# §14.3

Compute $\partial f/\partial x$, $\partial f/\partial y$

**Exercises 14.3**

**Calculating First-Order Partial Derivatives**

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$
2. $f(x, y) = x^2 - xy + y^2$
3. $f(x, y) = (x^2 - 1)(y + 2)$
4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
5. $f(x, y) = (xy - 1)^2$
6. $f(x, y) = (2x - 3y)^3$
7. $f(x, y) = \sqrt{x^2 + y^2}$
8. $f(x, y) = (x^3 + (y/2))^{2/3}$
9. $f(x, y) = 1/(x + y)$
10. $f(x, y) = x/(x^2 + y^2)$
11. $f(x, y) = (x + y)/(xy - 1)$
12. $f(x, y) = \tan^{-1}(y/x)$
13. $f(x, y) = e^{(x+y+1)}$
14. $f(x, y) = e^{-x}\sin(x + y)$
15. $f(x, y) = \ln(x + y)$
16. $f(x, y) = e^{xy}\ln y$
17. $f(x, y) = \sin^2(x - 3y)$
18. $f(x, y) = \cos^2(3x - y^2)$
19. $f(x, y) = x^y$
20. $f(x, y) = \log_y x$

#3 $f(x,y) = (x^2-1)(y+2)$

$$\partial f/\partial x = 2x\underbrace{(y+2)}_{\text{constant}}$$

$$\partial f/\partial y = \underbrace{(x^2-1)}_{\text{constant}} * 1$$

#4. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$, $\partial f/\partial x = 5y - 14x + 3$ & $\partial f/\partial y = 5x - 2y - 6$

#6 $f(x,y) = (2x-3y)^3$, $\frac{\partial f}{\partial x} = 3(2x-3y)^2 * 2 = 6(2x-3y)^2$ $\quad \frac{\partial f}{\partial y} = 3(2x-3y)^2 * (-3) = -9(2x-3y)^2$

#8 $f(x,y) = (x^3 + (\frac{y}{2}))^{2/3}$, $\partial f/\partial x = \frac{2}{3}(x^3 + (\frac{y}{2}))^{-1/3} * 3x^2 = \frac{2x^2}{\sqrt[3]{x^3+y/2}}$ $\quad \frac{\partial f}{\partial y} = \frac{2}{3}(x^3 + \frac{y}{2})^{-1/3} * \frac{1}{2} = \frac{1}{3\sqrt[3]{x^3+y/2}}$

#10 $f(x,y) = \frac{x}{x^2+y^2}$, $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$$\partial f/\partial y = \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1}\right) = -x(x^2+y^2)^{-2} * (2y) = \frac{-2xy}{(x^2+y^2)^2}$$

#12 $f(x,y) = \tan^{-1}(y/x)$, $\frac{\partial f}{\partial x} = \frac{1}{(\frac{y}{x})^2+1} * \left(\frac{-y}{x^2}\right) = \frac{1}{\frac{y^2}{x^2}+1} * \frac{-y}{x^2} = \frac{-y}{y^2+x^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{(\frac{y}{x})^2+1} * \left(\frac{1}{x}\right) = \frac{1}{\frac{y^2}{x^2}+1} * \frac{1}{x} = \frac{1}{y^2/x + x} = \frac{1}{\frac{1}{x}(y^2+x^2)} = \frac{x}{y^2+x^2}$$

#14 $f(x,y) = e^{-x}\sin(x+y)$, $\frac{\partial f}{\partial x} = e^{-x} * \cos(x+y) * (1) + -e^{-x}\sin(x+y) = e^{-x}(\cos(x+y) - \sin(x+y))$ $\quad \frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$

#16 $f(x,y) = e^{xy} * \ln(y)$, $\frac{\partial f}{\partial x} = ye^{xy} * \ln y$ & $\frac{\partial f}{\partial y} = xe^{xy}\ln y + e^{xy} * \frac{1}{y} = e^{xy}(x\ln y + \frac{1}{y})$

#18 $f(x,y) = \cos^2(3x-y^2)$, $\frac{\partial f}{\partial x} = 2\cos(3x-y^2) * (-\sin(3x-y^2)*3) = -6\cos(3x-y^2)\sin(3x-y^2)$

$$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) * (-\sin(3x-y^2) * (-2y)) = 4y\cos(3x-y^2)\sin(3x-y^2)$$

#20 $f(x,y) = \log_y(x) = \frac{\ln x}{\ln y}$, $\frac{\partial f}{\partial x} = \frac{1}{\ln y} * \frac{1}{x} = \frac{1}{x\ln y}$ & $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\ln x * (\ln y)^{-1}\right] = \ln x * -(\ln y)^{-2} * \frac{1}{y} = \frac{-\ln x}{y(\ln y)^2}$

base change formula $\log_b(x) = \frac{\ln(x)}{\ln b}$

### Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

**21.** $f(x, y) = \int_x^y g(t)\,dt$ ($g$ continuous for all $t$)

**22.** $f(x, y) = \sum_{n=0}^{\infty} (xy)^n$ ($|xy| < 1$)

#21 $f(x,y) = \int_x^y g(t)\,dt$

$$\partial f/\partial x = \partial/\partial x\left(\underbrace{\int_a^y g(t)\,dt}_{\text{does not depend on } x} + \int_x^a g(t)\,dt\right) = 0 + \partial/\partial x\left(-\int_a^x g(t)\,dt\right)$$

$$= \boxed{-g(x)} \text{ by FTC.}$$

$$\partial f/\partial y = \partial/\partial y\left(\int_a^y g(t)\,dt + \underbrace{\int_x^a g(t)\,dt}_{\text{does not depend on } y}\right) = \boxed{g(y)} \text{ by FTC.}$$

#22 $f(x,y) = \sum_{n=0}^{\infty} (xy)^n$, $|xy| < 1$.

