§14.1

Exercises 14.1

In Exercises 5–12, find and sketch the domain for each function.

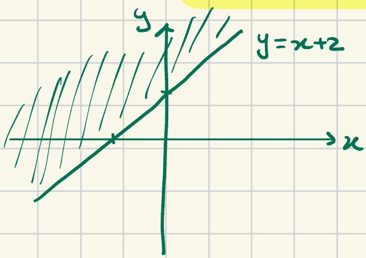
5. $f(x, y) = \sqrt{y - x - 2}$
6. $f(x, y) = \ln(x^2 + y^2 - 4)$
7. $f(x, y) = \dfrac{(x-1)(y+2)}{(y-x)(y-x^3)}$
8. $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$
9. $f(x, y) = \cos^{-1}(y - x^2)$
10. $f(x, y) = \ln(xy + x - y - 1)$
11. $f(x, y) = \sqrt{(x^2-4)(y^2-9)}$
12. $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

Find the domain

#5 $f(x,y) = \sqrt{y-x-2}$

D: $y - x - 2 \geq 0$

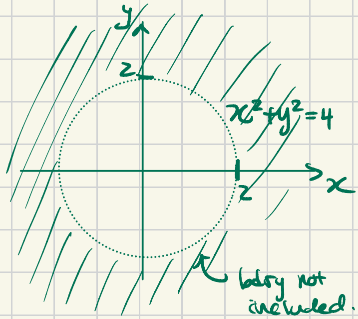
$y \geq x + 2$



#6 $f(x,y) = \ln(x^2+y^2-4)$

D: $x^2+y^2-4 > 0$

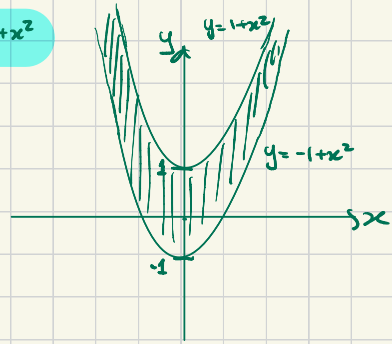
$x^2+y^2 < 4$



#9 $f(x,y) = \cos^{-1}(y - x^2)$

D: $-1 \leq y - x^2 \leq 1$

So $y \geq -1 + x^2$ and $y \leq 1 + x^2$



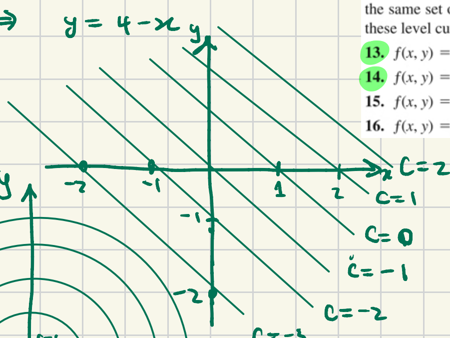
#13 $f(x,y) = x+y-1$, $c = -3, -2, \ldots, 1, 2, 3$

$c=-3$ $\quad -3 = x+y-1 \Rightarrow y = -2-x$

$c=-2$ $\quad -2 = x+y-1 \Rightarrow y = -1-x$

$c=0$ $\quad 0 = x+y-1 \Rightarrow y = 1-x$

$c=3$ $\quad 3 = x+y-1 \Rightarrow y = 4-x$

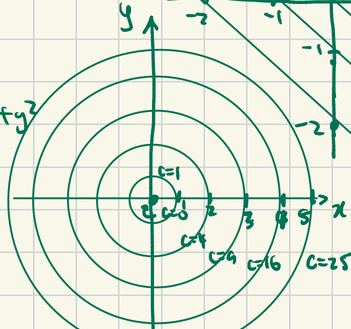


In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1$, $c = -3, -2, -1, 0, 1, 2, 3$
14. $f(x, y) = x^2 + y^2$, $c = 0, 1, 4, 9, 16, 25$
15. $f(x, y) = xy$, $c = -9, -4, -1, 0, 1, 4, 9$
16. $f(x, y) = \sqrt{25 - x^2 - y^2}$, $c = 0, 1, 2, 3, 4$

#14 $f(x,y) = x^2+y^2$

$c = 0, 1, 4, 9, 16, 25$



In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

17. $f(x, y) = y - x$ **18.** $f(x, y) = \sqrt{y - x}$

19. $f(x, y) = 4x^2 + 9y^2$ **20.** $f(x, y) = x^2 - y^2$

#17 $f(x,y) = y-x$

(a) D: $\mathbb{R}^2$

(b) R: $\mathbb{R}$

(c) level curves are: lines

(d) bdry: no bdry

(e) domain is $\mathbb{R}^2$, (f) unbounded, open and closed.

