometric Interpretations of Equations

In Exercises 1–16, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

1. $x = 2, \quad y = 3$
2. $x = -1, \quad z = 0$
3. $y = 0, \quad z = 0$
4. $x = 1, \quad y = 0$
5. $x^2 + y^2 = 4, \quad z = 0$
6. $x^2 + y^2 = 4, \quad z = -2$
7. $x^2 + z^2 = 4, \quad y = 0$
8. $y^2 + z^2 = 1, \quad x = 0$
9. $x^2 + y^2 + z^2 = 1, \quad x = 0$
10. $x^2 + y^2 + z^2 = 25, \quad y = -4$
11. $x^2 + y^2 + (z + 3)^2 = 25, \quad z = 0$
12. $x^2 + (y - 1)^2 + z^2 = 4, \quad y = 0$
13. $x^2 + y^2 = 4, \quad z = y$
14. $x^2 + y^2 + z^2 = 4, \quad y = x$
15. $y = x^2, \quad z = 0$
16. $z = y^2, \quad x = 1$

#1 $x=2$ is a plane
$y=3$ is a plane

$x=2, y=3$ is a line.

#5 $x^2+y^2=4$ is a cylinder
$z=0$ is a plane

$x^2+y^2=4$, $z=0$ is a circle

#9 $x^2+y^2+z^2=1$ is a sphere
$x=0$ is a plane

$x^2+y^2+z^2=1$, $x=0$ is a circle.

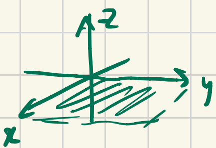
Geometric Interpretations of Inequalities and Equations

In Exercises 17–24, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

17. a. $x \ge 0, \quad y \ge 0, \quad z = 0$ b. $x \ge 0, \quad y \le 0, \quad z = 0$
18. a. $0 \le x \le 1$ b. $0 \le x \le 1, \quad 0 \le y \le 1$
    c. $0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1$
19. a. $x^2 + y^2 + z^2 \le 1$ b. $x^2 + y^2 + z^2 > 1$
20. a. $x^2 + y^2 \le 1, \quad z = 0$ b. $x^2 + y^2 \le 1, \quad z = 3$
    c. $x^2 + y^2 \le 1$, no restriction on z
21. a. $1 \le x^2 + y^2 + z^2 \le 4$
    b. $x^2 + y^2 + z^2 \le 1, \quad z \ge 0$
22. a. $x = y, \quad z = 0$ b. $x = y$, no restriction on z
23. a. $y \ge x^2, \quad z \ge 0$ b. $x \le y^2, \quad 0 \le z \le 2$
24. a. $z = 1 - y$, no restriction on x
    b. $z = y^3, \quad x = 2$

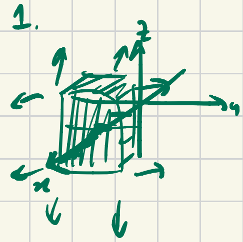
#17 a. $x\ge 0, y\ge 0, z=0$ is a quarter plane

b. $x\ge 0$, $y\le 0$, $z=0$ is the other half of the half plane.

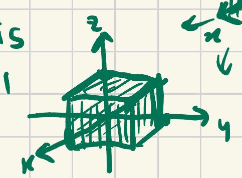


#18 a. $0\le x\le 1$ is an infinite box of width 1.

b. $0\le x\le 1$, $0\le y\le 1$ is an infinitely tall cube

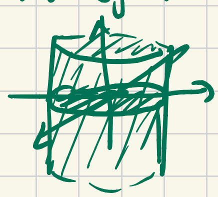


c. $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ is a cube of side length 1 with a vertex at $(0,0,0)$



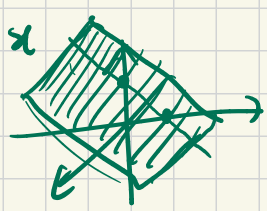
#20 c. $x^2+y^2\le 1$, no restriction on z

is an infinitely tall, filled cyllinder.



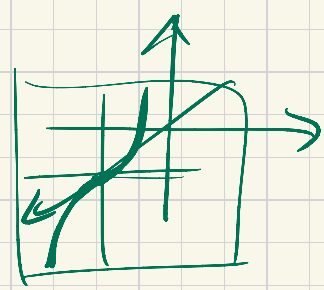
#24 a. $z=1-y$, no restriction on x

is a plane



#24 b. $z=y^3$, $x=2$

is a cubic curve in the $x=2$ plane



In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

25. The plane perpendicular to the
a. x-axis at $(3, 0, 0)$ **b.** y-axis at $(0, -1, 0)$
c. z-axis at $(0, 0, -2)$

26. The plane through the point $(3, -1, 2)$ perpendicular to the
a. x-axis **b.** y-axis **c.** z-axis

27. The plane through the point $(3, -1, 1)$ parallel to the
a. xy-plane **b.** yz-plane **c.** xz-plane

28. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the
a. xy-plane **b.** yz-plane **c.** xz-plane

29. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the
a. xy-plane **b.** yz-plane **c.** plane $y = 2$

30. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the
a. xy-plane **b.** yz-plane **c.** xz-plane

31. The line through the point $(1, 3, -1)$ parallel to the
a. x-axis **b.** y-axis **c.** z-axis

32. The set of points in space equidistant from the origin and the point $(0, 2, 0)$

33. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z-axis meets the sphere of radius 5 centered at the origin

34. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

#25 a. $x=3$ b. $y=-1$ c. $z=-2$

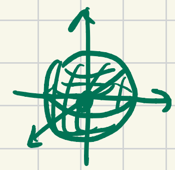
#26. a. $x=3$ b. $y=-1$ c. $z=2$ (lol)

#27. a. $z=1$ b. $x=3$ c. $y=-1$

#28 a. $x^2+y^2+z^2=4,\ z=0$

b. $x^2+y^2+z^2=4,\ x=0$

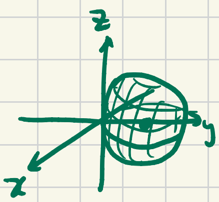
c. $x^2+y^2+z^2=4,\ y=0$



#29 a. $x^2+(y-2)^2+z^2=4,\ z=0$

b. $x^2+(y-2)^2+z^2=4,\ x=0$

c. $x^2+(y-2)^2+z^2=4,\ y=2$



QUIZ #1 on day (1 or 2?)

