§13.1

Exercises 13.1

Motion in the Plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j}, \quad t = 1$
2. $\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = -\frac{1}{2}$
3. $\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$
4. $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}, \quad t = 0$

#1. $r(t) = \langle t+1, t^2-1 \rangle$ at $t=1$.

eqn. $x = t+1 \Rightarrow t = 1-x$
$y = t^2 - 1$

So $y = (1-x)^2 + 1 = x^2 - 2x + 1 - 1 \Rightarrow y = x^2 - 2x$

$r'(t) = \langle 1, 2t \rangle = \vec{i} + 2t\vec{j}$ @ $t=1$ $r'(1) = \vec{i} + 2\vec{j}$

$r''(t) = \langle 0, 2 \rangle = 2\vec{j}$ @ $t=1$ $r''(1) = 2\vec{j}$

#2 $r(t) = \left\langle \frac{t}{t+1}, \frac{1}{t} \right\rangle$ $t = -\frac{1}{2}$

eqn. $x = \frac{t}{t+1}$, $y = \frac{1}{t} \rightarrow t = \frac{1}{y}$

So $x = \frac{1/y}{1/y+1} = \frac{1}{y} \cdot \frac{1}{1/y+1} = \frac{1}{1+y}$

So $1+y = \frac{1}{x} \Rightarrow y = \frac{1}{x} - 1$

$r'(t) = \frac{d}{dt}\left(\frac{t}{t+1}\right)\vec{i} + \frac{d}{dt}\left(\frac{1}{t}\right)\vec{j}$

$= \frac{(t+1)-t}{(t+1)^2}\vec{i} + \frac{-1}{t^2}\vec{j}$

$= \frac{1}{(t+1)^2}\vec{i} - \frac{1}{t^2}\vec{j}$

$r''(t) = \frac{-2}{(t+1)^3}\vec{i} + \frac{2}{t^3}\vec{j}$

@ $t = -1/2$ $r'(-\frac{1}{2}) = \frac{1}{(1/2)^2}\vec{i} - \frac{1}{(1/4)}\vec{j}$

$= 4\vec{i} - 4\vec{j}$

$r''(-\frac{1}{2}) = \frac{-2}{(1/2)^3}\vec{i} + \frac{2}{(-1/2)^3}\vec{j} = -16\vec{i} - 16\vec{j}$

#3 $r(t) = \langle e^t, \frac{2}{9}e^{2t} \rangle$ $t = \ln 3$

eqn. $x = e^t$, $y = \frac{2}{9}e^{2t} = \frac{2}{9}(e^t)^2 = \frac{2}{9}x^2$ so $y = \frac{2}{9}x^2$

$r'(t) = \langle e^t, \frac{2}{9} \cdot 2e^{2t} \rangle$ @ $t = \ln 3$ $r''(\ln 3) = \langle 3, 4 \rangle$

$r''(t) = \langle e^t, \frac{2}{9} \cdot 4e^{2t} \rangle$ $r''(\ln 3) = \langle 3, 8 \rangle$

note $e^{\ln 3} = 3$

so $e^{2\ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

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1. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j}, \quad t = 1$
2. $\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = -\frac{1}{2}$
3. $\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$
4. $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}, \quad t = 0$

#4. $r(t) = \langle \cos 2t, 3\sin 2t\rangle \quad t=0$

eqn. $x = \cos 2t$
$y = 3\sin 2t$

& $\cos^2\theta + \sin^2\theta = 1$ so

$\cos^2 2t + \sin^2 2t = 1$

& $(\cos 2t)^2 + \frac{1}{9}(3\sin 2t)^2 = 1 \Rightarrow x^2 + \frac{1}{9}y^2 = 1.$

$r'(t) = \langle -2\sin 2t, 6\cos 2t\rangle$ @ $t=0$ $\quad r'(0) = \langle 0, 6\rangle$

$r''(t) = \langle -4\cos 2t, -12\sin 2t\rangle \quad r''(0) = \langle -4, 0\rangle$

Motion in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$
10. $\mathbf{r}(t) = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t = 1$
11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$
12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k}, \quad t = \pi/6$
13. $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad t = 1$
14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$

In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

15. $\mathbf{r}(t) = (3t+1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$
16. $\mathbf{r}(t) = \left(\frac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$
17. $\mathbf{r}(t) = (\ln(t^2+1))\mathbf{i} + (\tan^{-1}t)\mathbf{j} + \sqrt{t^2+1}\,\mathbf{k}$
18. $\mathbf{r}(t) = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k}$

#9 $r(t) = \langle t+1, t^2-1, 2t\rangle \quad t=1$

$r'(t) = \langle 1, 2t, 2\rangle$ velocity vector $\frac{dr}{dt} = r'(t)$

$r''(t) = \langle 0, 2, 0\rangle$ acc. vector $\frac{d^2r}{dt^2} = r''(t)$

direction of motion at $t=t_0$ is $\frac{r'(t_0)}{|r'(t_0)|} = \frac{\vec{v}}{|\vec{v}|}$

at $t=1$

$\frac{r'(1)}{|r'(1)|} = \frac{1}{\sqrt{9}}\langle 1,2,2\rangle = \langle 1/3, 2/3, 2/3\rangle$

and speed is $|r'(1)| = 3$

& so velocity vector is (speed) $*$ (direction of motion)

$3 * \langle 1/3, 2/3, 2/3\rangle$

#10. $r(t) = \langle 1+t, t^2/\sqrt{2}, t^3/3\rangle \quad t=1$

vel. $r'(t) = \langle 1, \sqrt{2}t, t^2\rangle$ @ $t=1$ direction of motion is $\frac{r'(t_0)}{|r'(t_0)|} = \frac{1}{\sqrt{4}}\langle 1\sqrt{2}, 1\rangle$

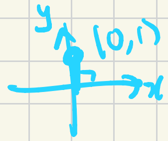
acc. $r''(t) = \langle 0, \sqrt{2}, 2t\rangle$

d.o.m. $= \langle 1/2, 1/\sqrt{2}, 1/2\rangle$

speed $= 2$

vel. vector is $2 * \langle 1/2, 1/\sqrt{2}, 1/2\rangle$

#11 $r(t) = \langle 2\cos t, 3\sin t, 4t \rangle \quad t = \pi/2$



vel. $r'(t) = \langle -2\sin t, 3\cos t, 4 \rangle$

acc. $r''(t) = \langle -2\cos t, -3\sin t, 0 \rangle$

@ $t = \pi/2$ d.o.m. is $\dfrac{r'(\pi/2)}{|r'(\pi/2)|} = \dfrac{1}{\sqrt{20}} \langle -2, 0, 4 \rangle$

$= \dfrac{1}{2\sqrt{5}} \langle -2, 0, 4 \rangle$

d.o.m. is $\langle -1/\sqrt{5}, 0, 2/\sqrt{5} \rangle$

speed is $2\sqrt{5}$

vel. vector $2\sqrt{5} * \langle -1/\sqrt{5}, 0, 2/\sqrt{5} \rangle$

Motion in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$

10. $\mathbf{r}(t) = (1+t)\mathbf{i} + \dfrac{t^2}{\sqrt{2}}\mathbf{j} + \dfrac{t^3}{3}\mathbf{k}, \quad t = 1$

11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$

12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \dfrac{4}{3}t\mathbf{k}, \quad t = \pi/6$

13. $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}, \quad t = 1$

14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$

In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

15. $\mathbf{r}(t) = (3t+1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$

16. $\mathbf{r}(t) = \left(\dfrac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\dfrac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$

17. $\mathbf{r}(t) = (\ln(t^2+1))\mathbf{i} + (\tan^{-1} t)\mathbf{j} + \sqrt{t^2+1}\,\mathbf{k}$

18. $\mathbf{r}(t) = \dfrac{4}{9}(1+t)^{3/2}\mathbf{i} + \dfrac{4}{9}(1-t)^{3/2}\mathbf{j} + \dfrac{1}{3}t\mathbf{k}$

$\sin^2\theta + \cos^2\theta = 1$
$1 + \sec^2\theta = \tan^2\theta$

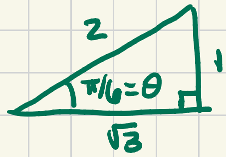
#12. $r(t) = \langle \sec t, \tan t, \frac{4}{3}t \rangle \quad t = \pi/6$

$(\sec t \tan t)' = \sec t(\sec^2 t) + (\sec t \tan t)\tan t$
$= \sec^3 t + \sec t \tan^2 t$

vel. $r'(t) = \langle \sec t \tan t, \sec^2 t, 4/3 \rangle$

acc. $r''(t) = \langle \sec^3 t + \sec t \tan^2 t, 2\sec^2 t \tan t, 0 \rangle$

@ $t = \pi/6$



$\sec\theta = \dfrac{hyp}{adj} = \dfrac{2}{\sqrt{3}}$

$\tan\theta = \dfrac{opp}{adj} = \dfrac{1}{\sqrt{3}}$

$r'(\frac{\pi}{6}) = \langle (\frac{2}{\sqrt{3}})(\frac{1}{\sqrt{3}}), (\frac{2}{\sqrt{3}})^2, 4/3 \rangle$