Recall geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$.

$$f(x,y) = \frac{1}{1-xy} \qquad \frac{\partial f}{\partial x} = \frac{-1}{(1-xy)^2} * (-y) = \boxed{\frac{y}{(1-xy)^2}} \qquad \frac{\partial f}{\partial y} = \boxed{\frac{x}{(1-xy)^2}}$$

### Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

**41.** $f(x, y) = x + y + xy$

**42.** $f(x, y) = \sin xy$

**43.** $g(x, y) = x^2y + \cos y + y \sin x$

**44.** $h(x, y) = xe^y + y + 1$

**45.** $r(x, y) = \ln(x + y)$

**46.** $s(x, y) = \tan^{-1}(y/x)$

**47.** $w = x^2 \tan(xy)$

**48.** $w = ye^{x^2 - y}$

**49.** $w = x \sin(x^2y)$

**50.** $w = \dfrac{x - y}{x^2 + y}$

#42 $f(x,y) = \sin xy$

$$f \xrightarrow{\partial/\partial x} f_x = y\cos xy \qquad f \xrightarrow{\partial/\partial y} f_y = x\cos xy$$

$$f_x \xrightarrow{\partial/\partial x} \boxed{f_{xx} = -y^2 \sin xy}$$

$$f_x \xrightarrow{\partial/\partial y} \boxed{f_{xy} = \cos xy - xy\sin xy} = f_{yx} \qquad f_y \xrightarrow{\partial/\partial x} f_{yx}$$

✓ $f_{xy} = f_{yx}$

$$f_y \xrightarrow{\partial/\partial y} \boxed{f_{yy} = -x^2 \sin xy}$$

#44 $h(x,y) = xe^y + y + 1$

$h_x = e^y$

$h_y = xe^y + 1$

$$\boxed{\begin{aligned} h_{xx} &= 0 \\ h_{xy} &= e^y = h_{yx} \\ h_{yy} &= xe^y \end{aligned}}$$

#46 $s(x,y) = \tan^{-1}(y/x)$

$$s_x = \frac{1}{1+(y/x)^2} * \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2} * -y/x^2 = \frac{-y}{x^2(1+y^2/x^2)} = \frac{-y}{x^2+y^2}$$

$$s_y = \frac{1}{1+(y/x)^2} * \frac{\partial}{\partial y}(y/x) = \frac{1}{1+y^2/x^2} * \frac{1}{x} = \frac{1}{1/x^2(x^2+y^2)} * \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$s_{xx} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \boxed{\frac{2xy}{(x^2+y^2)^2}}$$

$$s_{xy} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \boxed{\frac{-x^2+y^2}{(x^2+y^2)^2}} = s_{yx}$$

$$s_{yy} = \frac{(x^2+y^2)(0) - (x)(2y)}{(x^2+y^2)^2} = \boxed{\frac{-2xy}{(x^2+y^2)^2}}$$

## Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

**41.** $f(x, y) = x + y + xy$
**42.** $f(x, y) = \sin xy$
**43.** $g(x, y) = x^2y + \cos y + y \sin x$
**44.** $h(x, y) = xe^y + y + 1$
**45.** $r(x, y) = \ln(x + y)$
**46.** $s(x, y) = \tan^{-1}(y/x)$
**47.** $w = x^2 \tan(xy)$
**48.** $w = ye^{x^2 - y}$
**49.** $w = x \sin(x^2 y)$
**50.** $w = \dfrac{x - y}{x^2 + y}$

## Mixed Partial Derivatives

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

**51.** $w = \ln(2x + 3y)$
**52.** $w = e^x + x \ln y + y \ln x$
**53.** $w = xy^2 + x^2y^3 + x^3y^4$
**54.** $w = x \sin y + y \sin x + xy$

**55.** Which order of differentiation will calculate $f_{xy}$ faster: $x$ first or $y$ first? Try to answer without writing anything down.

- **a.** $f(x, y) = x \sin y + e^y$
- **b.** $f(x, y) = 1/x$
- **c.** $f(x, y) = y + (x/y)$
- **d.** $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$
- **e.** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
- **f.** $f(x, y) = x \ln xy$

**56.** The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: $x$ or $y$? Try to answer without writing anything down.

- **a.** $f(x, y) = y^2x^4e^x + 2$
- **b.** $f(x, y) = y^2 + y(\sin x - x^4)$
- **c.** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
- **d.** $f(x, y) = xe^{y^2/2}$

#48 $w = ye^{x^2-y}$

$w_x = ye^{x^2-y} * (2x) = 2xye^{x^2-y}$

$w_y = e^{x^2-y} + ye^{x^2-y} * (-1) = e^{x^2-y}(1-y)$

$w_{xx} = 2ye^{x^2-y} + 2xye^{x^2-y}(2x)$

$= 2ye^{x^2-y}(1+2x^2)$

$w_{xy} = 2xe^{x^2-y} + 2xye^{x^2-y}(-1)$

$= 2xe^{x^2-y}(1-y) = w_{yx}$

$w_{yy} = -e^{x^2-y}(1-y) + e^{x^2-y}(-1)$

$= -e^{x^2-y}(1-y+1) = -e^{x^2-y}(2-y)$

#52 $w = e^x + x\ln y + y\ln x$

$w_x = e^x + \ln y + \frac{y}{x}$

$w_y = \frac{x}{y} + \ln x$

$w_{xy} = 0 + \frac{1}{y} + \frac{1}{x}$

$w_{yx} = \frac{1}{y} + \frac{1}{x}$

equal ✓

#54 $w = x\sin y + y\sin x + xy$

$w_x = \sin y + y\cos x + y$

$w_y = x\cos y + \sin x + x$

$w_{xy} = \cos y + \cos x + 1$

$w_{yx} = \cos y + \cos x + 1$

equal ✓

**Mixed Partial Derivatives**

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

**51.** $w = \ln(2x + 3y)$

**52.** $w = e^x + x\ln y + y\ln x$

**53.** $w = xy^2 + x^2y^3 + x^3y^4$

**54.** $w = x\sin y + y\sin x + xy$

**55.** Which order of differentiation will calculate $f_{xy}$ faster: $x$ first or $y$ first? Try to answer without writing anything down.

**a.** $f(x, y) = x\sin y + e^y$

**b.** $f(x, y) = 1/x$

**c.** $f(x, y) = y + (x/y)$

**d.** $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$

**e.** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

**f.** $f(x, y) = x\ln xy$

**56.** The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: $x$ or $y$? Try to answer without writing anything down.

**a.** $f(x, y) = y^2x^4e^x + 2$

**b.** $f(x, y) = y^2 + y(\sin x - x^4)$

**c.** $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

**d.** $f(x, y) = xe^{y^2/2}$

#55 (a) $f(x,y) = x\sin y + e^y$

$f_{xy}$ is faster to compute. b/c $f_x = \sin y$ has only one term while $f_y = x\cos y + e^y$ is similar complexity to $f(x,y)$ itself.

(b) $f(x,y) = \frac{1}{x}$

$f_{yx}$ is faster to compute. b/c $f_y = 0$ in one step since $1/x$ does not depend on $y$.

(c) $f(x,y) = y + \frac{x}{y}$

$f_{xy}$ is faster? $f_x = 0 + \frac{1}{y} = \frac{1}{y}$ ← b/c. simple

$f_y = 1 + \frac{-x}{y^2}$ ← more complicated?