#30b #31a

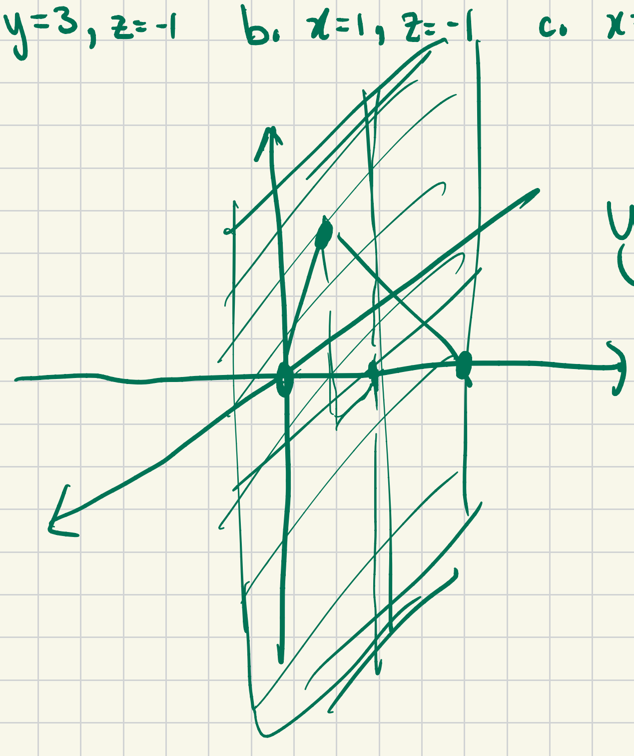
#30 a. $(x+3)^2+(y-4)^2+(z-1)^2=1,\ z=1$

b. $(x+3)^2+(y-4)^2+(z-1)^2=1,\ x=-3$

c. $(x+3)^2+(y-4)^2+(z-1)^2=1,\ y=4$

#31 a. $y=3,\ z=-1$ b. $x=1,\ z=-1$ c. $x=1,\ y=3$

#32★



$y=1$?

Suppose (x_0, y_0, z_0) is equidistant to $(0,0,0)$ and $(0,2,0)$.

Then $x^2+y^2+z^2 = x^2+(y-2)^2+z^2$

$\Rightarrow y^2=(y-2)^2$

$\Rightarrow y^2 = y^2-4y+4$

$\Rightarrow 4-4y=0 \Rightarrow 4(y-1)=0$

$\Rightarrow y=1$ ✓

In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

25. The plane perpendicular to the
a. x-axis at $(3, 0, 0)$ **b.** y-axis at $(0, -1, 0)$
c. z-axis at $(0, 0, -2)$

26. The plane through the point $(3, -1, 2)$ perpendicular to the
a. x-axis **b.** y-axis **c.** z-axis

27. The plane through the point $(3, -1, 1)$ parallel to the
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28. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the
a. xy-plane **b.** yz-plane **c.** xz-plane

29. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the
a. xy-plane **b.** yz-plane **c.** plane $y = 2$

30. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the
a. xy-plane **b.** yz-plane **c.** xz-plane

31. The line through the point $(1, 3, -1)$ parallel to the
a. x-axis **b.** y-axis **c.** z-axis

32. The set of points in space equidistant from the origin and the point $(0, 2, 0)$

33. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z-axis meets the sphere of radius 5 centered at the origin

34. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

#33 plane perpendicular to z-axis through the point (1,1,3) is

$z=3$

and the sphere centered at (0,0,0) with radius 5 is

$x^2+y^2+z^2=25$

So either works

(1) $z=3,\ x^2+y^2+z^2=25$

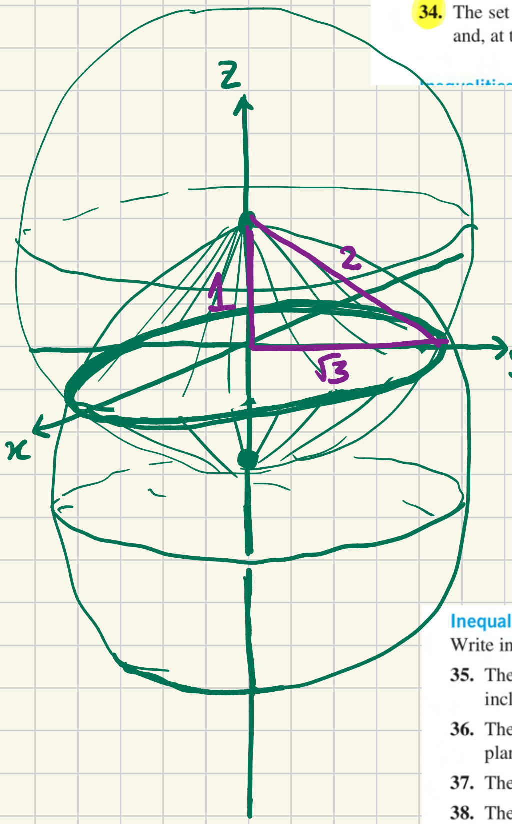
or

(2) $x^2+y^2=16,\ z=3$

#34 $x^2+y^2+(z-1)^2=4,$
$x^2+y^2+(z+1)^2=4$

or just

$x^2+y^2=3,\ z=0$



Inequalities to Describe Sets of Points

Write inequalities to describe the sets in Exercises 35–40.

35. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included)

36. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

37. The half-space consisting of the points on and below the xy-plane

38. The upper hemisphere of the sphere of radius 1 centered at the origin

39. The **(a)** interior and **(b)** exterior of the sphere of radius 1 centered at the point $(1, 1, 1)$

40. The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin. (*Closed* means the spheres are to be included. Had we wanted the spheres left out, we would have asked for the *open* region bounded by the spheres. This is analogous to the way we use *closed* and *open* to describe intervals: *closed* means endpoints included, *open* means endpoints left out. Closed sets include boundaries; open sets leave them out.)

§12.4

Exercises 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

1. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
2. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$
3. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$
5. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$
6. $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

#1 $\vec{u} = 2\vec{i} - 2\vec{j} - \vec{k}$, $\vec{v} = \vec{i} - \vec{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix}$$

$$= 2\vec{i} - (-2+1)\vec{j} + 2\vec{k} = 2\vec{i} + \vec{j} + 2\vec{k}$$

check $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 4-2-2=0$ ✓ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2-2=0$ ✓

In Exercises 9–14, sketch the coordinate axes and then include the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$
10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$
11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

#5 $\vec{u} = 2\vec{i}$, $\vec{v} = -3\vec{j}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} - 6\vec{k} = -6\vec{k}$$

#6 $\vec{u} = \vec{i} \times \vec{j}$ $\vec{v} = \vec{j} \times \vec{k}$

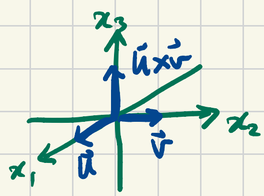
$\overset{*}{\Rightarrow} \vec{u} = \vec{k}$ $\overset{*}{\Rightarrow} \vec{v} = \vec{i}$