$= \langle \frac{2}{3}, \frac{4}{3}, \frac{4}{3} \rangle$

$|r'(\pi/6)| = \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{16}{9}} = \sqrt{\frac{36}{9}} = 6/3$

$= 2$

d.o.m is $\dfrac{r'(\pi/6)}{|r'(\pi/6)|} = \langle 1/3, 2/3, 2/3 \rangle$

speed is 2, $\vec{v} = r'(\pi/6) = 2 * \langle 1/3, 2/3, 2/3 \rangle$

#14. $r\langle t \rangle = \langle e^{-t}, 2\cos 3t, 2\sin 3t \rangle \quad t = 0$

vel. $r'(t) = \langle -e^{-t}, -6\sin 3t, 6\cos 3t \rangle$

acc. $r''(t) = \langle e^{-t}, -18\cos 3t, -18\cos 3t \rangle$

@ $t = 0$ $r'(0) = \langle -1, 0, 6 \rangle$

$|r'(0)| = \sqrt{37}$

d.o.m is $\langle -1/\sqrt{37}, 0, 6/\sqrt{37} \rangle$

speed is $\sqrt{37}$

$\vec{v} = r'(0) = \sqrt{37} * \langle -1/\sqrt{37}, 0, 6/\sqrt{37} \rangle$

Motion in Space

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10. $\mathbf{r}(t) = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t = 1$

11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$

12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k}, \quad t = \pi/6$

13. $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad t = 1$

14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$

In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

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17. $\mathbf{r}(t) = (\ln(t^2+1))\mathbf{i} + (\tan^{-1}t)\mathbf{j} + \sqrt{t^2+1}\,\mathbf{k}$

18. $\mathbf{r}(t) = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k}$

The Angle Between Two Nonzero Vectors u and v

§12.3 p.719 $\quad \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

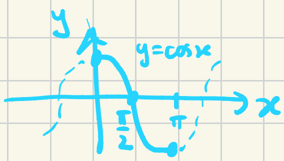
#15. $r(t) = \langle 3t+1, \sqrt{3}t, t^2 \rangle \quad t=0$

$r'(t) = \langle 3, \sqrt{3}, 2t \rangle \qquad \vec{v} = r'(0) = \langle 3, \sqrt{3}, 0 \rangle$

$r''(t) = \langle 0, 0, 2 \rangle \qquad \vec{a} = r''(0) = \langle 0, 0, 2 \rangle$

$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}| * |\vec{a}|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{16}\cdot 2}\right)$

$= \cos^{-1}(0)$



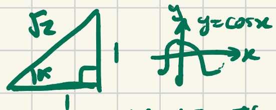
$\boxed{= \pi/2}$

#16. $r(t) = \langle \frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t - 16t^2, 0 \rangle$ $\quad$ r(t) "in space"

$r'(t) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - 32t, 0 \rangle \qquad \vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$

$r''(t) = \langle 0, -32, 0 \rangle \qquad \vec{a} = \langle 0, -32, 0 \rangle$

$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}| * |\vec{a}|}\right) = \cos^{-1}\left(\frac{-32/\sqrt{2}}{1 * 32}\right)$



$\theta = \cos^{-1}(-1/\sqrt{2}) = \boxed{-\pi/4}$

$\alpha = 45° = \pi/4$

#17. $r(t) = \langle \ln(t^2+1), \tan^{-1}t, \sqrt{t^2+1} \rangle \quad t=0$

$r'(t) = \langle \frac{2t}{t^2+1}, \frac{1}{t^2+1}, \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2+1}} \rangle$

$r''(t) = \langle \frac{-2t^2+2}{(t^2+1)^2}, \frac{-2t}{(t^2+1)^2}, \frac{t^2-t+1}{(t^2+1)^{3/2}} \rangle$

$\left(\frac{2t}{t^2+1}\right)' = \frac{(t^2+1)2 - 2t(2t)}{(t^2+1)^2}$

$= \frac{2t^2+2-4t^2}{(t^2+1)^2} = \frac{-2t^2+2}{(t^2+1)^2}$

$\left(\frac{t}{\sqrt{t^2+1}}\right)' = \frac{\sqrt{t^2+1}(1) - t\cdot\frac{1}{\sqrt{t^2+1}}}{(\sqrt{t^2+1})^2}$

$= \frac{t^2-t+1}{(\sqrt{t^2+1})^3}$

$\vec{v} = r'(0) = \langle 0, 1, 0 \rangle$

$\vec{a} = r''(0) = \langle 2, 0, 1 \rangle$

$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|}\right)$

$\theta = \cos^{-1} 0 = \boxed{\pi/2}$

#18. $r(t) = \langle \frac{4}{9}(1+t)^{3/2}, \frac{4}{9}(1-t)^{3/2}, \frac{1}{3}t \rangle \quad t=0$

$r'(t) = \langle \frac{2}{3}(1+t)^{1/2}, \frac{-2}{3}(1-t)^{1/2}, \frac{1}{3} \rangle \qquad @\ t=0 \qquad \vec{v} = \langle 2/3, -2/3, 1/3 \rangle$

$r''(t) = \langle \frac{1}{3}(1+t)^{-1/2}, \frac{+1}{3}(1-t)^{-1/2}, 0 \rangle \qquad \vec{a} = \langle 1/3, 1/3, 0 \rangle$

lol

$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|}\right) = \cos^{-1}\left(\frac{2/9 - 2/9}{1 * \frac{1}{3\sqrt{2}}}\right) = \cos^{-1}(0) = \boxed{\pi/2}$

Tangents to Curves

As mentioned in the text, the **tangent line** to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0. In Exercises 19–22, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

19. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

20. $\mathbf{r}(t) = t^2\mathbf{i} + (2t - 1)\mathbf{j} + t^3\mathbf{k}, \quad t_0 = 2$

21. $\mathbf{r}(t) = \ln t\,\mathbf{i} + \frac{t-1}{t+2}\mathbf{j} + t\ln t\,\mathbf{k}, \quad t_0 = 1$

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \frac{\pi}{2}$

#19. $r(t) = \langle \sin t,\ t^2 - \cos t,\ e^t \rangle \quad t_0 = 0$

$r'(t) = \langle \cos t,\ 2t + \sin t,\ e^t \rangle$

$\vec{v} = r'(t_0) = \langle 1, 0, 1 \rangle$

$P_0(0, -1, 1)$ ie $r(t_0)$

So tangent line is $\boxed{\begin{cases} x = t \\ y = -1 \\ z = 1 + t \end{cases}}$ line through P_0 and parallel to $\vec{v}$

#20. $r(t) = \langle t^2, 2t - 1, t^3 \rangle \quad t_0 = 2$

$r'(t) = \langle 2t, 2, 3t^2 \rangle$

$\vec{v} = r'(t_0) = \langle 4, 2, 12 \rangle$

$P_0(4, 3, 8)$

So line is $\boxed{\begin{cases} x = 4 + 4t \\ y = 3 + 2t \\ z = 8 + 12t \end{cases}}$

#21. $r(t) = \langle \ln t, \frac{t-1}{t+2}, t\ln t \rangle \quad t_0 = 1$

$r'(t) = \langle \frac{1}{t}, \frac{3}{(t+2)^2}, \ln t + 1 \rangle$

$\vec{v} = r'(t_0) = \langle 1, \frac{1}{3}, 1 \rangle$

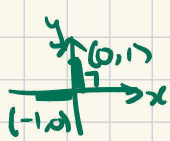
$P_0(0, 0, 0)$

$\left(\frac{t-1}{t+2}\right)' = \frac{(t+2)(1) - (t-1)(1)}{(t+2)^2} = \frac{3}{(t+2)^2}$

$(t\ln t)' = (1)\ln t + t \cdot \frac{1}{t} = \ln t + 1$

So line is $\boxed{\begin{cases} x = t \\ y = \frac{1}{3}t \\ z = t \end{cases}}$

#22 $r(t) = \langle \cos t, \sin t, \sin 2t \rangle \quad t_0 = \pi/2$



$r'(t) = \langle -\sin t, \cos t, 2\cos 2t \rangle$

$\vec{v} = r'(t_0) = \langle -1, 0, -2 \rangle$

$P_0(0, 1, 0)$

So line is $\boxed{\begin{cases} x = -t \\ y = 1 \\ z = -2t \end{cases}}$

Theory and Examples

23. Motion along a circle Each of the following equations in parts (a)–(e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in parts (a)–(e) is the same, the behavior, or "dynamics," of each particle is different. For each particle, answer the following questions.

i) Does the particle have constant speed? If so, what is its constant speed?

ii) Is the particle's acceleration vector always orthogonal to its velocity vector?

iii) Does the particle move clockwise or counterclockwise around the circle?

iv) Does the particle begin at the point (1, 0)?

a. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \ge 0$

b. $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}, \quad t \ge 0$

c. $\mathbf{r}(t) = \cos(t - \pi/2)\mathbf{i} + \sin(t - \pi/2)\mathbf{j}, \quad t \ge 0$

d. $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad t \ge 0$

e. $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}, \quad t \ge 0$

16

#23(a) $r(t) = \langle \cos t, \sin t \rangle \quad t \ge 0$

$r'(t) = \langle -\sin t, \cos t \rangle$

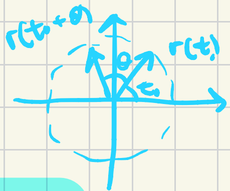
speed is $|r'(t)| = (-\sin t)^2 + \cos^2 t \equiv 1$ ✓ (i) yes