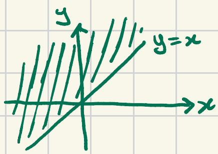
(pg. 808)

Defn: a region in $\mathbb{R}^2$ is bounded if it is contained inside a disk of radius r. Otherwise, the region is unbounded.

#18 $f(x,y) = \sqrt{y-x}$

(a) D: $y - x \geq 0$

$y \geq x$



(b) R: $\mathbb{R}$

(c) level curves are: lines $\sqrt{y-x} = c \Rightarrow y - x = c^2 \Rightarrow y = x + c^2$

(d) bdry: the line $y=x$

(e) domain is closed, and (f) unbounded

#19 $f(x,y) = 4x^2 + 9y^2$

(a) D: $\mathbb{R}^2$

(b) R: $[0,\infty)$

(c) level curves are: ellipses, $4x^2+9y^2 = c^2$

(d) bdry: no bdry

(e) domain is open & closed and (f) unbounded.

#20. $f(x,y) = x^2 - y^2$

(a) D: $\mathbb{R}^2$

(b) R: $\mathbb{R}$

(c) level curves are: (name?) of the form $y = \pm\sqrt{x^2-c}$

(d), (e), (f) same as #17, #19.

	graph	equation
#31	f	h
#32	e	l
#33	a	i
#34	c	k
#35	d	j
#36	b	g

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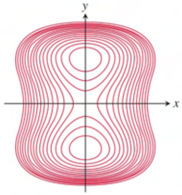
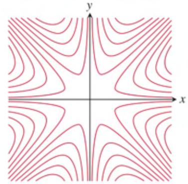
Matching Surfaces with Level Curves

Exercises 31–36 show level curves for the functions graphed in (a)–(f) on the following page. Match each set of curves with the appropriate function.

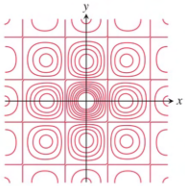
31.



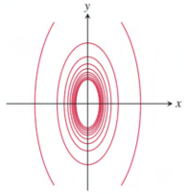
32.



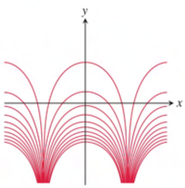
33.



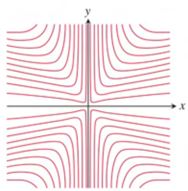
34.



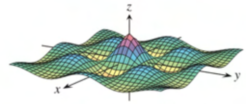
35.



36.

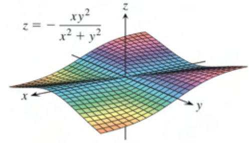


a.



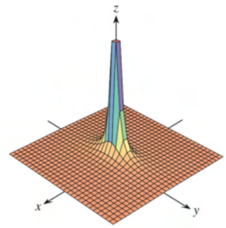
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2+y^2}/4}$

b.



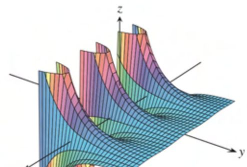
$z = -\dfrac{xy^2}{x^2+y^2}$

c.



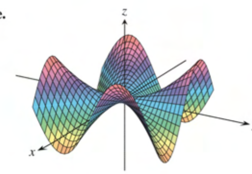
$z = \dfrac{1}{4x^2+y^2}$

d.



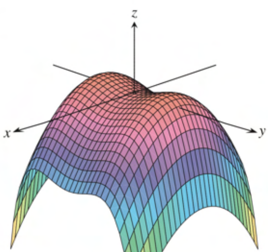
$z = e^{-y}\cos x$

e.