(*) $\vec{i} \to \vec{j} \to \vec{k} \to \vec{i}$

so $\vec{u} \times \vec{v} = \vec{k} \times \vec{i} = \vec{j}$

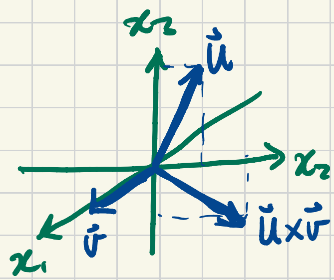
#9 $\vec{u} = \vec{i}$, $\vec{v} = \vec{j}$

$\vec{u} \times \vec{v} = \vec{i} \times \vec{j} = \vec{k}$



#14 $\vec{u} = \vec{j} + 2\vec{k}$, $\vec{v} = \vec{i}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0\vec{i} - (-2)\vec{j} - \vec{k} = 2\vec{j} - \vec{k}$$



check $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = 0$ ✓ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = 0$ ✓

#15 $P(1,-1,2),\ Q(2,0,-1),\ R(0,2,1)$

Triangles in Space

In Exercises 15–18,

a. Find the area of the triangle determined by the points P, Q, and R.

b. Find a unit vector perpendicular to plane PQR.

15. $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$
16. $P(1,1,1)$, $Q(2,1,3)$, $R(3,-1,1)$
17. $P(2,-2,1)$, $Q(3,-1,2)$, $R(3,-1,1)$
18. $P(-2,2,0)$, $Q(0,1,-1)$, $R(-1,2,-2)$

Area of parallelogram

$$\vec{PQ}\times\vec{PR} = \left(\begin{bmatrix}2\\0\\-1\end{bmatrix}-\begin{bmatrix}1\\-1\\2\end{bmatrix}\right)\times\left(\begin{bmatrix}0\\2\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\2\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\1\\-3\end{bmatrix}\times\begin{bmatrix}-1\\3\\-1\end{bmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}$$

$$= \vec{i}\begin{vmatrix}1 & -3\\ 3 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}1 & -3\\ -1 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}$$

$$= (-1+9)\vec{i} - (-1-3)\vec{j} + (3+1)\vec{k}$$

$$= 8\vec{i} + 4\vec{j} + 4\vec{k}$$

check $\begin{bmatrix}1\\1\\-3\end{bmatrix}\cdot\begin{bmatrix}8\\4\\4\end{bmatrix} = 8+4-12=0$ ✓

$\begin{bmatrix}-1\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}8\\4\\4\end{bmatrix} = -8+12-4=0$ ✓

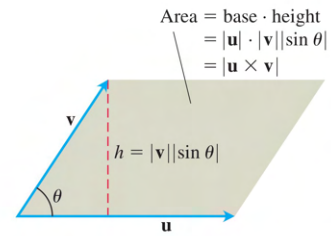


FIGURE 12.30 The parallelogram determined by **u** and **v**.

$$|\vec{PQ}\times\vec{PR}| = \left\|\begin{bmatrix}8\\4\\4\end{bmatrix}\right\| = \sqrt{8^2+4^2+4^2}$$

$$= \sqrt{64+16+16} = \sqrt{96} = \sqrt{2^5 * 3}$$

$$= \sqrt{2^4} * \sqrt{6}$$

$$= 4\sqrt{6}$$

$96 = 2^5 * 3$

96 → 2, 48 → 2, 24 → 4, 6 → 2, 3

Area of triangle is ½ area of parallelogram.

$$\frac{1}{2}\cdot 4\cdot\sqrt{6} = 2\sqrt{6}$$

Unit vector perp to PQR plane is

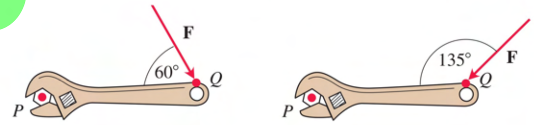
$$\frac{1}{\left\|\begin{bmatrix}8\\4\\4\end{bmatrix}\right\|}\begin{bmatrix}8\\4\\4\end{bmatrix} = \frac{1}{4\sqrt{6}}\begin{pmatrix}8\\4\\4\end{pmatrix} = \begin{pmatrix}2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$$

#25 Magnitude of Torque vector

$$= |r| * |F| * \sin\theta = \bigstar$$

In Exercises 25 and 26, find the magnitude of the torque exerted by **F** on the bolt at P if $|\vec{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25. 26.



$|r| = |\vec{PQ}| = 8$ in. $= \frac{8}{12}$ ft. $= \frac{3}{4}$ ft.

$|F| = 30$ lbs.

$\theta = 60° = \pi/3$

$$\bigstar = \frac{3}{4}\cdot 30\cdot\frac{\pi}{3} = \frac{30\pi}{4} = \frac{15}{2}\pi$$

7.5π ft. lbs.

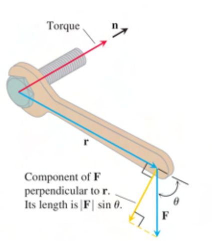


FIGURE 12.32 The torque vector describes the tendency of the force **F** to drive the bolt forward.

$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}|\sin\theta,$$

or $|\mathbf{r}\times\mathbf{F}|$. If we let $\mathbf{n}$ be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r}\times\mathbf{F}$, or

$$\text{Torque vector} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$$

27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

a. $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

b. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|$

c. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$

d. $\mathbf{u}\times(-\mathbf{u}) = \mathbf{0}$

e. $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$

f. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$

g. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$

h. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ b. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

c. $(-\mathbf{u})\times\mathbf{v} = -(\mathbf{u}\times\mathbf{v})$

#27 a. true $\left\|\begin{bmatrix}a\\b\\c\end{bmatrix}\right\|^2 = a^2+b^2+c^2 = \begin{bmatrix}a\\b\\c\end{bmatrix}\cdot\begin{bmatrix}a\\b\\c\end{bmatrix}$ ✓

b. false e.g. $\begin{bmatrix}2\\0\\0\end{bmatrix}\cdot\begin{bmatrix}2\\0\\0\end{bmatrix} \neq 2 = \left\|\begin{bmatrix}2\\0\\0\end{bmatrix}\right\|$ ✓

c. true, defn of $\vec{u}\times\vec{v}$ ✓

d. true $\begin{bmatrix}a\\b\\c\end{bmatrix}\times\begin{bmatrix}-a\\-b\\-c\end{bmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ -a & -b & -c\end{vmatrix} = 0$ ✓

e. false, $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ b/c properties of determinant

f. true, $\vec{u}\times(\vec{v}+\vec{w}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1+w_1 & v_2+w_2 & v_3+w_3\end{vmatrix}$