$r''(t) = \langle -\cos t, -\sin t \rangle$

$r'(t_0) \cdot r''(t_0) = 0$ for every $t = t_0$

b/c $(-\sin t)(-\cos t) + (\cos t)(-\sin t) \equiv 0$ ✓ (ii) yes

(iii) CCW



$r(0) = \langle 1, 0 \rangle$ (iv) yes

#23(d) $r(t) = \langle \cos t, -\sin t \rangle \quad t \ge 0$

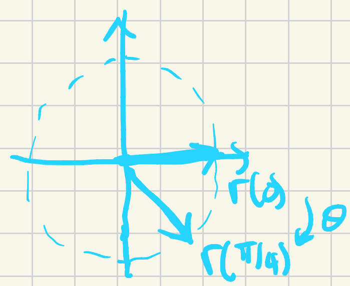
$r'(t) = \langle -\sin t, -\cos t \rangle \qquad |r'(t)| = (-\sin t)^2 + (-\cos t)^2 \equiv 1$ for all t (i) yes

$r''(t) = \langle -\cos t, \sin t \rangle \qquad \vec{a} \cdot \vec{v} = r'(t) \cdot r''(t) = \sin t \cos t - \cos t \sin t \equiv 0$

(ii) yes

$r(0) = \langle 1, 0 \rangle$

$r(\pi/4) = \langle \cos \pi/4, -\sin \pi/4 \rangle = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$



(iii) CW

(iv) yes

#23(e) $r(t) = \langle \cos(t^2), \sin(t^2) \rangle \quad t \ge 0$

$r'(t) = \langle -2t\sin(t^2), 2t\cos(t^2) \rangle \qquad |r'(t)| = 4t^2 \sin^2 t^2 + 4t^2 \cos^2 t^2 = 4t^2 * 1$

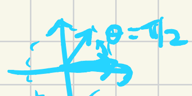
$r''(t) = \langle -2\sin(t^2) - 4t^2\cos(t^2), 2\cos(t^2) - 4t^2\sin(t^2) \rangle$ (i) NO

$\vec{a} \cdot \vec{v} = r'(t) \cdot r''(t) = -2t\sin t^2 (-2\sin t^2 - 4t^2\cos t^2) + 2t\cos t^2 (2\cos t^2 - 4t^2 \sin t^2)$

$= -4t\sin^2 t^2 + \cancel{8t^3 \sin t^2 \cos t^2} + 4t\cos^2 t^2 - \cancel{8t^3 \sin t^2 \cos t^2} \ne 0$ for some t's

e.g. $t = 0$

(ii) NO



$r(0) = \langle 1, 0 \rangle$

$r(\pi/2) = \langle \cos \pi/4, \sin \pi/4 \rangle = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

(iii) CCW

(iv) yes

§13.2

Exercises 13.2

Integrating Vector-Valued Functions

Evaluate the integrals in Exercises 1–10.

1. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt$
2. $\int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right]dt$
3. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1+\cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}]\,dt$
4. $\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$
5. $\int_1^4 \left[\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right]dt$
6. $\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right]dt$
7. $\int_0^1 [te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}]\,dt$
8. $\int_1^{\ln 3} [te^t\mathbf{i} + e^t\mathbf{j} + \ln t\,\mathbf{k}]\,dt$
9. $\int_0^{\pi/2} [\cos t\,\mathbf{i} - \sin 2t\,\mathbf{j} + \sin^2 t\,\mathbf{k}]\,dt$
10. $\int_0^{\pi/4} [\sec t\,\mathbf{i} + \tan^2 t\,\mathbf{j} - t\sin t\,\mathbf{k}]\,dt$

#1 $r(t) = \langle t^3, 7, t+1\rangle$

$$\int_0^1 r(t)\,d = \left\langle \tfrac{1}{4}t^4, 7t, \tfrac{1}{2}t^2+t\right\rangle\Big|_0^1$$

$$= \left\langle \tfrac{1}{4}-0,\ 7-0,\ \left(\tfrac{1}{2}+1\right)-0\right\rangle = \boxed{\left\langle \tfrac{1}{4}, 7, \tfrac{3}{2}\right\rangle}$$

#2. $r(t) = \langle 6-6t, 3\sqrt{t}, \frac{4}{t^2}\rangle$

$\int t^{1/2}\,dt = \frac{2}{3}t^{3/2}+C$

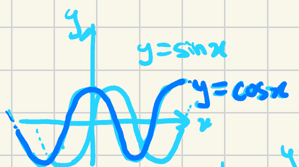
$\int t^{-2}\,dt = -t^{-1}+C$

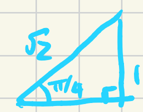
$$\int_1^2 r(t)\,dt = \left\langle 6t-3t^2,\ \tfrac{2}{3}\cdot 3t^{3/2},\ -\tfrac{4}{t}\right\rangle\Big|_1^2$$

$$= \left\langle (12-12)-(6-3),\ 2\cdot\sqrt{8}-2,\ -2-(-4)\right\rangle$$

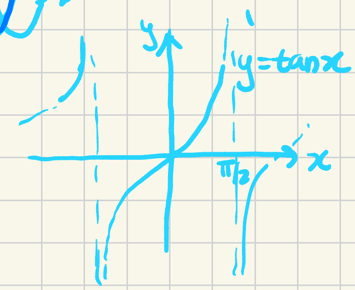
$$= \boxed{\langle -3, 2\sqrt{8}-2, 2\rangle}$$

#3. $r(t) = \langle \sin t, 1+\cos t, \sec^2 t\rangle$





$$\int_{-\pi/4}^{\pi/4} r(t)\,dt = \langle -\cos t,\ t+\sin t,\ \tan t\rangle\Big|_{-\pi/4}^{\pi/4}$$



$$= \left\langle -\tfrac{1}{\sqrt{2}} - \tfrac{-1}{\sqrt{2}},\ \left(\tfrac{\pi}{4}+\tfrac{1}{\sqrt{2}}\right)-\left(\tfrac{-\pi}{4}-\tfrac{1}{\sqrt{2}}\right),\ 1-(-1)\right\rangle$$

$$= \left\langle 0,\ 2\cdot\tfrac{\pi}{4}+2/\sqrt{2},\ 2\right\rangle = \boxed{\left\langle 0, \tfrac{\pi}{2}+\sqrt{2}, 2\right\rangle}$$

#4. $r(t) = \langle \sec t\tan t, \tan t, 2\sin t\cos t\rangle$

$$\int \tan t\,dt = \int \frac{\sin t}{\cos t}\,dt \qquad \boxed{\begin{array}{l} u = \cos t \\ du = -\sin t\,dt\end{array}}$$

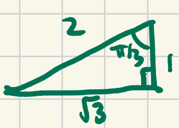
$$= \int -\tfrac{1}{u}\,du = -\ln|u|+C$$

$$= -\ln|\cos t|+C$$

$$= \ln|\sec t|+C$$

$$\int_0^{\pi/3} r(t)\,dt = \langle \sec t,\ \ln|\sec t|,\ \sin^2 t\rangle\Big|_0^{\pi/3}$$

$$= \left\langle 2-1,\ \ln 1,\ (\sqrt{3}/2)^2 - 0\right\rangle$$



$\theta = \frac{\pi}{3}$

$$\boxed{= \langle 1, 0, 3/4\rangle}$$

$$\boxed{\begin{array}{l} u = \sin t \\ du = \cos t\,dt\end{array}}$$

$$\int \sin t\cos t\,dt = \int u\,du = \tfrac{1}{2}u^2+C$$

$$= \tfrac{1}{2}\sin^2 t + C$$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{2}{1} = 2 \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

Exercises 13.2

Integrating Vector-Valued Functions

Evaluate the integrals in Exercises 1–10.

1. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt$
2. $\int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right]dt$
3. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1+\cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}]\,dt$
4. $\int_0^{\pi/3} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$
5. $\int_1^4 \left[\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right]dt$
6. $\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right]dt$
7. $\int_0^1 [te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}]\,dt$
8. $\int_1^{\ln 3} [te^t\mathbf{i} + e^t\mathbf{j} + \ln t\,\mathbf{k}]\,dt$
9. $\int_0^{\pi/2} [\cos t\,\mathbf{i} - \sin 2t\,\mathbf{j} + \sin^2 t\,\mathbf{k}]\,dt$
10. $\int_0^{\pi/4} [\sec t\,\mathbf{i} + \tan^2 t\,\mathbf{j} - t\sin t\,\mathbf{k}]\,dt$

#5 $r(t) = \langle \frac{1}{t}, \frac{1}{5-t}, \frac{1}{2t} \rangle$

$$\int_1^4 r(t)\,dt = \langle \ln|t|, -\ln|5-t|, \tfrac{1}{2}\ln|t| \rangle \Big|_1^4$$

$\frac{1}{2}\ln(4) = \ln(4^{1/2}) = \ln 2$

$$= \langle \ln 4 - 0, -0-(-\ln 4), \tfrac{1}{2}\ln 4 - \tfrac{1}{2}\cdot 0 \rangle$$

$$= \langle \ln 4, \ln 4, \ln 2 \rangle$$

#6. $r(t) = \langle \frac{2}{\sqrt{1-t^2}}, \frac{\sqrt{3}}{1+t^2} \rangle$

$\int \frac{1}{\sqrt{1-t^2}}\,dt = \sin^{-1} t + C$

$\int \frac{1}{1+t^2}\,dt = \tan^{-1} t + C$

$$\int_0^1 r(t)\,dt = \langle 2\sin^{-1} t, \sqrt{3}\tan^{-1} t \rangle \Big|_0^1$$

$\sin(\pi/2) = 1$

$\sin(0) = 0$

$$= \langle 2 * \tfrac{\pi}{2} - 0, \sqrt{3} * \tfrac{\pi}{4} - 0 \rangle$$

$\tan(\pi/4) = 1$

$\tan(0) = 0$

$$= \langle \pi, \sqrt{3}\pi/4 \rangle$$

#7. $r(t) = \langle te^{t^2}, e^{-t}, 1 \rangle$

$\int te^{t^2}\,dt = \frac{1}{2}e^{t^2} + C$

$\boxed{u = t^2 \quad du = 2t\,dt}$

$$\int_0^1 r(t)\,dt = \langle \tfrac{1}{2}e^{t^2}, -e^{-t}, t \rangle \Big|_0^1$$

$$= \langle \tfrac{1}{2}e - \tfrac{1}{2}, -\tfrac{1}{e} - (-1), 1-0 \rangle$$

$\boxed{u = t \quad dv = e^t\,dt \quad du = dt \quad v = e^t}$

$$= \langle \tfrac{1}{2}(e-1), 1 - \tfrac{1}{e}, 1 \rangle$$

$\int te^t\,dt = \underbrace{te^t - \int e^t\,dt}_{uv - \int v\,du} = te^t - e^t + C$

#8. $r(t) = \langle te^t, e^t, \ln t \rangle$

$\int \ln t\,dt = t\ln t - \int t * \frac{1}{t}\,dt = t\ln t - t + C$

$$\int_1^{\ln 3} r(t)\,dt$$

$$= \langle e^t(t-1), e^t, t(\ln t - 1) \rangle \Big|_1^{\ln 3}$$

$\boxed{u = \ln t \quad dv = dt \quad du = \frac{1}{t}\,dt \quad v = t}$

$$= \langle 3(\ln 3 - 1) - 0, 3 - e, \ln 3(\ln(\ln 3) - 1) - (1)(-1) \rangle$$

$$= \langle 3\ln 3 - 3, 3 - e, \ln 3 * \ln(\ln 3) - \ln 3 + 1 \rangle$$

#9. $r(t) = \langle \cos t, -\sin 2t, \sin^2 t \rangle$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

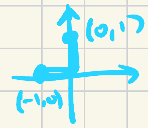
$$\int_0^{\pi/2} r(t)\,dt$$

So $\displaystyle\int \sin^2 t\,dt = \int \frac{1}{2} - \frac{1}{2}\cos 2t\,dt$

$$= \left\langle \sin t, \tfrac{1}{2}\cos 2t, \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t \right\rangle \Big|_0^{\pi/2}$$

$= \frac{1}{2}t - \frac{1}{4}\sin 2t + C$

$$= \left\langle 1-0, \tfrac{1}{2}(-1) - \tfrac{1}{2}(1), \left(\tfrac{\pi}{4} - 0\right) - (0-0) \right\rangle$$



$$= \boxed{\langle 1, -1, \pi/4 \rangle}$$

Exercises 13.2

Integrating Vector-Valued Functions

Evaluate the integrals in Exercises 1–10.

1. $\displaystyle\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt$
2. $\displaystyle\int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right]dt$
3. $\displaystyle\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1+\cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}]\,dt$
4. $\displaystyle\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$
5. $\displaystyle\int_1^4 \left[\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right]dt$
6. $\displaystyle\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right]dt$
7. $\displaystyle\int_0^1 [te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}]\,dt$
8. $\displaystyle\int_1^{\ln 3} [te^t\mathbf{i} + e^t\mathbf{j} + \ln t\,\mathbf{k}]\,dt$
9. $\displaystyle\int_0^{\pi/2} [\cos t\,\mathbf{i} - \sin 2t\,\mathbf{j} + \sin^2 t\,\mathbf{k}]\,dt$
10. $\displaystyle\int_0^{\pi/4} [\sec t\,\mathbf{i} + \tan^2 t\,\mathbf{j} - t\sin t\,\mathbf{k}]\,dt$

#10. $r(t) = \langle \sec t, \tan^2 t, t\sin t \rangle$

$\displaystyle\int \sec t\,dt = \int \sec t\left(\frac{\sec t + \tan t}{\sec t + \tan t}\right)dt$

$$\int_0^{\pi/4} r(t)\,dt$$

$\displaystyle= \int \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\,dt$

$u = \sec t + \tan t$
$du = \sec t\tan t + \sec^2 t\,dt$

$\displaystyle= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sec t + \tan t| + C$

$$= \left\langle \ln|\sec t + \tan t|, \tan t - t, -t\cos t + \sin t \right\rangle \Big|_0^{\pi/4}$$

$\displaystyle\int \tan^2 t\,dt = \int \sec^2 t - 1\,dt = \tan t - t + C$

$\sin^2\theta + \cos^2\theta = 1$
$\to \tan^2\theta + 1 = \sec^2\theta$

$$= \left\langle \ln(\sqrt{2}+1) - \ln(1+0), \left(1 - \tfrac{\pi}{4}\right) - 0, \left(-\tfrac{\pi}{4}\cdot\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - 0 \right\rangle$$

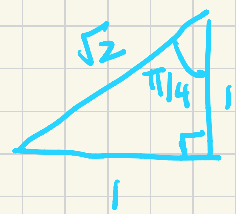
$$= \boxed{\left\langle \ln(\sqrt{2}+1), -3\pi/4, \tfrac{1}{4\sqrt{2}}(4-\pi) \right\rangle}$$

$\displaystyle\int t\sin t\,dt = -t\cos t - \int -\cos t\,dt$

$u = t \quad dv = \sin t\,dt$
$du = dt \quad v = -\cos t$

$= -t\cos t + \sin t + C$

$\theta = \pi/4$



$\sec\theta = \dfrac{hyp}{adj} = \sqrt{2}$

$\tan\theta = \dfrac{opp}{adj} = 1$

$\sec 0 = 1$
$\tan 0 = 0$

$\ln 1 = 0$

Initial Value Problems

Solve the initial value problems in Exercises 11–16 for $\mathbf{r}$ as a vector function of t.

11. Differential equation: $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

12. Differential equation: $\frac{d\mathbf{r}}{dt} = (180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}$

Initial condition: $\mathbf{r}(0) = 100\mathbf{j}$

13. Differential equation: $\frac{d\mathbf{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\mathbf{i} + e^{-t}\mathbf{j} + \frac{1}{t+1}\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{k}$

14. Differential equation: $\frac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

15. Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$

Initial conditions: $\mathbf{r}(0) = 100\mathbf{k}$ and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}$

#11. $\frac{dr}{dt} = \langle -t, -t, -t \rangle \quad r(0) = \langle 1, 2, 3 \rangle$

$f_n'(t) = -t \Rightarrow f_n(t) = -\frac{1}{2}t^2 + C, \quad n = 1,2,3$

$f_1(0) = 1 \Rightarrow C = 1$, sim $f_n(t) = -\frac{1}{2}t^2 + n, \ n = 1,2,3$

$$r(t) = \langle -\tfrac{1}{2}t^2 + 1, -\tfrac{1}{2}t^2 + 2, -\tfrac{1}{2}t^2 + 3 \rangle$$

#12. $\frac{dr}{dt} = \langle 180t, 180t - 16t^2 \rangle \quad r(0) = \langle 0, 100 \rangle$

$f'(t) = 180t \Rightarrow f(t) = 90t^2 + C$

since $f(0) = 0 \Rightarrow C = 0$, so $f(t) = 90t^2$

$g'(t) = 180t - 16t^2 \Rightarrow g(t) = 90t^2 - \frac{16}{3}t^3 + C, \quad g(0) = 100 \Rightarrow C = 100.$

$$r(t) = \langle 90t^2, 90t^2 - \tfrac{16}{3}t^3 + 100 \rangle$$

#13. $\frac{dr}{dt} = \langle (t+1)^{1/2}, e^{-t}, \frac{1}{t+1} \rangle \quad r(0) = \langle 0, 0, 1 \rangle$

$$r(t) = \tfrac{2}{3}(t+1)^{3/2}\vec{i} - e^{-t}\vec{j} + (\ln|t+1| + 1)\vec{k}$$

#14. $\frac{dr}{dt} = (t^3 + 4t)\vec{i} + t\vec{j} + 2t^2\vec{k} \quad r(0) = \vec{i} + \vec{j}.$

$$r(t) = (\tfrac{1}{4}t^4 + 2t^2 + 1)\vec{i} + (\tfrac{1}{2}t^2 + 1)\vec{j} + \tfrac{2}{3}t^3\vec{k}$$

#15. $\frac{d^2r}{dt^2} = -32\vec{k} \quad r(0) \overset{*}{=} 100\vec{k}$ and $\left.\frac{dr}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$

$\frac{dr}{dt} = c_1\vec{i} + c_2\vec{j} + (-32t + c_3)\vec{k} \qquad c_1 = 8, c_2 = 8, c_3 = 0.$

$r(t) = (8t + c_1)\vec{i} + (8t + c_2)\vec{j} + (-16t^2 + c_3)\vec{k}, \quad * \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 100 \end{cases}$

$$r(t) = 8t\vec{i} + 8t\vec{j} + (-16t^2 + 100)\vec{k}$$

Motion Along a Straight Line

17. At time $t = 0$, a particle is located at the point (1, 2, 3). It travels in a straight line to the point (4, 1, 4), has speed 2 at (1, 2, 3) and constant acceleration $3\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find an equation for the position vector $\mathbf{r}(t)$ of the particle at time t.