$z = \dfrac{xy(x^2-y^2)}{x^2+y^2}$

f.



$z = y^2 - y^4 - x^2$

§14.2

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$
2. $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

#1 $\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^3 + 5}{x^2 + y^2 + 2} = \frac{5}{2}$

#2 $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = 0$

3. $\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$
4. $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$
5. $\lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$
6. $\lim_{(x,y)\to(0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

#3 $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{24} = 2\sqrt{6}$

#4 $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left(\frac{1}{2}+\frac{-1}{3}\right)^2 = \left(\frac{3}{6}-\frac{2}{6}\right)^2 = \frac{1}{36}$

#5 $\lim_{(x,y)\to(0,\pi/4)} \sec x \tan x = \sec(0)\tan(\pi/4) = 1 * 1 = 1$

7. $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$
8. $\lim_{(x,y)\to(1,1)} \ln|1 + x^2 y^2|$
9. $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$
10. $\lim_{(x,y)\to(1/27,\pi^3)} \cos \sqrt[3]{xy}$
11. $\lim_{(x,y)\to(1,\pi/6)} \frac{x \sin y}{x^2 + 1}$
12. $\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y + 1}{y - \sin x}$

#6 $\lim_{(x,y)\to(0,0)} \cos \frac{x^2+y^3}{x+y+1} = \cos(0) = 1$

#7 $\lim_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = (e^{\ln 2})^{-1} = 2^{-1} = \frac{1}{2}$

#8 $\lim_{(x,y)\to(1,1)} \ln|1+x^2y^2| = \ln 2$

① $\lim_{y\to 0} e^y = 1$

② $\lim_{x\to 0} \frac{\sin x}{x} = 1$

#9 $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$, note @$(0,0)$ $\frac{e^0 \sin(0)}{0} = \frac{0}{0}$ is <u>undefined</u>.

but define $f(x,y) = e^y$, $g(x,y) = \frac{\sin x}{x}$. Then by Property 4 (Product Rule) on page 816

$$\lim_{(x,y)\to(0,0)} f(x,y)g(x,y) = \lim_{(x,y)\to(0,0)} f(x,y) * \lim_{(x,y)\to(0,0)} g(x,y) \overset{①\&②}{=} 1 * 1 = 1$$

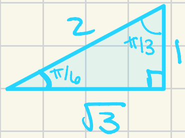
$$\#10 \quad \lim_{(x,y)\to\left(\frac{1}{27},\pi^3\right)} \cos\left(\sqrt[3]{xy}\right) = \cos\left(\sqrt[3]{\tfrac{1}{27}\pi^3}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

7. $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$ 8. $\lim_{(x,y)\to(1,1)} \ln\left|1+x^2y^2\right|$

9. $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$ 10. $\lim_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy}$

11. $\lim_{(x,y)\to(1,\pi/6)} \frac{x\sin y}{x^2+1}$ 12. $\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y+1}{y-\sin x}$



$\cos\theta = \frac{adj}{hyp}$

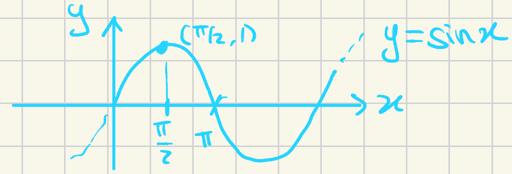
$\sin\theta = \frac{opp}{hyp}$

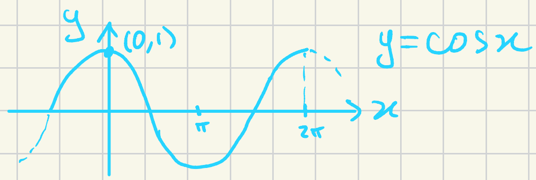
$$\#11 \quad \lim_{(x,y)\to(1,\pi/6)} \frac{x\sin y}{x^2+1} = \frac{1*\sin(\pi/6)}{2} = \frac{1*1/2}{2}$$

$$= \frac{1}{2}*\frac{1}{2} = \frac{1}{4}$$

$$\#12 \quad \lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{\cos(0)+1}{0-\sin(\pi/2)} = \frac{1+1}{0-1} = \frac{2}{-1}$$

$$= -2$$





$$\#13 \quad \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim \frac{(x-y)\cancel{(x-y)}}{\cancel{x-y}}$$

$$= \lim x-y = 1-1 = 0$$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$ 14. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y}$

15. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1}$

16. $\lim_{\substack{(x,y)\to(2,-4)\\ x\neq -4,\, x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x}$

17. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

18. $\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}$ 19. $\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4}$

20. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

21. $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ 22. $\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$

23. $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ 24. $\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

$$\#16 \quad \lim_{\substack{(x,y)\to(2,-4)\\ x\neq 4,\, x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim \frac{\cancel{y+4}}{(x^2-x)\cancel{(y+4)}}$$

$$= \lim \frac{1}{x^2-x}$$

$$x^2y-xy+4x^2-4x$$

$$= x^2(y+4)-x(y+4) \qquad = \frac{1}{2^2-2} = \frac{1}{2}$$

$$= (x^2-x)(y+4)$$

$$\#18 \quad \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim \sqrt{x+y}+2 = \sqrt{2+2}+2$$

$$= \sqrt{4}+2 = 4$$

$$\frac{x+y-4}{\sqrt{x+y}-2} * \frac{\sqrt{x+y}+2}{\sqrt{x+y}+2} = \frac{\cancel{(x+y-4)}\left(\sqrt{x+y}+2\right)}{\cancel{(x+y)-4}}$$

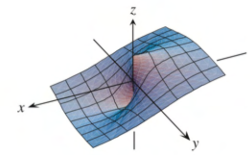
$$= \left(\sqrt{x+y}+2\right)$$

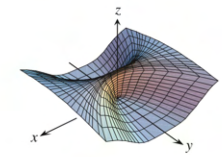
Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE

No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$ **42.** $f(x, y) = \dfrac{x^4}{x^4 + y^2}$





43. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$ **44.** $f(x, y) = \dfrac{xy}{|xy|}$

45. $g(x, y) = \dfrac{x - y}{x + y}$ **46.** $g(x, y) = \dfrac{x^2 - y}{x - y}$

47. $h(x, y) = \dfrac{x^2 + y}{y}$ **48.** $h(x, y) = \dfrac{x^2 y}{x^4 + y^2}$

#41 $f(x,y) = \dfrac{-x}{\sqrt{x^2+y^2}}$ Let $y = mx$

Then $f(x, mx) = \dfrac{-x}{\sqrt{x^2+m^2x^2}} = \dfrac{-x}{|x|\sqrt{1+m^2}} = \pm\dfrac{1}{\sqrt{1+m^2}}$

So $f(x,y)\Big|_{y=mx} = \dfrac{\pm 1}{\sqrt{1+m^2}}$

and hence $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \pm\dfrac{1}{\sqrt{1+m^2}}$ ← value of the limit depends on m

So by the Two-path test, the limit is DNE ///.

#42 $f(x,y) = \dfrac{x^4}{x^4+y^2}$ Let $y = mx^2$

Then $f(x, mx^2) = \dfrac{x^4}{x^4+m^2x^4} = \dfrac{1}{1+m^2}$. So $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} f(x,y) = \dfrac{1}{1+m^2}$.

So, by Two-path test the limit is DNE. ///

#44 $f(x,y) = \dfrac{xy}{|xy|}$ Let $y = mx$

Then $f(x, mx) = \dfrac{mx^2}{|m|x^2} = \dfrac{m}{|m|} = \begin{cases} 1 & \text{if } m>0 \\ -1 & \text{if } m<0. \end{cases}$

So $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \begin{cases} 1 & \text{if } m=1 \\ -1 & \text{if } m=-1. \end{cases}$ By the Two-path test, the limit is DNE ///

#46 $f(x,y) = \dfrac{x^2-y}{x-y}$ Let $y = mx$

Then $f(x,y) = \dfrac{x^2-mx}{x-mx} = \dfrac{x(x-1)}{x(1-m)} \overset{x\neq 0}{=} \dfrac{x-1}{1-m}$,

So $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \dfrac{0-1}{1-m} = \dfrac{1}{m-1}$. So by the Two-path test, the limit is DNE. ///