(d) $f_{xy}$ faster b/c $f_x$ has only one term.

(e) $f_{yx}$ faster b/c $f_y$ has only one term

(f) $f_{yx}$ faster b/c $f_y$ has only one term, while $f_x$ requires the PROD rule.

#56 (a) $f(x,y) = y^2x^4e^x + 2$, first do $\frac{\partial^3 f}{\partial y^3}$ since $f(k,y) = Cy^2 + 2$ is quadratic in $y$ w/ fixed $x$-value.

$C$ is a constant which depends on the value of $k$

(b) Same answer and reasoing as part (a).

(c) $f(k,y) = C_1y + C_2$ is linear in $y$ when $x = k$ is a constant, so do $\frac{\partial^3 f}{\partial y^3}$ first

$C_1, C_2$ constants that depend on the value of $k$.

(d) $f(x,k) = Cx$, so do $\frac{\partial^2 f}{\partial x^2}$ first.

constant which depends on $k$

#74 Show $f(x,y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

where

$$f(x,y) = 2z^3 - 3(x^2+y^2)z$$

$$f_{xx} = \frac{\partial}{\partial x}(0 - 6xz) = -6z$$

$$f_{yy} = \frac{\partial}{\partial y}(0 - 6yz) = -6z$$

$$f_{zz} = \frac{\partial}{\partial z}(6z^2 - 3(x^2+y^2)) = 12z + 0.$$

Check? 

$$f_{xx} + f_{yy} + f_{zz} = -6z - 6z + 12z$$
$$= 0 \checkmark$$

The **three-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

The **two-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

Show that each function in Exercises 73–80 satisfies a Laplace equation.

**73.** $f(x, y, z) = x^2 + y^2 - 2z^2$

**74.** $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

**75.** $f(x, y) = e^{-2y} \cos 2x$

**76.** $f(x, y) = \ln\sqrt{x^2 + y^2}$

**77.** $f(x, y) = 3x + 2y - 4$

**78.** $f(x, y) = \tan^{-1} \frac{x}{y}$

**79.** $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

**80.** $f(x, y, z) = e^{3x+4y} \cos 5z$

#76 Check that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

where

$$f(x,y) = \ln\sqrt{x^2+y^2}$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}} * \frac{1}{2}(x^2+y^2)^{-1/2} * 2x = \frac{x}{x^2+y^2}$$

$$f_{xx} = \frac{(x^2+y^2) - x*2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

and by symmetry $f_{yy} = \frac{-y^2+x^2}{(x^2+y^2)^2}$.

So $f_{xx} + f_{yy} = \frac{1}{(x^2+y^2)^2}(-\cancel{x^2} + \cancel{y^2} - \cancel{y^2} + \cancel{x^2}) = 0 \checkmark$

#78 $f(x,y) = \tan^{-1}(x/y)$

$$f_x = \frac{1}{1 + x^2/y^2} * (1) = \frac{y^2}{y^2+x^2}$$

$$f_{xx} = \frac{-y^2}{(y^2+x^2)^2} * 2x = \frac{-2xy}{(y^2+x^2)^2}$$

$$f_y = \frac{1}{1+x^2/y^2} * \left(\frac{-x}{y^2}\right) = \frac{-x}{y^2+x^2}$$

$$f_{yy} = \frac{(y^2+x^2)(0) - (-x)(2y)}{(y^2+x^2)^2} = \frac{+2xy}{(y^2+x^2)^2}$$

So $f_{xx} + f_{yy} = 0 \checkmark$

# §14.4

**Exercises 14.4**

**Chain Rule: One Independent Variable**

In Exercises 1–6, **(a)** express $dw/dt$ as a function of $t$, both by using the Chain Rule and by expressing $w$ in terms of $t$ and differentiating directly with respect to $t$. Then **(b)** evaluate $dw/dt$ at the given value of $t$.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$
2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$
3. $w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$
4. $w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}; \quad t = 3$

CHAIN rule formula

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

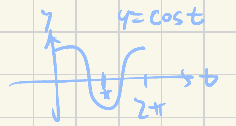
#2 $\quad w = x^2 + y^2 \quad x = \cos t + \sin t$

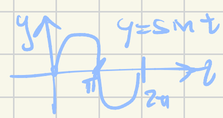
$y = \sin t$

@ $t = \pi$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

So $\boxed{\frac{\partial w}{\partial t} = 2x(-\sin t + \cos t) + 2y(\cos t)}$




@ $t = \pi$, $\left.\frac{\partial w}{\partial t}\right|_{t=\pi} = 2(-1)(0-1) + 2(0)(-1)$

$x(\pi) = 0 - 1 = -1$ $\qquad = 2 + 0 = 2$

$y(\pi) = 0$

Check. $w(t) = (\cos t + \sin t)^2 + \sin^2 t = \cos^2 t + 2\cos t \sin t + 2\sin^2 t$

$w'(t) = 2\cos t(-\sin t) + 2\cos^2 t - 2\sin^2 t + 4\sin t \cos t$

$= 2\cos^2 t - 2\sin^2 t + 2\sin t \cos t$

$w'(\pi) = 2(-1)^2 - 2(0) + 2(0)(-1) = 2$

#3 $\quad w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t \quad$ @ $t = 3 \quad x(3) = \cos^2(3)$

$y = \sin^2 t \qquad y(t) = \sin^2(t)$

$z = 1/t \qquad z(t) = 1/3$

CHAIN

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial t} = \frac{1}{z} * \cancel{2\cos t}(-\sin t) + \frac{1}{z} * \cancel{2\sin t}\cos t + \left(\frac{-x}{z^2} - \frac{y}{z^2}\right)\frac{-1}{t^2}$$

$$\frac{\partial w}{\partial t} = \frac{x+y}{t^2 z^2} \qquad @\ t = 3 \qquad \left.\frac{\partial w}{\partial t}\right|_{t=3} = \frac{\cos^2(3) + \sin^2(3)}{3^2(1/3)^2} = \frac{1}{1} = 1$$

Check: $w(t) = \frac{\cos^2 t}{1/t} + \frac{\sin^2 t}{1/t} = t * (\cos^2 t + \sin^2 t) = t$

$\frac{\partial w}{\partial t} = 1$. So $\left.\frac{\partial w}{\partial t}\right|_{t=3} = 1$ ✓

#4 $w=\ln(x^2+y^2+z^2)$, $x=\cos t$, $y=\sin t$, $z=4\sqrt{t}$ @ $t=3$

$x(3)=\cos(3)$

$y(3)=\sin(3)$

$z(3)=4\sqrt{3}$

CHAIN

$$\boxed{\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}}$$

## Exercises 14.4

### Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express $dw/dt$ as a function of $t$, both by using the Chain Rule and by expressing $w$ in terms of $t$ and differentiating directly with respect to $t$. Then **(b)** evaluate $dw/dt$ at the given value of $t$.