18. A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point (3, 0, 3) with constant acceleration $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

#17. $P_0(1,2,3)$, $P_1(4,1,4)$, $\left\|\left(\frac{dr}{dt}\Big|_{t=0}\right)\right\| = 2$

$\vec{v} = \langle 3,-1,1\rangle$ $\qquad \frac{d^2r}{dt^2} = 3\vec{i} - \vec{j} + \vec{k}$

$|\vec{v}| = \sqrt{9+1+1} = \sqrt{11}$

$\frac{dr}{dt}\Big|_{t=t_0} = C * \left\langle \frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right\rangle$, has length $\left|\frac{dr}{dt}\right| = C$.

$\frac{dr}{dt} = \int \frac{d^2r}{dt^2}\,dt = (3t+c_1)\vec{i} - (t+c_2)\vec{j} + (t+c_3)\vec{k}$, $\frac{dr}{dt}\Big|_{t=0}$ has length 2

So $\frac{dr}{dt} = \left(3t+\frac{6}{\sqrt{11}}\right)\vec{i} - \left(t+\frac{2}{\sqrt{11}}\right)\vec{j} + \left(t+\frac{2}{\sqrt{11}}\right)\vec{k}$

$r(t) = \int \frac{dr}{dt}\,dt = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + C_1\right)\vec{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + C_2\right)\vec{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + C_3\right)\vec{k}$

$r(0) = \langle 1,2,3\rangle$ so

$r(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\vec{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right) + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\vec{k}$

$= \underbrace{\left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)}_{s}\underbrace{\left[3\vec{i} - \vec{j} + \vec{k}\right]}_{\vec{v}} + \underbrace{\vec{i} + 2\vec{j} + 3\vec{k}}_{P_0(1,2,3)}$

$s \;*\; \vec{v} \;+\; P_0(1,2,3)$

speed * direction + starting location

#18 $P_0(1,-1,2)$ $\quad P_1(3,0,3)$ $\qquad \left\|\frac{dr}{dt}\Big|_{t=0}\right\| = 2$ $\qquad$ d.o.m. $\frac{1}{\sqrt{6}}\langle 2,1,1\rangle$

$\vec{v} = \langle 2,1,1\rangle$

$|\vec{v}| = \sqrt{6}$ $\qquad \frac{d^2r}{dt^2} = 2\vec{i} + \vec{j} + \vec{k}$

$\frac{dr}{dt} = \int \frac{d^2r}{dt^2}\,dt = (2t+c_1)\vec{i} + (t+c_2)\vec{j} + (t+c_3)\vec{k}$ and $\left\|\frac{dr}{dt}\Big|_{t=0}\right\| = 2$ so

$\langle c_1,c_2,c_3\rangle = 2 * \frac{1}{\sqrt{6}}\langle 2,1,1\rangle$

$r(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + c_1\right)\vec{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + c_2\right) + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + c_3\right)\vec{k}$

$r(0) = \langle 1,-1,2\rangle = \langle c_1,c_2,c_3\rangle$

$r(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\vec{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\vec{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\vec{k}$

$= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\langle 2,1,1\rangle + \langle 1,-1,2\rangle$

Formula

$r(t) = \left(\frac{1}{2}t^2 + \frac{s}{|\vec{v}|}t\right)\vec{v} + \vec{P_0}$

$s = \left\|\frac{dr}{dt}\Big|_{t=0}\right\|$ & $\vec{v} = \overrightarrow{P_0P_1}$

#1. $r(t) = \langle 2\cos t, 2\sin t, \sqrt{5}t \rangle, 0 \le \pi \le t$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ unit tangent vector, $\vec{v} = \frac{dr}{dt}$

§13.3

Exercises 13.3

Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$
6. $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \le t \le 2$
7. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$
8. $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \le t \le 2$
9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

$\vec{v} = \frac{dr}{dt} = \langle -2\sin t, 2\cos t, \sqrt{5} \rangle \qquad |\vec{v}|^2 = 4\sin^2 t + 4\cos^2 t + 5$

$= 4(1) + 5 = 9$

$$\vec{T} = \left\langle -\frac{2}{3}\sin t, \frac{2}{3}\cos t, \frac{\sqrt{5}}{3} \right\rangle$$

$$L = \int_0^{\pi} |\vec{v}|\, dt = \int_0^{\pi} \sqrt{9}\, dt = 3t\Big|_0^{\pi} = 3\pi$$

#2. $r(t) = \langle 6\sin 2t, 6\cos 2t, 5t \rangle \qquad 0 \le t \le \pi$

$\vec{v} = \frac{dr}{dt} = \langle 12\cos 2t, -12\sin 2t, 5 \rangle \qquad |\vec{v}|^2 = 144(\cos^2 2t + \sin^2 2t) + 25 = 169$

$|\vec{v}| = 13$

$$\vec{T} = \left\langle \frac{12}{13}\cos 2t, -\frac{12}{13}\sin 2t, \frac{5}{13} \right\rangle$$

$$L = \int_0^{\pi} |v|\, dt = \int_0^{\pi} 13\, dt = 13t\Big|_0^{\pi} = 13\pi$$

#3 $r(t) = \langle t, 0, \frac{2}{3}t^{3/2} \rangle \qquad 0 \le t \le 8$

$\vec{v} = \frac{dr}{dt} = \langle 1, 0, t^{1/2} \rangle \qquad |\vec{v}|^2 = 1 + t, \quad |v| = \sqrt{1+t}$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{1+t}}, 0, \frac{t^{1/2}}{(1+t)^{1/2}} \right\rangle$$

$\boxed{u = 1+t \quad du = dt}$

$$L = \int_0^8 |\vec{v}|\, dt = \int_0^8 \sqrt{1+t}\, dt = \int_*^* u^{1/2} du = \frac{2}{3}u^{3/2}\Big|_*^* = \frac{2}{3}(1+t)^{3/2}\Big|_0^8$$

$$= \frac{2}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{2}{3}(27 - 1) = 52/3$$

Exercises 13.3

Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$
6. $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \le t \le 2$
7. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$
8. $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \le t \le 2$
9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

#4. $r(t) = \langle 2+t, -(t+1), t\rangle \quad 0 \le t \le 3$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} \qquad \vec{v} = \frac{dr}{dt} = \langle 1, -1, 1\rangle \quad |\vec{v}| = \sqrt{3}$

$$\vec{T} = \langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\rangle$$

$$L = \int_0^3 |\vec{v}|\,dt = \sqrt{3}t\Big|_0^3 = 3\sqrt{3}$$

#5. $r(t) = \langle 0, \cos^3 t, \sin^3 t\rangle \quad 0 \le t \le \pi/2$

$\vec{v} = \langle 0, 3\cos^2 t(-\sin t), 3\sin^2 t(\cos t)\rangle$

$|\vec{v}|^2 = 9\cos^4 t\sin^2 t + 9\sin^4\cos^2 t = 9\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t) = 9\sin^2 t\cos^2 t$

$\Rightarrow |\vec{v}| = 3\sin t\cos t$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \langle 0, -\cos t, \sin t\rangle$$

! wow!

So much nicer than $\vec{v}$! just by normalizing ♡!

$$L = \int_0^{\pi/2} |\vec{v}|\,dt = \int_0^{\pi/2} 3\sin t\cos t\,dt = \int_*^* 3u\,du = \frac{3}{2}u^2\Big|_*^* = \frac{3}{2}\sin^2 t\Big|_0^{\pi/2}$$

$\boxed{\begin{aligned} u &= \sin t \\ du &= \cos t\,dt \end{aligned}}$

$\sin \pi/2 = 1$
$\sin 0 = 0$

$$= \frac{3}{2}\left[1^2 - 0^2\right] = \frac{3}{2}$$

#6. $r(t) = \langle 6t^3, -2t^3, -3t^3\rangle \quad 1 \le t \le 2$

$\frac{dr}{dt} = \vec{v} = \langle 18t^2, -6t^2, -9t^2\rangle \qquad |\vec{v}|^2 = t^4(18^2 + 6^2 + 9^2)$

$= 441t^4$

!?

$|\vec{v}| = 21t^2$

$18^2 = (9 \cdot 2)^2 = 81 \cdot 4 = 324$

$324 + 36 + 81$

$= 360 + 81$

$= 441$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{6}{7}, \frac{-2}{7}, \frac{3}{7}\right\rangle$$

a unit vector! holy moly how'd they come up with these lol!