$$= \vec{i}\left(u_2(v_3+w_3) - u_3(v_2+w_2)\right) - \vec{j}\left(u_1(v_3+w_3) - u_3(v_1+w_1)\right) + \vec{k}\left(u_1(v_2+w_2) - u_2(v_1+w_1)\right)$$

$$= \vec{i}\left(u_2v_3 - u_3v_2 + u_2w_3 - u_3w_2\right) - \vec{j}\left(u_1v_3 - u_3v_1 + u_1w_3 - u_3w_1\right) + \vec{k}\left(u_1v_2 - u_2v_1 + u_1w_2 - u_2w_1\right)$$

$$= \vec{i}(u_2v_3 - u_3v_2) - \vec{j}(u_1v_3 - u_3v_1) + \vec{k}(u_1v_2 - u_2v_1) + \vec{i}(u_2w_3 - u_3w_2) - \vec{j}(u_1w_3 - u_3w_1) + \vec{k}(u_1w_2 - u_2w_1)$$

$$= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} + \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\end{vmatrix} = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$$ ✓

g. true $(\vec{u}\times\vec{v})\cdot\vec{v} = \begin{bmatrix}u_2v_3 - u_3v_2\\ -u_1v_3 + u_3v_1\\ u_1v_2 - u_2v_1\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$

$$= \cancel{u_2v_1v_3} - \cancel{u_3v_1v_2} - \cancel{u_1v_2v_3} + \cancel{u_3v_1v_2} + \cancel{u_1v_2v_3} - \cancel{u_2v_1v_3} = 0$$ ✓

h. true, two row swaps

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$ ✓

Lin Alg

$$u\cdot(v\times w) = (v\times w)\cdot u = \begin{vmatrix}v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\\ u_1 & u_2 & u_3\end{vmatrix} = (-1)\begin{vmatrix}u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\\ v_1 & v_2 & v_3\end{vmatrix} = (-1)^2\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

a. $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

b. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|$

c. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$

d. $\mathbf{u}\times(-\mathbf{u}) = \mathbf{0}$

e. $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$

f. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$

g. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$

h. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ b. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

c. $(-\mathbf{u})\times\mathbf{v} = -(\mathbf{u}\times\mathbf{v})$

#28 a. true, $\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$

$= \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ ✓

b. true, $\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} = -\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix}$ ✓

c. true, $\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -u_1 & -u_2 & -u_3\\ v_1 & v_2 & v_3\end{vmatrix} = -\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$

T/F $(c\vec{u})\times\vec{v} \stackrel{?}{=} c(u\times v)$?

29. Given nonzero vectors **u**, **v**, and **w**, use dot product and cross product notation, as appropriate, to describe the following.

a. The vector projection of **u** onto **v**

b. A vector orthogonal to **u** and **v**

c. A vector orthogonal to $\mathbf{u}\times\mathbf{v}$ and **w**

d. The volume of the parallelepiped determined by **u**, **v**, and **w**

e. A vector orthogonal to $\mathbf{u}\times\mathbf{v}$ and $\mathbf{u}\times\mathbf{w}$

f. A vector of length $|\mathbf{u}|$ in the direction of **v**

30. Compute $(\mathbf{i}\times\mathbf{j})\times\mathbf{j}$ and $\mathbf{i}\times(\mathbf{j}\times\mathbf{j})$. What can you conclude about the associativity of the cross product?

31. Let **u**, **v**, and **w** be vectors. Which of the following make sense, and which do not? Give reasons for your answers.

a. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

b. $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$

c. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$

d. $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$

Properties of the Cross Product

If **u**, **v**, and **w** are any vectors and r, s are scalars, then

1. $(r\mathbf{u})\times(s\mathbf{v}) = (rs)(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
3. $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$
4. $(\mathbf{v}+\mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u} + \mathbf{w}\times\mathbf{u}$
5. $\mathbf{0}\times\mathbf{u} = \mathbf{0}$
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

(*)

#29 a. $\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} * \vec{v}$

b. $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ & $\vec{v}$.

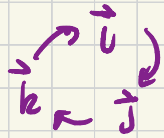
c. $(\vec{u}\times\vec{v})\times\vec{w}$ is orthogonal to $\vec{u}\times\vec{v}$ and $\vec{w}$.

d. $\left|(\vec{u}\times\vec{v})\cdot\vec{w}\right|$ the abs value of the triple scalar prod. is the volume of parallelpiped.

e. $(\vec{u}\times\vec{v})\times(\vec{u}\times\vec{w})$

$\stackrel{*}{=} [(\vec{u}\times\vec{v})\cdot\vec{w}]*\vec{u} - [\cancelto{0}{(\vec{u}\times\vec{v})\cdot\vec{u}}]*\vec{w}$

So $\vec{u}$ is orthogonal to both $\vec{u}\times\vec{v}$ and $\vec{u}\times\vec{w}$.



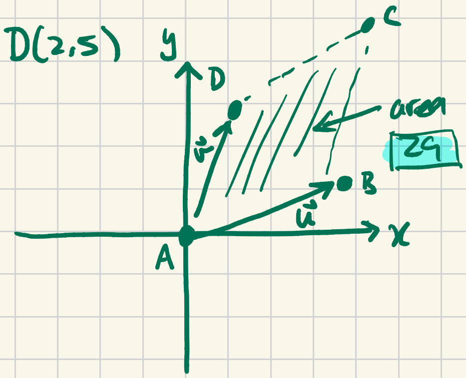
Sanity check $(\vec{i}\times\vec{j})\times(\vec{i}\times\vec{k})$

$= \vec{k}\times -\vec{j} = \vec{i}$ ✓

f. $\frac{|\vec{u}|}{|\vec{v}|} * \vec{v} = \frac{\vec{u}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}$ has same length as $\vec{u}$ ✓ & in direction of $\vec{v}$ ✓.

#31. a&c defined, b&d not defined

#36. $A(0,0)$, $B(7,3)$, $C(9,8)$, $D(2,5)$



$\vec{u} = \vec{AB} = \langle 7,3,0\rangle$

$\vec{v} = \vec{AD} = \langle 2,5,0\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix}$$

$$= \langle 0,0,35-6\rangle = \langle 0,0,29\rangle$$

$$|\vec{u}\times\vec{v}| = 29$$

Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$

36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$

37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$

38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$

39. $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$

40. $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

41. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$

42. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$

43. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$

44. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$

45. $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$

46. $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$

47. $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$

$|\mathbf{u}\times\mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u}\times\mathbf{v}$ is

$$\boxed{|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.}$$

This is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ (Figure 12.30), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}||\sin\theta|$ the height.