1. $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t = \pi$
2. $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$; $t = 0$
3. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = 1/t$; $t = 3$
4. $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$; $t = 3$

So

$$\frac{\partial w}{\partial t}=\frac{2x}{x^2+y^2+z^2}(-\sin t)+\frac{2y}{x^2+y^2+z^2}(\cos t)+\frac{2z}{x^2+y^2+z^2}\left(\frac{4}{2\sqrt{t}}\right)$$

and

$$\left.\frac{\partial w}{\partial t}\right|_{t=3}=\frac{-2\cancel{\cos(3)\sin(3)}+2\cancel{\sin(3)\cos(3)}}{\cos^2(3)+\sin^2(3)+(4\sqrt{3})^2}+\frac{\cancel{2}*4\cancel{\sqrt{3}}}{\underbrace{\cos^2(3)+\sin^2(3)}_{1}+(4\sqrt{3})^2}*\left(\frac{4}{\cancel{2\sqrt{3}}}\right)$$

$$=\frac{16}{1+16*3}=\boxed{\frac{16}{49}}$$

Check: $w(t)=\ln(\underbrace{\cos^2t+\sin^2t}_{1}+(4\sqrt{t})^2)=\ln(1+16t)$

$\frac{dw}{dt}=\frac{16}{1+16t}$ @ $t=3$ $\quad\frac{dw}{dt}(3)=\frac{16}{1+16*3}=\frac{16}{49}$ ✓

CHAIN rule formula

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}.$$

### Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of $u$ and $v$ both by using the Chain Rule and by expressing $z$ directly in terms of $u$ and $v$ before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point $(u, v)$.

7. $z = 4e^x \ln y$, $x = \ln(u \cos v)$, $y = u \sin v$; $(u, v) = (2, \pi/4)$
8. $z = \tan^{-1}(x/y)$, $x = u \cos v$, $y = u \sin v$; $(u, v) = (1.3, \pi/6)$

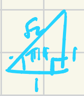
In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of $u$ and $v$ both by using the Chain Rule and by expressing $w$ directly in terms of $u$ and $v$ before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point $(u, v)$.

9. $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$; $(u, v) = (1/2, 1)$
10. $w = \ln(x^2 + y^2 + z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$; $(u, v) = (-2, 0)$

#7 $z=4e^x\ln y$, $x=\ln(u\cos v)$, $y=u\sin v$, $(u,v)=(2,\pi/4)$

@$(2,\pi/4)$ $x=\ln(2*\frac{1}{\sqrt{2}})=\ln(\sqrt{2})$, $y=\sqrt{2}$

$$\boxed{\frac{dz}{du}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$



CHAIN

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\quad\text{and}\quad\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

$$\frac{\partial z}{\partial u}=4\ln y\,e^x*\frac{1}{u\cancel{\cos v}}\cancel{\cos v}+\frac{4e^x}{y}*\sin v$$

$$\frac{\partial z}{\partial u}=\frac{4\ln y\,e^x}{u}+\frac{4e^x\sin v}{y}$$

$$\frac{\partial z}{\partial v}=4\ln y\,e^x*\frac{1}{\cancel{u}\cos v}*-\cancel{u}\sin v+\frac{4e^x}{y}*u\cos v$$

$$\frac{\partial z}{\partial v}=-4\ln y\,e^x\tan v+\frac{4e^x}{y}u\cos v$$

$e^{\ln(t)}=t$

@$(2,\pi/4)$ $$\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(2,\pi/4)}=4\ln(\sqrt{2})e^{\ln(\sqrt{2})}*\frac{1}{2}+\frac{4e^{\ln(\sqrt{2})}}{\sqrt{2}}*1/\sqrt{2}$$

$$=\cancel{4}*\cancel{\tfrac{1}{2}}\ln(2)*\sqrt{2}*\cancel{\tfrac{1}{2}}+\frac{4}{2}*\sqrt{2}$$

$$=\boxed{\sqrt{2}\ln(2)+2\sqrt{2}}$$

@$(2,\pi/4)$ $$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(2,\pi/4)}=-4\ln(\sqrt{2})e^{\ln(\sqrt{2})}*1+\frac{4e^{\ln(\sqrt{2})}}{\cancel{\sqrt{2}}}*\cancel{\sqrt{2}}$$

$$=-4*\tfrac{1}{2}\ln 2*\sqrt{2}+4\sqrt{2}=\boxed{-2\sqrt{2}\ln 2+4\sqrt{2}}$$

also $$z=4e^{\ln(u\cos v)}\ln(u\sin v)$$

$$=4u\cos v\ln(u\sin v)$$

So $$\frac{\partial z}{\partial u}=4\cos v\left(\ln(u\sin v)+\frac{\cancel{u}}{\cancel{u\sin v}}*\cancel{\sin v}\right)$$

$$\boxed{=4\cos v(\ln(u\sin v)+1)}$$

and $$\frac{\partial z}{\partial v}=4u\left(-\sin v\ln(u\sin v)+\frac{\cos v}{\cancel{u}\sin v}*\cancel{u}\cos v\right)$$

$$\boxed{=4u\left(-\sin v\ln(u\sin v)+\frac{\cos^2 v}{\sin v}\right)}$$

**8.** $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u\sin v;$
$(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of $u$ and $v$ both by using the Chain Rule and by expressing $w$ directly in terms of $u$ and $v$ before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point $(u, v)$.