$\frac{18}{21} = \frac{6}{7}, \quad \frac{6}{21} = \frac{2}{7}, \quad \frac{9}{21} = \frac{3}{7}$

$6^2 + 2^2 + 3^2 = 49 = 7^2$

$$L = \int_1^2 |\vec{v}|\,dt = \int_1^2 21t^2\,dt = 7t^3\Big|_1^2 = 7(8-1) = 7^2 = 49$$

!!!

Exercises 13.3

Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$
6. $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \le t \le 2$
7. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$
8. $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \le t \le 2$
9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

#7. $r(t) = \langle t\cos t, t\sin t, \frac{2\sqrt{2}}{3}t^{3/2}\rangle \qquad 0 \le t \le \pi$

$$\frac{dr}{dt} = \vec{v} = \langle \cos t - t\sin t, \sin t + t\cos t, \sqrt{2}t^{1/2}\rangle$$

$$|\vec{v}|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t$$

$$= \cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t$$
$$+ \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t + 2t$$

$$= 1 + t^2 + 2t$$

They're just.... too good. just accept it and try not to rethink all your life choices

$$|\vec{v}| = \sqrt{1+2t+t^2} = \sqrt{(1+t^2)} = 1+t$$

?! Spoke too soon!

$$\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{1+t}\langle \cos t - t\sin t, \sin t + t\cos t, \sqrt{2}t^{1/2}\rangle}$$

$$L = \int_0^\pi |\vec{v}|\,dt = \int_0^\pi 1+t\,dt = t + \tfrac{1}{2}t^2\Big|_0^\pi = \boxed{\pi + \tfrac{1}{2}\pi^2}$$

#8. $r(t) = \langle t\sin t + \cos t, t\cos t - \sin t, 0\rangle \qquad \sqrt{2} \le t \le 2$

$$\frac{dr}{dt} = \vec{v} = \langle \cancel{\sin t} + t\cos t - \cancel{\sin t}, \cancel{\cos t} - t\sin t - \cancel{\cos t}, 0\rangle = \langle t\cos t, -t\sin t, 0\rangle$$

$$|\vec{v}|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 \quad \text{so} \quad |\vec{v}| = t$$

$$\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \langle \cos t, -\sin t, 0\rangle}$$

$$L = \int_{\sqrt{2}}^2 |\vec{v}|\,dt = \int_{\sqrt{2}}^2 t\,dt = \tfrac{1}{2}t^2\Big|_{\sqrt{2}}^2 = \tfrac{1}{2}(4-2) = \tfrac{1}{2} * 2 = \boxed{1}$$

#9. $r(t) = \langle 5\sin t, 5\cos t, 12t\rangle$ Find t_0 s.t. $r(t_0)$ is 26π units along the curve from $r(0) = \langle 0,5,0\rangle$.

That is, find t_0 s.t. $L = 26\pi$. Note $\vec{v} = \frac{dr}{dt} = \langle 5\cos t, -5\sin t, 12\rangle$

$$26\pi = L = \int_0^{t_0} |\vec{v}|\,dt. \quad \text{Note } |\vec{v}|^2 = 25 + 144 = 169 \Rightarrow |v| = 13$$

So $L = \int_0^{t_0} 13\,dt = 13t\Big|_0^{t_0} = 13t_0 = 26\pi \Rightarrow t_0 = 2\pi$ & $\boxed{r(t_0) = \langle 0, 5, 24\pi\rangle}$

Arc Length Parameter

In Exercises 11–14, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\, d\tau$$

from Equation (3). Then find the length of the indicated portion of the curve.

11. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le \pi/2$

12. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad \pi/2 \le t \le \pi$

13. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + e^t\mathbf{k}, \quad -\ln 4 \le t \le 0$

14. $\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k}, \quad -1 \le t \le 0$

#11. $r(t) = \langle 4\cos t, 4\sin t, 3t\rangle \quad 0 \le t \le \pi/2$

$\vec{v} = \frac{dr}{dt} = \langle -4\sin t, 4\cos t, 3\rangle \quad |\vec{v}|^2 = 16 + 9 = 25$

so $|\vec{v}| = 5$

$$S = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t 5\, d\tau = 5\tau\Big|_0^t$$

$$\boxed{S = 5t}$$

$$L = \int_0^{\pi/2} |\vec{v}|\, d\tau = S\Big|_{t=\pi/2} = \boxed{\frac{5}{2}\pi}$$

#12. $r(t) = \langle \cos t + t\sin t, \sin t - t\cos t, 0\rangle \quad \pi/2 \le t \le \pi$

$\frac{dr}{dt} = \vec{v} = \langle \cancel{-\sin t} + \cancel{\sin t} + t\cos t, \cancel{\cos t} - \cancel{\cos t} + t\sin t, 0\rangle = \langle t\cos t, t\sin t, 0\rangle$

$|\vec{v}|^2 = t^2 \Rightarrow |\vec{v}| = t$ so $S = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t \tau\, d\tau = \frac{1}{2}\tau^2\Big|_0^t$

$= \frac{1}{2}t^2 - 0 \Rightarrow \boxed{s(t) = \frac{1}{2}t^2}$

$$L = \int_{\pi/2}^{\pi} |\vec{v}|\, d\tau = \int_0^{\pi} |\vec{v}|\, d\tau - \int_0^{\pi/2} |\vec{v}|\, d\tau = S(\pi) - S(\pi/2)$$

$$= \frac{1}{2}\pi^2 - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{1}{2}\left(\pi^2 - \frac{\pi^2}{4}\right) = \boxed{\frac{3\pi^2}{4}}$$

#13. $r(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle \quad -\ln 4 \le t \le 0$

$\vec{v} = \frac{dr}{dt} = \langle -e^t\sin t + e^t\cos t, e^t\cos t + e^t\sin t, e^t\rangle = e^t\langle -\sin t + \cos t, \sin t + \cos t, 1\rangle$

$|\vec{v}|^2 = e^{2t}\left(\overbrace{\sin^2 t - 2\cancel{\sin t}\cos t + \cos^2 t}^{=1} + \overbrace{\sin^2 t + 2\cancel{\sin t}\cos t + \cos^2 t}^{=1} + 1\right) = 3e^{2t}$

$|\vec{v}| = \sqrt{3}e^t$ so

$$S = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t \sqrt{3}e^{\tau}\, d\tau = \boxed{\sqrt{3}e^t - \sqrt{3}}$$

$$L = \int_{-\ln 4}^{0} |\vec{v}(\tau)|\, dt = -\int_0^{-\ln 4} |\vec{v}(\tau)|\, d\tau = -S\Big|_{t=-\ln 4} = -\left(\sqrt{3}e^{(-\ln 4)} - \sqrt{3}\right)$$

$$= -\left(\frac{\sqrt{3}}{4} - \sqrt{3}\right) = \boxed{\frac{3\sqrt{3}}{4}}$$

§13.4

Exercises 13.4

Plane Curves

Find **T**, **N**, and κ for the plane curves in Exercises 1–4.

1. $\mathbf{r}(t) = t\mathbf{i} + (\ln \cos t)\mathbf{j}, \quad -\pi/2 < t < \pi/2$
2. $\mathbf{r}(t) = (\ln \sec t)\mathbf{i} + t\mathbf{j}, \quad -\pi/2 < t < \pi/2$
3. $\mathbf{r}(t) = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j}$
4. $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0$

#1. $r(t) = \langle t, \ln(\cos t)\rangle$

$-\pi/2 < t < \pi/2$

$T = \frac{\vec{v}}{|\vec{v}|}$ $\quad \vec{v} = \frac{dr}{dt} = \langle 1, \frac{1}{\cos t} * -\sin t\rangle = \langle 1, -\tan t\rangle$

$|\vec{v}|^2 = 1 + \tan^2 t = \sec^2 t$

$\Rightarrow |\vec{v}| = \sec t$

So $T = \langle \frac{1}{\sec t}, \frac{-\tan t}{\sec t}\rangle = \langle \cos t, \frac{-\sin t/\cos t}{1/\cos t}\rangle = \langle \cos t, -\sin t\rangle$

$N = \frac{dT/dt}{|dT/dt|}$ $\quad \frac{dT}{dt} = \langle -\sin t, -\cos t\rangle$ & $\left|\frac{dT}{dt}\right|^2 = \sin^2 t + \cos^2 t = 1$ ✓

So $N = \langle -\sin t, -\cos t\rangle$, $K = \frac{1}{|\vec{v}|} * \left|\frac{dT}{dt}\right| = \frac{1}{\sec t} * 1 = \cos t$

see #6

#2 $r(t) = \langle \ln(\sec t), t\rangle$ $\quad -\pi/2 < t < \pi/2$

$\vec{v} = \frac{dr}{dt} = \langle \frac{1}{\sec t} * \sec t \tan t, 1\rangle = \langle \tan t, 1\rangle$ $\quad |\vec{v}|^2 = \tan^2 t + 1 = \sec^2 t$

$|\vec{v}| = \sec t$

$T = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{\tan t}{\sec t}, \frac{1}{\sec t}\rangle = \langle \sin t, \cos t\rangle$