#48. $A(0,0,0)$, $B(1,2,0)$, $C(0,-3,2)$, $D(3,-4,5)$

$\vec{u} = \vec{AB} = \langle 1,2,0\rangle$

$\vec{v} = \vec{AC} = \langle 0,-3,2\rangle$

$\vec{w} = \vec{AD} = \langle 3,-4,5\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & -3 & 2 \end{vmatrix} = \langle 4,2,-3\rangle$$

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \langle 4,2,-3\rangle\cdot\langle 3,-4,5\rangle$$

$$= 12-12-15 = -15$$

so $|(\vec{u}\times\vec{v})\cdot\vec{w}| = 15$

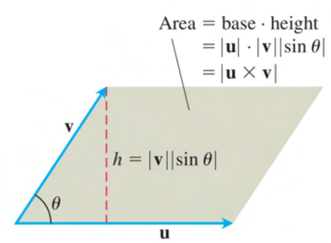


FIGURE 12.30 The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

48. Find the volume of a parallelepiped if four of its eight vertices are $A(0, 0, 0)$, $B(1, 2, 0)$, $C(0, -3, 2)$, and $D(3, -4, 5)$.

49. Triangle area Find a 2×2 determinant formula for the area of the triangle in the xy-plane with vertices at $(0, 0)$, (a_1, a_2), and (b_1, b_2). Explain your work.

Triple Scalar or Box Product

The product $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is called the **triple scalar product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in that order). As you can see from the formula

$$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = |\mathbf{u}\times\mathbf{v}||\mathbf{w}||\cos\theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (Figure 12.34). The number $|\mathbf{u}\times\mathbf{v}|$ is the area of the base

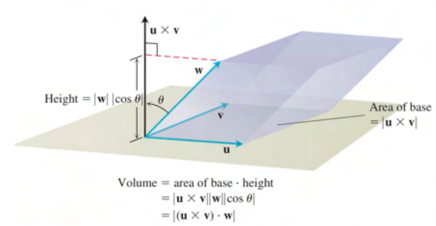


FIGURE 12.34 The number $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$ is the volume of a parallelepiped.

§12.5

Exercises 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$

2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$

3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$

4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$

5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$

6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t$, $y = 2 - t$, $z = 3t$

7. The line through $(1, 1, 1)$ parallel to the z-axis

8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

#1. $x = 3 + t,\ y = -4 + t,\ z = -1 + t$

#2. $\vec{PQ} = (-1-1)\vec{i} + (-2)\vec{j} + (1-(-1))\vec{k}$

$= -2\vec{i} - 2\vec{j} + 2\vec{k}$

so eqn of line is $x_1 = 1 - 2t,\ y = 2 - 2t,\ z = -1 + 2t$

9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$

10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

11. The x-axis

12. The z-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$

14. $(0, 0, 0)$, $(1, 0, 0)$

15. $(1, 0, 0)$, $(1, 1, 0)$

16. $(1, 1, 0)$, $(1, 1, 1)$

17. $(0, 1, 1)$, $(0, -1, 1)$

18. $(0, 2, 0)$, $(3, 0, 0)$

19. $(2, 0, 2)$, $(0, 2, 0)$

20. $(1, 0, -1)$, $(0, 3, 0)$

#5. $x = 0,\ y = 2t,\ z = t$

#6. $x = 3 + 2t,\ y = -2 - t,\ z = 1 + 3t$

#7. $x = 1,\ y = 1,\ z = 1 + t$

* #8. since $\begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ whenever (x, y, z) is a solution to

$3x + 7y - 5z = 0 \quad (*)$

So $\begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$ is perpendicular to the plane $3x + 7y - 5z = 21 \quad (\dagger)$ which is a translate of $(*)$,

and so $x = 2 + 3t,\ y = 4 + 7t,\ z = 5 - 5t$ is perp. to $(\dagger)$ and passes through $(2, 4, 5)$.

#10 $\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = (10-12)\vec{i} - (5-9)\vec{j} + (4-6)\vec{k} = \langle -2, 4, -2 \rangle$

Check $\langle 1,2,3 \rangle \cdot \langle -2, 4, -2 \rangle = -2 + 8 - 6 = 0$ ✓

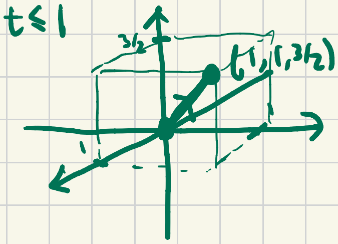
$\langle 4,5,6 \rangle \cdot \langle -2, 4, -2 \rangle = -8 + 20 - 12 = 0$ ✓

So line is $x = 2 - 2t,\ y = 3 + 4t,\ z = -2t$

#11 $x = t,\ y = 0,\ z = 0$

#12 $x = 0,\ y = 0,\ z = t$

#13. $x = t,\ y = t,\ z = \frac{3}{2}t,\ 0 \le t \le 1$

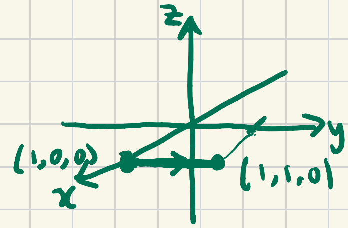


9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$
10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
11. The x-axis
12. The z-axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0)$, $(1, 1, 3/2)$
14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$
16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$
18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$
20. $(1, 0, -1)$, $(0, 3, 0)$

#15. $x = 1,\ y = t,\ z = 0,$
$0 \le t \le 1$

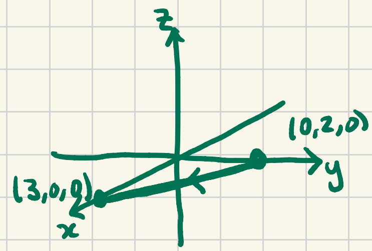


Planes

Find equations for the planes in Exercises 21–26.