**9.** $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

**10.** $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

#8 $z = \tan^{-1}(x/y)$, $x = u\cos v$, $y = u\sin v$ and $(u,v) = (1.3, \pi/6)$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

@ $(1.3, \pi/6)$ $\quad x = 1.3 * \frac{\sqrt{3}}{2} = .65 * \sqrt{3}$
$y = 1.3 * \frac{1}{2} = .65$


$\cos(\pi/6) = \sqrt{3}/2$
$\sin(\pi/6) = 1/2$

$$\frac{\partial z}{\partial u} = \frac{1}{1+\frac{x^2}{y^2}} * \frac{1}{y} * \cos v + \frac{1}{1+\frac{x^2}{y^2}} * \frac{-x}{y^2} * \sin v$$

$$= \frac{y\cos v}{y^2+x^2} + \frac{-x\sin v}{y^2+x^2}$$

@ $(1.3, \pi/6)$ $$\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1.3,\pi/6)} = \frac{.65 * \cancel{\sqrt{3}/2}}{(.65)^2 + (.65\sqrt{3})^2} - \frac{.65\sqrt{3} * \cancel{\frac{1}{2}}}{(.65)^2 + (.65\sqrt{3})^2} = 0$$

$$\frac{\partial z}{\partial v} = \frac{1}{1+x^2/y^2} * \frac{1}{y} * (-u\sin v) + \frac{1}{1+x^2/y^2} * \frac{-x}{y^2} * u\cos v$$

$$= \frac{-yu\sin v}{y^2+x^2} + \frac{-xu\cos v}{y^2+x^2}$$

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1.3,\pi/6)} = \frac{-.65 * 1.3 * 1/2}{(.65)^2 + (.65\sqrt{3})^2} - \frac{.65\sqrt{3} * 1.3 * \sqrt{3}/2}{(.65)^2 + (.65\sqrt{3})^2}$$

$$= \frac{.65(1.3) * \frac{1}{2}}{(.65)^2 + (.65\sqrt{3})^2}\left[-1 - 3\right]$$

$$= \frac{\cancel{(.65)} 1.3(-2)}{(.65)^{\cancel{2}}(1+3)} = \frac{-2(1.3)}{4(.65)} = -1.$$

$$z = \tan^{-1}\left(\frac{u\cos v}{u\sin v}\right) = \tan^{-1}(\cot v)$$

$$\frac{\partial z}{\partial u} = 0$$

$\sin^2 t + \cos^2 t = 1$

$1 + \cot^2 t = \csc^2 t$

$$\frac{\partial z}{\partial v} = \frac{1}{1+\cot^2 v} * (-\csc^2 v)$$

$$= \frac{1}{\csc^2 v}(-\csc^2 v) = -1.$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

#9 $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$ @ $(u,v) = (1/2, 1)$

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}}$$

**Chain Rule**

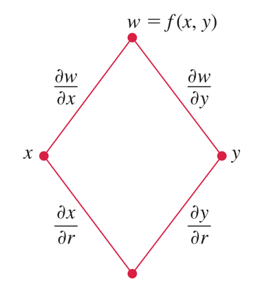



**FIGURE 14.23** Branch diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

(a) $$\frac{\partial w}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v)$$

$$= x + y + 2z + xv + yv$$

$$= u + \cancel{v} + u - \cancel{v} + 2uv + uv + \cancel{v^2} + uv - \cancel{v^2} = 2u + 4uv$$

@ $(1/2, 1)$
$x(1/2, 1) = 3/2$
$y(1/2, 1) = -1/2$
$z(1/2, 1) = 1/2$

$$\frac{\partial w}{\partial v} = (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$= -x + y + xu + yu = -\cancel{u} - v + \cancel{u} - v + u^2 + \cancel{uv} + u^2 - \cancel{uv}$$

$$= -2v + 2u^2$$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

(b) @ $(1/2, 1)$

$$\left.\frac{\partial w}{\partial u}\right|_{@(1/2,1)} = \frac{3}{2} + \frac{-1}{2} + 2\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)(1) + \left(\frac{-1}{2}\right)(1) = 3$$

$$\left.\frac{\partial w}{\partial v}\right|_{@(1/2,1)} = \frac{-3}{2} + \frac{-1}{2} + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) = -1.5$$

Check $\frac{\partial w}{\partial u}$ & $\frac{\partial w}{\partial v}$ @ $(1/2, 1)$

$$\left.\frac{\partial w}{\partial u}\right|_{@(1/2,1)} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1)$$
$$= 1 + 2 = 3 \checkmark$$

$$\left.\frac{\partial w}{\partial v}\right|_{@(1/2,1)} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -2 + \frac{1}{2} = -\frac{3}{2} \checkmark$$

#9 $w = xy + yz + xz, \quad x = u+v, \quad y = u-v, \quad z = uv \quad @(u,v) = (1/2, 1)$

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}}$$

(a)

$$\frac{\partial w}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v)$$

$$= x + y + 2z + xv + yv$$

$$= u + \cancel{v} + u - \cancel{v} + 2uv + uv + \cancel{v^2} + uv - \cancel{v^2} = 2u + 4uv$$

@$(1/2, 1)$

$x(1/2,1) = 3/2$

$y(1/2,1) = -1/2$

$z(1/2,1) = 1/2$

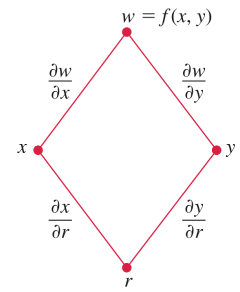
✱

$$\frac{\partial w}{\partial v} = (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$= -x + y + xu + yu = -\cancel{u} - v + \cancel{u} - v + u^2 + \cancel{uv} + u^2 - \cancel{uv}$$

$$= -2v + 2u^2$$

**Chain Rule**



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

**FIGURE 14.23** Branch diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

Check $\frac{\partial w}{\partial u}$ & $\frac{\partial w}{\partial v}$ @$(1/2,1)$

$$\left.\frac{\partial w}{\partial u}\right|_{@(1/2,1)} = 2\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{2}\right)(1) = 1 + 2 = 3 \checkmark$$

$$\left.\frac{\partial w}{\partial v}\right|_{@(1/2,1)} = -2(1) + 2\left(\tfrac{1}{2}\right)^2 = -2 + \tfrac{1}{2} = -3/2 \checkmark$$

$$w(u,v) = (u+v)(u-v) + (u-v)(uv) + (u+v)(uv)$$

$$= u^2 - v^2 + u^2v - \cancel{uv^2} + u^2v + \cancel{uv^2}$$

$$= u^2 - v^2 + 2u^2v$$

$$\frac{\partial w}{\partial u} = 2u + 4uv \checkmark$$

$$\frac{\partial w}{\partial v} = -2v + 2u^2 \checkmark$$

**8.** $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u \sin v;$ $(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of $u$ and $v$ both by using the Chain Rule and by expressing $w$ directly in terms of $u$ and $v$ before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point $(u, v)$.