$N = \frac{dT/dt}{|dT/dt|} = \frac{1}{\cos^2 t + \sin^2 t}\langle \cos t, -\sin t\rangle = \langle \cos t, -\sin t\rangle$

$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right| = \frac{1}{\sec t} * 1 = \cos t$

$\left((1+t^2)^{-1/2}\right)' = -\frac{1}{2}(1+t^2)^{-3/2} * 2t$

$\left(\frac{-t}{\sqrt{1+t^2}}\right)' = \frac{\sqrt{1+t^2}(-1) - (-t)\frac{2t}{2\sqrt{1+t^2}}}{(\sqrt{1+t^2})^2}$

#3. $r(t) = \langle 2t+3, 5-t^2\rangle$

$\vec{v} = \frac{dr}{dt} = \langle 2, -2t\rangle$ $\quad |\vec{v}|^2 = 4 + 4t^2 = 4(1+t^2)$ so $|\vec{v}| = 2\sqrt{1+t^2}$

$T = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{2\sqrt{1+t^2}}\langle 2, -2t\rangle = \langle \frac{1}{\sqrt{1+t^2}}, \frac{-t}{\sqrt{1+t^2}}\rangle = \frac{1}{\sqrt{1+t^2}}\langle 1, -t\rangle$

$N = \frac{dT/dt}{|dT/dt|}$ (next page)

#3. (cont.)

$$T = \frac{\vec{v}}{|\vec{v}|} = \boxed{\left\langle \frac{1}{\sqrt{1+t^2}}, \frac{-t}{\sqrt{1+t^2}} \right\rangle}$$

$$N = \frac{dT/dt}{|dT/dt|} = \frac{1}{|dT/dt|}\left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{-1}{(1+t^2)^{3/2}} \right\rangle$$

$$\left|\frac{dT}{dt}\right|^2 = \frac{1}{(1+t^2)^{3/2}}(1+t^2) = \frac{1}{\sqrt{1+t^2}}$$

$$\text{So} \quad N = \boxed{\left\langle \frac{-t}{(1+t^2)^{1/2}}, \frac{-1}{(1+t^2)^{1/2}} \right\rangle}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right| = \frac{1}{2\sqrt{1+t^2}}\left|\left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{-1}{(1+t^2)^{3/2}} \right\rangle\right|$$

$$= \left|\left\langle \frac{-t}{2(1+t^2)^2}, \frac{-1}{2(1+t^2)^2} \right\rangle\right| = \frac{1}{2(1+t^2)^2}\sqrt{1+t^2} = \boxed{\frac{1}{2\sqrt{1+t^2}}}$$

A NEW ERA begins...

BALL PEN!

Exercises 13.4

Plane Curves

Find **T**, **N**, and κ for the plane curves in Exercises 1–4.

1. $\mathbf{r}(t) = t\mathbf{i} + (\ln\cos t)\mathbf{j}, \quad -\pi/2 < t < \pi/2$
2. $\mathbf{r}(t) = (\ln\sec t)\mathbf{i} + t\mathbf{j}, \quad -\pi/2 < t < \pi/2$
3. $\mathbf{r}(t) = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j}$
4. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad t > 0$

$$\left(\frac{1}{\sqrt{1+t^2}}\right)' = \left[(1+t^2)^{-1/2}\right]' = -\frac{1}{2}(1+t^2)^{-3/2} * 2t = \frac{-t}{(1+t^2)^{3/2}}$$

$$\left(\frac{-t}{\sqrt{1+t^2}}\right)' = \frac{\sqrt{1+t^2}(-1) - (-t)\frac{1}{2}(1+t^2)^{-1/2} * 2t}{1+t^2} = \frac{(1+t^2)^{-1/2}\left[-(1+t^2) + \frac{t}{2} * 2t\right]}{1+t^2} = \frac{-1 - \cancel{t^2} + \cancel{t^2}}{(1+t^2)^{3/2}}$$

#4 $\quad r(t) = \langle \cos t + t\sin t, \sin t - t\cos t \rangle \quad t > 0$

$$\vec{v} = \frac{dr}{dt} = \langle \cancel{-\sin t} + t\cos t + \cancel{\sin t}, \cancel{\cos t} + (-t)(-\sin t) + \cancel{(-1)\cos t} \rangle$$

$$= \langle t\cos t, t\sin t \rangle \quad \text{So,} \quad |\vec{v}|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$$

$$\Rightarrow |\vec{v}| = t \quad (t>0)$$

$$T = \frac{\vec{v}}{|\vec{v}|} = \boxed{\langle \cos t, \sin t \rangle}$$

$$N = \frac{dT/dt}{|dT/dt|} = \frac{1}{\sqrt{(-\sin t)^2 + (\cos t)^2}}\langle -\sin t, \cos t \rangle = \boxed{\langle -\sin t, \cos t \rangle}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right| = \left\langle -\frac{1}{t}\sin t, \frac{1}{t}\cos t \right\rangle = \frac{1}{t}(\sin^2 t + \cos^2 t) = \boxed{\frac{1}{t}}$$

5. A formula for the curvature of the graph of a function in the *xy*-plane

a. The graph $y = f(x)$ in the xy-plane automatically has the parametrization $x = x, y = f(x)$, and the vector formula $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Use this formula to show that if f is a twice-differentiable function of x, then

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}.$$

b. Use the formula for κ in part (a) to find the curvature of $y = \ln(\cos x), -\pi/2 < x < \pi/2$. Compare your answer with the answer in Exercise 1.

c. Show that the curvature is zero at a point of inflection.

6. A formula for the curvature of a parametrized plane curve

a. Show that the curvature of a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

The dots in the formula denote differentiation with respect to t, one derivative for each dot. Apply the formula to find the curvatures of the following curves.

b. $\mathbf{r}(t) = t\mathbf{i} + (\ln \sin t)\mathbf{j}, \quad 0 < t < \pi$

c. $\mathbf{r}(t) = [\tan^{-1}(\sinh t)]\mathbf{i} + (\ln \cosh t)\mathbf{j}$

#5. $r(t) = \langle t, f(t) \rangle$

(a) $\vec{v} = \langle 1, f'(t) \rangle \quad |\vec{v}|^2 = 1 + (f'(t))^2$ so $|\vec{v}| = \sqrt{1+(f'(t))^2}$

$$T = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+(f'(t))^2}} \langle 1, f'(t) \rangle = \left\langle \frac{1}{\sqrt{1+(f'(t))^2}}, \frac{f'(t)}{\sqrt{1+(f'(t))^2}} \right\rangle$$

$$\left[\left(1+(f'(t))^2\right)^{-1/2}\right]' = -\frac{1}{2} * 2f'(t) * f''(t)\left(1+(f'(t))^2\right)^{-3/2} = \frac{f'(t)f''(t)}{[1+(f'(t))^2]^{3/2}}$$

$$\left[\frac{f'(t)}{\sqrt{1+(f'(t))^2}}\right]' = \frac{\sqrt{1+(f'(t))^2}\, f''(t) - f'(t) * \frac{1}{2}\left(1+(f'(t))^2\right)^{-1/2} * 2f'(t) * f''(t)}{1+(f'(t))^2}$$

$$= \frac{(1+(f'(t))^2) f''(t) - f''(t)(f'(t))^2}{(1+(f'(t))^2)^{3/2}} = \frac{f''(t)}{(1+(f'(t))^2)^{3/2}}$$

$$\frac{dT}{dt} = \left\langle \frac{f'(t)f''(t)}{(1+(f'(t))^2)^{3/2}}, \frac{f''(t)}{(1+(f'(t))^2)^{3/2}} \right\rangle$$

$$\left|\frac{dT}{dt}\right| = \frac{1}{(1+(f'(t))^2)^{3/2}} \sqrt{(f'(t))^2(f''(t))^2 + (f''(t))^2}$$

$$= \frac{\sqrt{1+(f'(t))^2}}{(1+(f'(t))^2)^{3/2}} \sqrt{(f''(t))^2} = \frac{|f''(t)|}{(1+(f'(t))^2)}$$

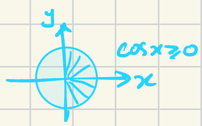
$$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right| = \frac{1}{\sqrt{1+(f'(t))^2}} \frac{|f''(t)|}{(1+(f'(t))^2)} = \boxed{\frac{|f''(t)|}{(1+(f'(t))^2)^{3/2}}}$$

Note: In general, if

$$T = \frac{1}{\sqrt{1+z^2}}\hat{i} + \frac{z}{\sqrt{1+z^2}}\hat{j}, \quad z = z(t) \qquad (\star)$$

Then $\dfrac{dT}{dt} = \dfrac{zz'}{(1+z^2)^{3/2}}\hat{i} + \dfrac{z'}{(1+z^2)^{3/2}}\hat{j}$

(b) $y = \ln(\cos x) \quad -\pi/2 < x < \pi/2$



$f(t) = \ln(\cos(t))$

$f'(t) = \frac{1}{\cos t} * (\cos t)' = \frac{-\sin t}{\cos t} = -\tan t$

$f''(t) = -\sec^2 t$

$$K = \frac{|f''(t)|}{(1+(f'(t))^2)^{3/2}} = \frac{\sec^2 t}{(1+\tan^2 t)^{3/2}} = \frac{\sec^2 t}{(\sec^2 t)^{3/2}}$$

$$K = \frac{1}{\sec t} = \boxed{\cos t}$$ same as #1 ✓

(c) If $f''(t) = 0$ then $K = 0$.

but inflection points have vanishing second derivative, so claim follows.