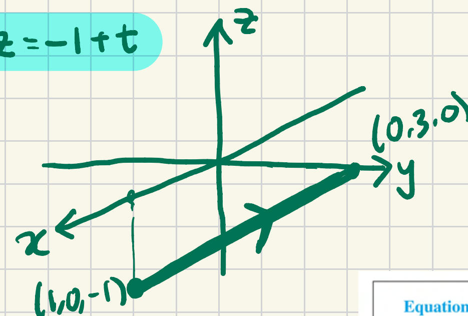
21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane
$$3x + y + z = 7$$
23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
25. The plane through $P_0(2, 4, 5)$ perpendicular to the line
$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$
26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

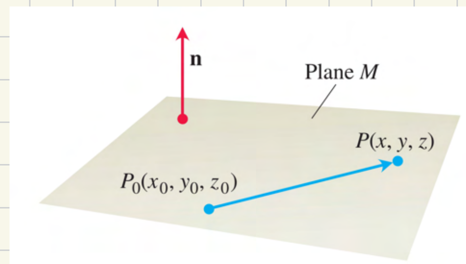
#18. $x = 3t,\ y = 2 - 2t,\ z = 0$
$0 \le t \le 1$



#20. $P(1, 0, -1)$, $Q(0, 3, 0)$ so $\vec{PQ} = \langle -1, -3, 1 \rangle$

$x = 1 - t,\ y = -3t,\ z = -1 + t$
$0 \le t \le 1$





Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

#21. $P_0(0, 2, -1)$ normal to $\vec{n} = \langle 3, -2, -1 \rangle$

vector equation: $\vec{n} \cdot \vec{P_0P} = 0$

$\langle 3, -2, -1 \rangle \cdot \langle x, y-2, z-1 \rangle = 0$

component equation: $3(x-0) - 2(y-2) - (z-1) = 0$

component eqn. simp.: $3x - 2y - z = -1$

$D = \vec{P_0} \times \vec{n}$
$D = \langle 0, 2, -1 \rangle \cdot \langle 3, -2, -1 \rangle$
$= -2 + 1 = -1$

#24. $P_0(2, 4, 5)$, $Q_0(1, 5, 7)$, $R_0(-1, 6, 8)$

$\vec{n} = \vec{P_0Q_0} \times \vec{P_0R_0} = \langle -1, 1, 2 \rangle \times \langle -3, 2, 3 \rangle$

$= \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = \langle -1, -3, 1 \rangle$

vector equation: $\vec{n} \cdot \vec{P_0P} = 0$

$\langle -1, -3, 1 \rangle \cdot \langle x-2, y-4, z-5 \rangle = 0$

comp eqn: $-(x-2) - 3(y-4) + (z-5) = 0$

comp eqn. simp: $-x - 3y + z = -9$

$D = \vec{P_0} \times \vec{n} = -2 - 12 + 5$
$= -9$

check: $\vec{n} \cdot \vec{P_0Q_0} = 1 - 3 + 2 = 0$ ✓
$\vec{n} \cdot \vec{P_0R_0} = 3 - 6 + 3 = 0$ ✓

#21. Component equation is

$$3(x-0)-2(y-2)-(z-(-1))=0$$

Simplifying we get

$$3x-2y+4-z-1=0$$

$$\Rightarrow \boxed{3x-2y-z=-3}$$

Check ✓
$3(0)-2(2)-(-1)=-3$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$

24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$

25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A

27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.

28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

#22. $3x+y+z=D$ where $D=3(1)+(-1)+3=5$

so $\boxed{3x+y+z=5}$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

#23. $P(1,1,-1)$, $Q(2,0,2)$, $R(0,-2,1)$

$$\vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 3\\ -1 & -3 & 2\end{vmatrix}=\langle -2+9, -(2+3), -3-1\rangle$$

$$\vec{n}=\langle 7,-5,-4\rangle$$

Check $\langle 7,-5,-4\rangle\cdot\langle 1,-1,3\rangle=7+5-12=0$ ✓
$\langle 7,-5,-4\rangle\cdot\langle -1,-3,2\rangle=-7+15-8=0$ ✓

So plane is $7x-5y-4z=D$, $D=7(1)-5(1)-4(-1)=7-5+4=6$

So $\boxed{7x-5y-4z=6}$

#26. $A(1,-2,1)$ & $\vec{n}=\langle 1,-2,1\rangle$

So plane is $x-2y+z=D$ where $D=1-2(-2)+1=6$

So $\boxed{x-2y+z=6}$

#27. line #1 $x=2t+1$, $y=3t+2$, $z=4t+3$ line #2 $x=s+2$, $y=2s+4$, $z=-4s-1$

So
$$\begin{cases}2t+1=s+2\\ 3t+2=2s+4\\ 4t+3=-4s-1\end{cases}\rightarrow\begin{cases}2t-s=1\\ 3t-2s=2\\ 4t+4s=-4\end{cases}$$

$$\left[\begin{array}{cc|c}2&-1&1\\3&-2&2\\4&4&-4\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&-1\\2&-1&1\\3&-2&2\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&-1\\0&-3&3\\0&-5&5\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&0\\0&1&-1\\0&0&0\end{array}\right]$$

$t=0, s=-1$

point of intersection when $t=0, s=-1$
@ $(1,2,3)$.

Check
#1 $(1,2,3)$
#2 $(1,2,3)$ ✓

$$\vec{n}=\langle 2,3,4\rangle\times\langle 1,2,-4\rangle=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&3&4\\1&2&-4\end{vmatrix}$$
$$=\langle -12-8, -(-8-4), 4-3\rangle=\langle -20,12,1\rangle \checkmark\checkmark$$

So $\boxed{-20x+12y+z=7}$ $(D=7)$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane
$$3x + y + z = 7$$
23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
25. The plane through $P_0(2, 4, 5)$ perpendicular to the line
$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$
26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

#26. $A(1,-2,1)$ $\quad \vec{n} = \vec{A} = \langle 1,-2,1\rangle \quad P = (x,y,z)$

vector equation: $\vec{n} \times \overrightarrow{AP} = 0$

$$\langle 1,-2,1\rangle \cdot \langle x-1, y+2, z-1\rangle = 0$$

comp eqn : $(x-1) - 2(y+2) + (z-1) = 0$

comp eqn simp: $x - 2y + z = 6$

$$D = \vec{A}\cdot\vec{n} = \vec{n}\cdot\vec{n} = |\vec{n}|^2 = 1+4+1 = 6$$

#27

line 1
$$\begin{cases} x = 2t+1 \\ y = 3t+2 \\ z = 4t+3 \end{cases}$$

line 2
$$\begin{cases} x = s+2 \\ y = 2s+4 \\ z = -4s-1 \end{cases}$$

(lol did them again oh well)

Find s, t such that (x_0, y_0, z_0) on both lines.