**9.** $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$ $(u, v) = (1/2, 1)$

**10.** $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$ $z = ue^v; \quad (u, v) = (-2, 0)$

(b)

@$(1/2, 1)$ plug into ✱

$$\left.\frac{\partial w}{\partial u}\right|_{@(1/2,1)} = \frac{3}{2} + \frac{-1}{2} + 2\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)(1) + \left(\frac{-1}{2}\right)(1) = 3$$

$$\left.\frac{\partial w}{\partial v}\right|_{@(1/2,1)} = \frac{-3}{2} + \frac{-1}{2} + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) = -1.5$$

#10 $w = \ln(x^2+y^2+z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$, @$(u,v) = (-2,0)$

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}}$$

(a) $$\frac{\partial w}{\partial u} = \frac{2x}{x^2+y^2+z^2}\left(e^v(\sin u + u\cos u)\right) + \frac{2y}{x^2+y^2+z^2}\left(e^v(\cos u - u\sin u)\right) + \frac{2z}{x^2+y^2+z^2}\left(e^v\right)$$

Notice

$x^2+y^2+z^2 =$

$= u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}$

$= u^2e^{2v}(\sin^2 u + \cos^2 u + 1)$

$= 2u^2e^{2v}$

also

$x^2+y^2 = z^2$ ✓

$$= \frac{1}{u^2e^{2v}}\left(ue^{2v}(\sin^2 u + u\sin u\cos u) + ue^{2v}(\cos^2 u - u\sin u\cos u) + ue^{2v}\right)$$

$$= \frac{1}{u^{\cancel{2}}\cancel{e^{2v}}} * u\cancel{e^{2v}}\left(\overbrace{\sin^2 u + u\cancel{\sin u\cos u} + \cos^2 u}^{1} - u\cancel{\sin u\cos u} + 1\right)$$

$$= \frac{1}{u} * 2 = 2/u$$

$x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$

$$\frac{\partial w}{\partial v} = \frac{2x}{x^2+y^2+z^2}\left(\overbrace{ue^v\sin u}^{x'=x}\right) + \frac{2y}{x^2+y^2+z^2}\left(\overbrace{ue^v\cos u}^{y'=y}\right) + \frac{2z}{x^2+y^2+z^2}\left(\overbrace{ue^v}^{z'=z}\right)$$

$$= \frac{1}{u^2e^{2v}}\left(\underbrace{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}_{x^2+y^2+z^2 = 2u^2e^{2v}}\right) = 2$$

8. $z = \tan^{-1}(x/y)$, $x = u\cos v$, $y = u\sin v$; $(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of $u$ and $v$ both by using the Chain Rule and by expressing $w$ directly in terms of $u$ and $v$ before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point $(u, v)$.

9. $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$; $(u, v) = (1/2, 1)$

10. $w = \ln(x^2 + y^2 + z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$; $(u, v) = (-2, 0)$

$$w(u,v) = \ln(x^2+y^2+z^2)$$
$$= \ln(u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v})$$
$$= \ln(u^2e^{2v}(\sin^2 u + \cos^2 u + 1))$$
$$= \ln(2u^2e^{2v}) = \ln(2) + 2\ln(u) + 2v$$

$\frac{\partial w}{\partial u} = \frac{2}{u}$ ✓, $\frac{\partial w}{\partial v} = 2$ ✓

(b) @$(-2,0)$ $\frac{\partial w}{\partial u} = -1/2$ and $\frac{\partial w}{\partial v} = 2$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of $x$, $y$, and $z$ both by using the Chain Rule and by expressing $u$ directly in terms of $x$, $y$, and $z$ before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point $(x, y, z)$.

**11.** $u = \frac{p-q}{q-r}$, $p = x+y+z$, $q = x-y+z$, $r = x+y-z$; $(x, y, z) = (\sqrt{3}, 2, 1)$

**12.** $u = e^{qr}\sin^{-1} p$, $p = \sin x$, $q = z^2 \ln y$, $r = 1/z$; $(x, y, z) = (\pi/4, 1/2, -1/2)$

#11 $u = \frac{p-q}{q-r}$, $p = x+y+z$, $q = x-y+z$, $r = x+y-z$

@ $(x,y,z) = (\sqrt{3}, 2, 1)$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}}$$

(a)

$$\frac{\partial u}{\partial x} = \frac{1}{q-r}(1) + \frac{\overbrace{(q-r)(-1)-(p-q)(1)}^{-q+r-p+q = r-p}}{(q-r)^2}(1) + \frac{\overbrace{-(p-q)(-1)}^{p-q}}{(q-r)^2}(1)$$

$$= \frac{1}{(q-r)^2}\left[\cancel{q}-\cancel{r} + \cancel{r}-\cancel{p} + \cancel{p}-\cancel{q}\right] = 0$$

$$\frac{\partial u}{\partial y} = \frac{1}{q-r}(1) + \frac{r-p}{(q-r)^2}(-1) + \frac{p-q}{(q-r)^2}(1)$$

$$= \frac{1}{(q-r)^2}\left[\cancel{q}-r-r+p+p-\cancel{q}\right] = \frac{2p-2r}{(q-r)^2}$$

So, $\frac{\partial u}{\partial y}(x,y,z) = \frac{2(2z)}{(-2y+2z)^2} = \frac{z}{(z-y)^2}$

$$\frac{\partial u}{\partial z} = \frac{1}{q-r}(1) + \frac{r-p}{(q-r)^2}(1) + \frac{p-q}{(q-r)^2}(-1)$$

$$= \frac{1}{q-r}\left(q-\cancel{r}+\cancel{r}-p-p+q\right) = \frac{2q-2p}{(q-r)^2}$$

So, $\frac{\partial u}{\partial z}(x,y,z) = \frac{2(-p+q)}{(q-r)^2} = 2\frac{-2y}{(-2y+2z)^2} = \frac{-y}{(z-y)^2}$

$u = \frac{p-q}{q-r}$, $p = x+y+z$, $q = x-y+z$, $r = x+y-z$

@ $(x,y,z) = (\sqrt{3}, 2, 1)$

$$u(x,y,z) = \frac{2y}{-2y+2z} = \frac{y}{z-y}$$

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = \frac{(z-y)(1) - y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2} \checkmark$$

$$\frac{\partial u}{\partial z} = \frac{-y}{(z-y)^2} \checkmark$$

(b) @ $(x,y,z) = (\sqrt{3}, 2, 1)$

$$\left.\frac{\partial u}{\partial x}\right|_{@(\sqrt{3},2,1)} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{@(\sqrt{3},2,1)} = \frac{1}{(1-2)^2} = 1$$

$$\left.\frac{\partial u}{\partial z}\right|_{@(\sqrt{3},2,1)} = \frac{-2}{(1-2)^2} = -2$$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of $x$, $y$, and $z$ both by using the Chain Rule and by expressing $u$ directly in terms of $x$, $y$, and $z$ before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point $(x, y, z)$.