5. A formula for the curvature of the graph of a function in the *xy*-plane

a. The graph $y = f(x)$ in the xy-plane automatically has the parametrization $x = x$, $y = f(x)$, and the vector formula $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Use this formula to show that if f is a twice-differentiable function of x, then

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}.$$

b. Use the formula for κ in part (a) to find the curvature of $y = \ln(\cos x)$, $-\pi/2 < x < \pi/2$. Compare your answer with the answer in Exercise 1.

c. Show that the curvature is zero at a point of inflection.

6. A formula for the curvature of a parametrized plane curve

a. Show that the curvature of a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula

$$\boxed{\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.}$$

The dots in the formula denote differentiation with respect to t, one derivative for each dot. Apply the formula to find the curvatures of the following curves.

b. $\mathbf{r}(t) = t\mathbf{i} + (\ln \sin t)\mathbf{j}$, $0 < t < \pi$

$\boxed{\textbf{c. } \mathbf{r}(t) = [\tan^{-1}(\sinh t)]\mathbf{i} + (\ln \cosh t)\mathbf{j}}$

next page

#6 $r(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$, $x = f(t)$ & $y = g(t)$

(a) $\vec{v} = \frac{dr}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, $|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}$ Note: $\left(\dot{z} := \frac{dz}{dt}\right)$

$$T = \frac{\vec{v}}{|\vec{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\hat{\imath} + \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\hat{\jmath}$$

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = \frac{\sqrt{\dot{x}^2+\dot{y}^2}\,\ddot{x} - \dot{x}\frac{1}{2}(\dot{x}^2+\dot{y}^2)^{-1/2}(2\dot{x}\ddot{x}+2\dot{y}\ddot{y})}{\dot{x}^2+\dot{y}^2}$$

$$= \frac{(\dot{x}^2+\dot{y}^2)\ddot{x} - \dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

$$= \frac{\cancel{\ddot{x}\dot{x}^2} + \ddot{x}\dot{y}^2 - \cancel{\ddot{x}\dot{x}^2} - \dot{x}\dot{y}\ddot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

$$= \frac{\dot{y}(\ddot{x}\dot{y} - \dot{x}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

So by symmetry $\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = \frac{\dot{x}(\ddot{y}\dot{x} - \dot{y}\ddot{x})}{(\dot{x}^2+\dot{y}^2)^{3/2}}$ ← (swap the roles of x & y)

$$\frac{dT}{dt} = \frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}\left[\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}\right]$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right| = \frac{1}{\cancel{\sqrt{\dot{x}^2+\dot{y}^2}}} * \frac{|\ddot{x}\dot{y} - \dot{x}\ddot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} * \cancel{\sqrt{\dot{y}^2+\dot{x}^2}},\qquad \boxed{K = \frac{|\ddot{x}\dot{y} - \dot{x}\ddot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}}$$

(b) $r(t) = t\hat{\imath} + \ln\sin t\,\hat{\jmath}$ $0 < t < \pi$

$\csc x \neq 0$ ✓

$x = t$
$y = \ln\sin t$

$\dot{x} = 1$ $\dot{y} = \frac{1}{\sin t}\cos t = \cot t$
$\ddot{x} = 0$ $\ddot{y} = -\csc^2 t$

$$K = \frac{|\csc^2 t|}{(1+\cot^2 t)^{3/2}} = \frac{\csc^2 t}{\csc^3 t}$$

$$K = \frac{1}{\csc t} = \boxed{\sin t}$$

$\sin^2 t + \cos^2 t = 1$

$\Rightarrow \frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$

$\Rightarrow 1 + \cot^2 t = \csc^2 t$

(c) $r(t) = \tan^{-1}(\sinh t)\hat{\imath} + \ln(\cosh t)\hat{\jmath}$

$x = \tan^{-1}(\sinh t)$
$y = \ln(\cosh t)$

Use the K formula!

(c) $r(t) = \tan^{-1}(\sinh t)\hat{i} + \ln(\cosh t)\hat{j}$

Use the κ formula!

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

$x = \tan^{-1}(\sinh t)$

$\dot{x} = \frac{1}{\sinh^2 t + 1}\cosh t = \frac{\cosh t}{\cosh^2 t} = \frac{1}{\cosh t} = (\cosh t)^{-1}$

$\ddot{x} = -(\cosh t)^{-2}\sinh t = \frac{-\sinh t}{\cosh^2 t} = \frac{-\tanh t}{\cosh t}$

$y = \ln(\cosh t)$

$\dot{y} = \frac{1}{\cosh t}\sinh t = \tanh t$

$\ddot{y} = \frac{1}{\cosh^2 t}$

$\frac{d}{dt}\tanh t = \left(\frac{\sinh t}{\cosh t}\right)' = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t}$

$\cosh^2 t - \sinh^2 t = \frac{1}{4}\left[(e^{2t} + e^{-2t} + 2) - (e^{2t} + e^{-2t} - 2)\right] = 1$ ✓

So $\sinh^2 t + 1 = \cosh^2 t$

$\sinh t = \frac{e^t - e^{-t}}{2}$

$\cosh t = \frac{e^t + e^{-t}}{2}$

$\tanh t = \frac{e^t + e^{-t}}{e^t - e^{-t}} = \frac{e^{2t} + 1}{e^{2t} - 1}$

$$\kappa = \frac{\left|\frac{1}{\cosh t}\frac{1}{\cosh^2 t} - \tanh t \frac{-\tanh t}{\cosh t}\right|}{\frac{1}{\cosh^2 t} + \frac{\sinh^2 t}{\cosh^2 t}} = \frac{\left|\frac{1}{\cosh^3 t} + \frac{\sinh^2 t}{\cosh^3 t}\right|}{\frac{1}{\cosh^2 t}(1 + \sinh^2 t)}$$

$$= \frac{\cosh^{\cancel{2}} t}{\cosh^3 t}\frac{\cancel{|1+\sinh^2 t|}}{\cancel{(1+\sinh^2 t)}} = \boxed{\kappa = \frac{1}{\cosh t}}$$

Space Curves

Find **T**, **N**, and κ for the space curves in Exercises 9–16.

9. $\mathbf{r}(t) = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k}$

10. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}$

11. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + 2\mathbf{k}$

12. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}$

13. $\mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j},\quad t > 0$

14. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j},\quad 0 < t < \pi/2$

15. $\mathbf{r}(t) = t\mathbf{i} + (a\cosh(t/a))\mathbf{j},\quad a > 0$

16. $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

#9 $r(t) = 3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4t\,\hat{k}$

$T = \frac{v}{|v|}$ $\quad v = 3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 4\hat{k}$

$|v|^2 = 9\cos^2 t + 9\sin^2 t + 16 = 9 + 16 = 25$

so $|v| = 5$

$N = \frac{dT/dt}{|dT/dt|}$

$$\boxed{T = \frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j} + \frac{4}{5}\hat{k}}$$

$\frac{dT}{dt} = -\frac{3}{5}\sin t\,\hat{i} - \frac{3}{5}\cos t\,\hat{j} + 0\hat{k}$

$\left|dT/dt\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t} = \sqrt{9/25} = \frac{3}{5}$

$$\boxed{N = -\sin t\,\hat{i} - \cos t\,\hat{j} + 0\hat{k}}$$

$\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{1}{5} * \frac{3}{5} = \boxed{\frac{3}{25}}$

Space Curves

Find **T**, **N**, and κ for the space curves in Exercises 9–16.

9. $\mathbf{r}(t) = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k}$
10. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}$
11. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + 2\mathbf{k}$
12. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}$
13. $\mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j}, \quad t > 0$
14. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 < t < \pi/2$
15. $\mathbf{r}(t) = t\mathbf{i} + (a\cosh(t/a))\mathbf{j}, \quad a > 0$
16. $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

#10 $r(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j} + 3\hat{k}$

$$v = \frac{dr}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\hat{i} + (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\hat{j} + 0\hat{k}$$

$$= t\cos t\,\hat{i} + t\sin t\,\hat{j} + 0\hat{k}$$

$|v|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$ so $|v| = |t|$

$$T = \frac{v}{|v|} = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\hat{k}$$

$\frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$ & $\left|\frac{dT}{dt}\right| = 1.$

so $N = \frac{dT}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\hat{k}$

$$K = \frac{1}{|t|}$$

More on Curvature

17. Show that the parabola $y = ax^2$, $a \neq 0$, has its largest curvature at its vertex and has no minimum curvature. (*Note:* Since the curvature of a curve remains the same if the curve is translated or rotated, this result is true for any parabola.)

#17 $y = ax^2$, Find K.

From problem #5

$$K(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

$y = f(x) = ax^2$

$y' = 2ax$

$y'' = 2a$

$K = \frac{2|a|}{[1 + 2a^2x^2]^{3/2}} > 0$ but $k \to 0$ as $x \to \pm\infty$

Max value of K when $x = 0$