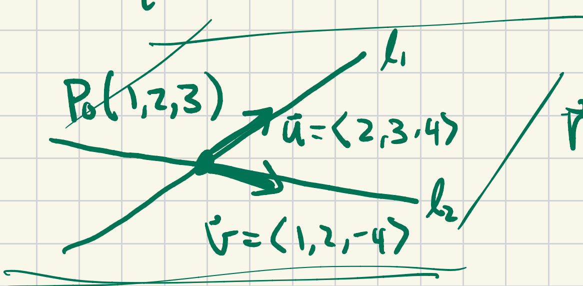
$$\begin{cases} 2t+1 = s+2 \\ 3t+2 = 2s+4 \\ 4t+3 = -4s-1 \end{cases} \rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \\ 4t + 4s = -4 \end{cases} \rightarrow$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 3 & -2 & 2 \\ 4 & 4 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 3 & -2 & 2 \\ 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -5 & 5 \\ 0 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$t = 0$, $s = -1$

check

$P_0 = (1,2,3)$ line 1 ✓ line 2 ✓



$$\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

$$= \langle -20, -(-12), 1\rangle$$

$$\vec{n} = \langle -20, 12, 1\rangle$$

check $\vec{u}\cdot\vec{n} = 0$ ✓
$\vec{v}\cdot\vec{n} = 0$ ✓

plane containing l_1, l_2

vector eqn: $\vec{n}\cdot\overrightarrow{P_0P} = 0$

$$\langle -20, 12, 1\rangle \cdot \langle x-1, y-2, z-3\rangle = 0$$

comp eqn: $-20(x-1) + 12(y-2) + (z-3) = 0$

comp eqn simp: $-20x + 12y + z = -5$

$$D = \vec{P_0}\cdot\vec{n} = -20 + 12 + 3 = -5$$

#29. L1: $\vec{u}_1 = \langle 1,1,-1\rangle$ $P_1(-1,2,1)$

L2: $\vec{u}_2 = \langle -4,2,-2\rangle$ $P_2(1,1,2)$

$$\vec{n} = \vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = \langle -2+2, -(-2-4), 2+4\rangle$$

$$\vec{n} = \langle 0, 6, 6\rangle \checkmark$$

so plane is $6y + 6z = 18$ $(D=18)$

In Exercises 29 and 30, find the plane containing the intersecting lines.

29. $L1: x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$
$L2: x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$

30. $L1: x = t, \quad y = 3 - 3t, \quad z = -2 - t; \quad -\infty < t < \infty$
$L2: x = 1 + s, \quad y = 4 + s, \quad z = -1 + s; \quad -\infty < s < \infty$

31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.

32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$. ? how?

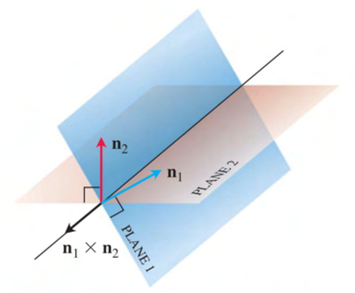


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

#31. $P_0(2,1,-1)$, $\vec{n}_1 = \langle 2,1,-1\rangle$, $\vec{n}_2 = \langle 1,2,1\rangle$

$$\vec{\ell} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \langle 1+2, -(2+1), 4-1\rangle$$

$$\vec{\ell} = \langle 3, -3, 3\rangle \checkmark$$

so plane is $3x - 3y + 3z = 0$ $\qquad D = 6 - 3 - 3 = 0$

#34. line $P(5,5,-3)$ $\vec{v} = \langle 3,4,-5\rangle$ $S(0,0,0)$

$$\vec{PS} = \langle -5,-5,3\rangle$$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \langle 25-12, -(25-9), -20+15\rangle$$

$$= \langle 13, -16, -5\rangle \quad \text{calc} \checkmark$$

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12); \quad x = 4t, \quad y = -2t, \quad z = 2t$

34. $(0, 0, 0); \quad x = 5 + 3t, \quad y = 5 + 4t, \quad z = -3 - 5t$

35. $(2, 1, 3); \quad x = 2 + 2t, \quad y = 1 + 6t, \quad z = 3$

36. $(2, 1, -1); \quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$

37. $(3, -1, 4); \quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

38. $(-1, 4, 3); \quad x = 10 + 4t, \quad y = -3, \quad z = 4t$

Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \qquad (5)$$

$$|\vec{PS} \times \vec{v}| = \sqrt{13^2 + 16^2 + 5^2} = \sqrt{169 + 256 + 25} = \sqrt{450}$$

$$\begin{array}{r} 169 \\ 256 \\ +\ 25 \\ \hline 450 \end{array}$$

$$|\vec{v}| = \sqrt{9 + 16 + 25} = \sqrt{50} \quad \text{so} \quad d = \sqrt{\frac{450}{50}} = \sqrt{9} = 3$$

use calculus?

minimize $d(t) = \sqrt{(5+3t)^2 + (5+4t)^2 + (-3-5t)^2}$

$\Leftrightarrow$ minimize $\tilde{d}(t) = (5+3t)^2 + (5+4t)^2 + (-3-5t)^2$

next $\tilde{d}'(t) = 2(5+3t)\cdot 3 + 2(5+4t)\cdot 4 + 2(-3-5t)\cdot(-5)$

$= 30 + 18t + 40 + 32t + 30 + 50t = 0$

$\tilde{d}'(t) = 100 + 100t = 0$

$\Rightarrow t = -1$

$\tilde{d}(-1) = 2^2 + 1^2 + 2^2 = 9$

$\Rightarrow d(-1) = \sqrt{9} = 3$ ✓

#35. $S(2,1,3)$ $P(2,1,3)$ $\vec{v} = \langle 2,6,0 \rangle$!? lol

$\vec{PS} = \langle 0,0,0 \rangle$ $\vec{PS} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 0 & 0 \\ 2 & 6 & 0 \end{vmatrix} = 0$

So distance is 0

Distances

In Exercises 33–38, find the distance from the point to the line.

33. (0, 0, 12); $x = 4t$, $y = -2t$, $z = 2t$

34. (0, 0, 0); $x = 5 + 3t$, $y = 5 + 4t$, $z = -3 - 5t$

35. (2, 1, 3); $x = 2 + 2t$, $y = 1 + 6t$, $z = 3$

36. (2, 1, −1); $x = 2t$, $y = 1 + 2t$, $z = 2t$

37. (3, −1, 4); $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$

38. (−1, 4, 3); $x = 10 + 4t$, $y = -3$, $z = 4t$

Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$

#36 $S(2,1,-1)$ $P(0,1,0)$ $\vec{v} = \langle 2,2,2 \rangle$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = \langle 2, -(4+2), 4 \rangle = \langle 2,-6,4 \rangle$$

$|\vec{v}| = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$ $|\vec{PS} \times \vec{v}| = \sqrt{4+36+16} = \sqrt{56} = 2\sqrt{14}$

So $d = \frac{2\sqrt{14}}{2\sqrt{3}} = \sqrt{14/3} \approx 2.16$

In Exercises 39–44, find the distance from the point to the plane.