**11.** $u = \dfrac{p-q}{q-r}$, $p = x + y + z$, $q = x - y + z$, $r = x + y - z$; $(x, y, z) = \left(\sqrt{3}, 2, 1\right)$

**12.** $u = e^{qr}\sin^{-1} p$, $p = \sin x$, $q = z^2 \ln y$, $r = 1/z$; $(x, y, z) = (\pi/4, 1/2, -1/2)$

#12 $u = e^{qr}\sin^{-1} p$, $p = \sin x$, $q = z^2 \ln y$, $r = 1/z$

$@ (x,y,z) = (\pi/4, 1/2, -1/2)$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}}$$

$$\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + \cancel{re^{qr}\sin^{-1}p(0)} + \cancel{qe^{qr}\sin^{-1}p(0)}$$

$$= \frac{e^{z\ln y}}{\sqrt{\underbrace{1-\sin^2 x}_{\cos x}}}\cancel{(\cos x)}$$

$e^{z\ln x} = (e^{\ln x})^z = x^z$

$= y^z$ (note: $\cos x > 0$ @ $x = \pi/4$)

$$\frac{\partial u}{\partial y} = \cancel{\frac{e^{qr}}{\sqrt{1-p^2}}(0)} + re^{qr}\sin^{-1}p\left(\frac{z^2}{y}\right) + \cancel{qe^{qr}\sin^{-1}(p)(0)}$$

$$= \frac{1}{z}e^{z\ln y}\underbrace{\sin^{-1}(\sin x)}_{x} * \frac{z^2}{y}$$

$$= \frac{zx}{y} * y^z$$

$$\frac{\partial u}{\partial z} = \cancel{\frac{e^{qr}}{\sqrt{1-p^2}}(0)} + re^{qr}\sin^{-1}(p)(2z\ln y) + qe^{qr}\left(\frac{-1}{z^2}\right)$$

$$= \cancel{\tfrac{1}{z}}e^{z\ln y}\underbrace{\sin^{-1}(\sin x)}_{x}\,2\cancel{z}\ln y + \cancel{z^2}\ln y\, e^{z\ln y} * \frac{-1}{\cancel{z^2}}$$

$$= 2y^z x\ln y - y^z\ln y$$

$$u(x,y,z) = e^{z\ln y}\sin^{-1}(\sin x) = xy^z$$

$\frac{\partial u}{\partial x} = y^z$ ✓

$\frac{\partial u}{\partial y} = x * zy^{z-1}$ ✓

$\frac{\partial u}{\partial z} = xy^z * \ln y$

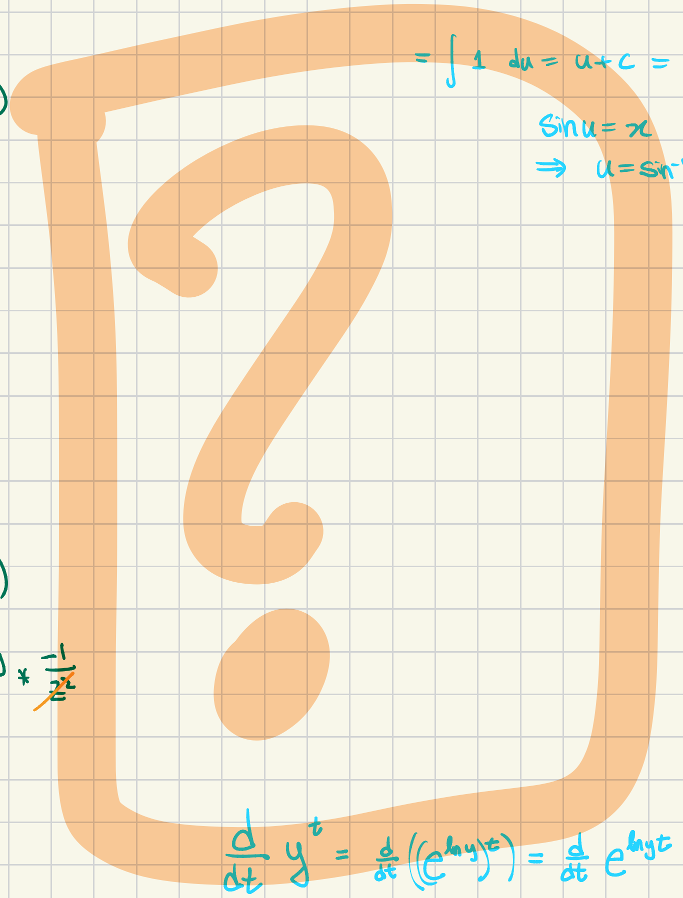
$$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$$

b/c $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{\cos^2 u}}\cos u\,du$

$\sin u = x$, $\cos u\,du = dx$; $1-x^2 = \cos^2 u$

$= \int 1\,du = u + C = \sin^{-1}x + C$ ✓

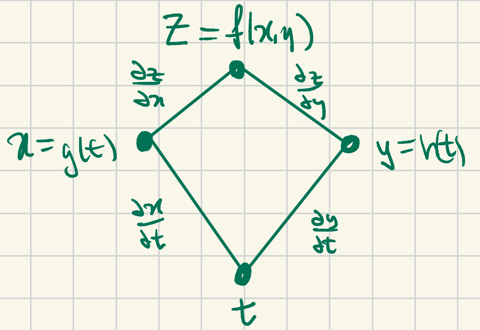
$\sin u = x \Rightarrow u = \sin^{-1}x$



$$\frac{d}{dt}y^t = \frac{d}{dt}\left((e^{\ln y})^t\right) = \frac{d}{dt}e^{\ln y\, t}$$

$$= \ln y\, e^{\ln y\, t} = \ln y * y^t$$

#13 $\frac{dz}{dt}$ for $z = f(x,y)$, $x = g(t)$, $y = h(t)$




$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

## Using a Branch Diagram

In Exercises 13–24, draw a branch diagram and write a Chain Rule formula for each derivative.