39. (2, −3, 4), $x + 2y + 2z = 13$

40. (0, 0, 0), $3x + 2y + 6z = 6$

41. (0, 1, 1), $4y + 3z = -12$

42. (2, 2, 3), $2x + y + 2z = 4$

43. (0, −1, 0), $2x + y + 2z = 4$

44. (1, 0, −1), $-4x + y + z = 4$

45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

46. Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

The Distance from a Point to a Plane

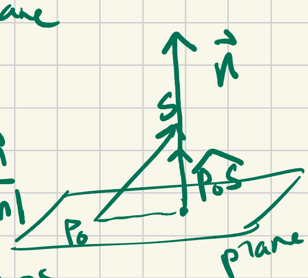
If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\vec{PS}$ onto $\mathbf{n}$. That is, the distance from S to the plane is

$$d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

#39. $S(2,-3,4)$ $x+2y+2z=13$ $\vec{n} = \langle 1,2,2 \rangle$

Find $P_0(x_0,y_0,z_0)$ on the plane
e.g. $P_0(13,0,0)$



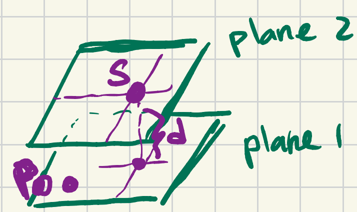
$$\widehat{P_0S} = \text{proj}_{\vec{n}}(\vec{P_0S}) = \underbrace{\left(\vec{P_0S} \cdot \frac{\vec{n}}{|\vec{n}|}\right)}_{d} * \frac{\vec{n}}{|\vec{n}|}$$

$$\pm d = \frac{1}{\sqrt{9}} \langle -11,-3,4 \rangle \cdot \langle 1,2,2 \rangle$$

$$= \frac{1}{3}(-11-6+8) = \frac{1}{3}(-9) = -3 \qquad d = |-3| = 3$$

#45 $x+2y+6z=1$ plane 1, $x+2y+6z=10$ plane 2

$P_0(1,0,0)$, $S = (10,0,0)$, $n = \langle 1,2,6 \rangle$



$$\pm d = \frac{1}{\sqrt{41}} \langle -9,0,0 \rangle \cdot \langle 1,2,6 \rangle$$

$$= \frac{1}{\sqrt{41}}(-9) = -9/\sqrt{41} \qquad d = 9/\sqrt{41}$$

if they don't intersect, then they are parallel.

$\vec{n} = \langle 1,2,6 \rangle$ for both ✓

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

#47. $x+y=1 \quad 2x+y-2z=2$

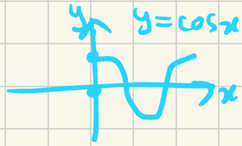
$\vec{n}_1 = \langle 1,1,0 \rangle \quad \vec{n}_2 = \langle 2,1,-2 \rangle$

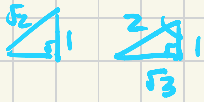
$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$\theta = \cos^{-1}\left(\frac{\langle 1,1,0\rangle \cdot \langle 2,1,-2\rangle}{\sqrt{2} * \sqrt{9}}\right) = \cos^{-1}(3/6)$$

$$\theta = \cos^{-1}(1/2) = 60^\circ$$

or $\boxed{\pi/3}$





$\theta = 60^\circ$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

54. $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

55. $x = 1 + 2t, \quad y = 1 + 5t, \quad z = 3t; \quad x + y + z = 2$

56. $x = -1 + 3t, \quad y = -2, \quad z = 5t; \quad 2x - 3z = 7$

#53 $\begin{cases} x = 1-t \\ y = 3t \\ z = 1+t \end{cases}$ line, $2x - y + 3z = 6$ plane

Find t s.t $P_0(x_0, y_0, z_0)$ is on both line and plane.

So,

$2(1-t) - (3t) + 3(1+t) = 6 \Rightarrow 2 - 2t - 3t + 3 + 3t = 6$

$\Rightarrow 5 - 2t = 6 \Rightarrow 2t = -1, \; \underline{\underline{t = -1/2}}$.

plug in $t = -1/2$ into line

$\boxed{P_0(3/2, -3/2, 1/2)}$ on both line & plane.

Check plane

$3 + 3/2 + 3/2 = 6$ ✓

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, \quad x + y = 2$

58. $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$

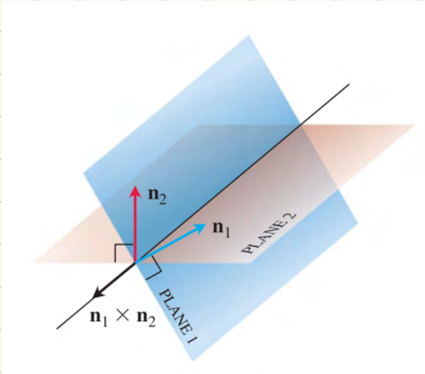
59. $x - 2y + 4z = 2, \quad x + y - 2z = 5$

60. $5x - 2y = 11, \quad 4y - 5z = -17$

#57 $x+y+z=1$ plane1, $x+y=2$ plane2

$\vec{n}_1 = \langle 1,1,1 \rangle, \; \vec{n}_2 = \langle 1,1,0 \rangle$

$$\vec{u} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, -(-1), 0 \rangle = \langle -1, 1, 0 \rangle$$



So line is in direction of $\vec{u}$ & passes through $P_0(x_0, y_0, z_0)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

t

$x = 2 - t$

$y = t$ (free)

$z = -1$

$P_0(2, 0, -1)$ & $\vec{u} = \langle -1, 1, 0 \rangle$

line of int. $\boxed{\begin{cases} x = 2-t \\ y = t \\ z = -1 \end{cases}}$

§12.6

1. d circle when z fixed

2. i

3. a doesn't depend on x

4. g circle for $x \neq 0$.

5. l $x = -z^2 + \text{const}$ when $y = \text{const}$

6. e circle when x fixed, $x < 0$

7. b doesn't depend on y

8. j

9. k

10. f oval when z fixed, $z < 0$

11. h circle for fixed $y \neq 0$

12. c oval when $x, y,$ or z fixed

Exercises 12.6

Matching Equations with Surfaces

In Exercises 1–12, match the equation with the surface it defines. Also, identify each surface by type (paraboloid, ellipsoid, etc.). The surfaces are labeled (a)–(l).

1. $x^2 + y^2 + 4z^2 = 10$
2. $z^2 + 4y^2 - 4x^2 = 4$
3. $9y^2 + z^2 = 16$
4. $y^2 + z^2 = x^2$
5. $x = y^2 - z^2$
6. $x = -y^2 - z^2$
7. $x^2 + 2z^2 = 8$
8. $z^2 + x^2 - y^2 = 1$
9. $x = z^2 - y^2$
10. $z = -4x^2 - y^2$
11. $x^2 + 4z^2 = y^2$
12. $9x^2 + 4y^2 + 2z^2 = 36$