**13.** $\frac{dz}{dt}$ for $z = f(x, y)$, $x = g(t)$, $y = h(t)$

**14.** $\frac{dz}{dt}$ for $z = f(u, v, w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$

**15.** $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x, y, z)$, $x = f(u, v)$, $y = g(u, v)$, $z = k(u, v)$

**16.** $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$

**17.** $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = g(x, y)$, $x = h(u, v)$, $y = k(u, v)$

**18.** $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = g(u, v)$, $u = h(x, y)$, $v = k(x, y)$

**19.** $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ for $z = f(x, y)$, $x = g(t, s)$, $y = h(t, s)$

**20.** $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r, s)$

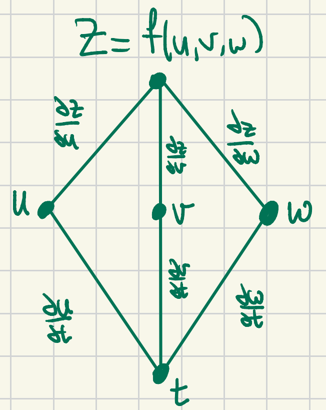
**21.** $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for $w = g(u)$, $u = h(s, t)$

**22.** $\frac{\partial w}{\partial p}$ for $w = f(x, y, z, v)$, $x = g(p, q)$, $y = h(p, q)$, $z = j(p, q)$, $v = k(p, q)$

**23.** $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ for $w = f(x, y)$, $x = g(r)$, $y = h(s)$

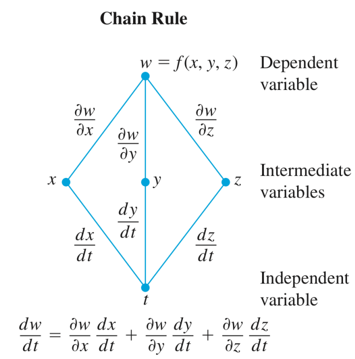
**24.** $\frac{\partial w}{\partial s}$ for $w = g(x, y)$, $x = h(r, s, t)$, $y = k(r, s, t)$

#14 $\frac{dz}{dt}$ for $z = f(u,v,w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$




$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t}$$

Here we have three routes from $w$ to $t$ instead of two, but finding $dw/dt$ is still the same. Read down each route, multiplying derivatives along the way; then add.



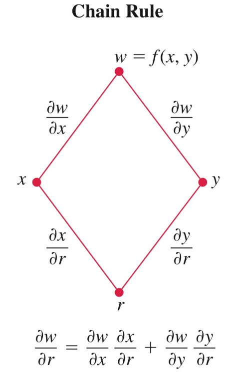
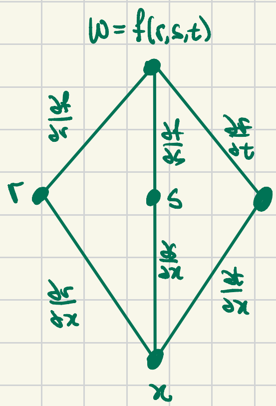
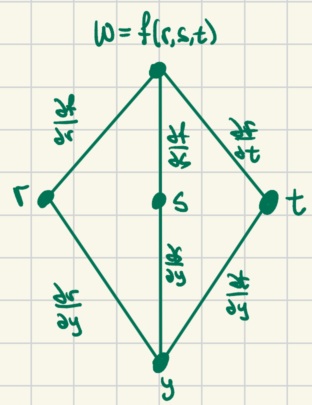




**FIGURE 14.23** Branch diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

#16 $\frac{dw}{dx}$ and $\frac{dw}{dy}$ for $w = f(r,s,t)$, $r = g(x,y)$, $s = h(x,y)$, $t = k(x,y)$




$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

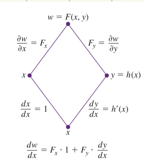



**FIGURE 14.25** Branch diagram for differentiating $w = F(x, y)$ with respect to $x$. Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

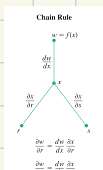
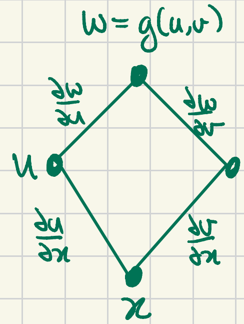
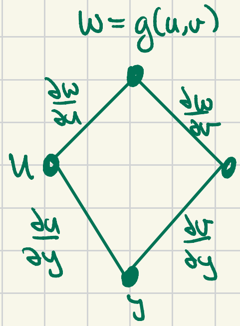



**FIGURE 14.24** Branch diagram for differentiating $f$ as a composite function of $r$ and $s$ with one intermediate variable.

#18 $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w=g(u,v)$, $u=h(x,y)$, and $v=k(x,y)$




$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

## Using a Branch Diagram

In Exercises 13–24, draw a branch diagram and write a Chain Rule formula for each derivative.

**13.** $\frac{dz}{dt}$ for $z = f(x, y)$, $x = g(t)$, $y = h(t)$

**14.** $\frac{dz}{dt}$ for $z = f(u, v, w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$

**15.** $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x, y, z)$, $x = f(u, v)$, $y = g(u, v)$, $z = k(u, v)$

**16.** $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$

**17.** $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = g(x, y)$, $x = h(u, v)$, $y = k(u, v)$

**18.** $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = g(u, v)$, $u = h(x, y)$, $v = k(x, y)$

**19.** $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ for $z = f(x, y)$, $x = g(t, s)$, $y = h(t, s)$

**20.** $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r, s)$

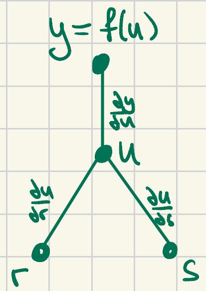
**21.** $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for $w = g(u)$, $u = h(s, t)$

**22.** $\frac{\partial w}{\partial p}$ for $w = f(x, y, z, v)$, $x = g(p, q)$, $y = h(p, q)$, $z = j(p, q)$, $v = k(p, q)$

**23.** $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ for $w = f(x, y)$, $x = g(r)$, $y = h(s)$

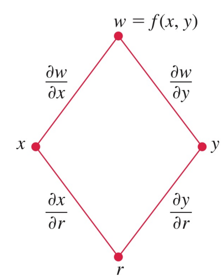
**24.** $\frac{\partial w}{\partial s}$ for $w = g(x, y)$, $x = h(r, s, t)$, $y = k(r, s, t)$

#20 $\frac{\partial y}{\partial r}$ for $y=f(u)$, $u=g(r,s)$




$$\frac{\partial y}{\partial r}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial r}$$

**Chain Rule**

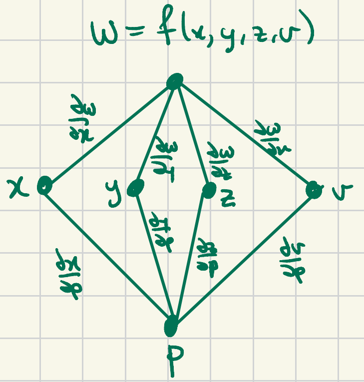



$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

**FIGURE 14.23** Branch diagram for the equation

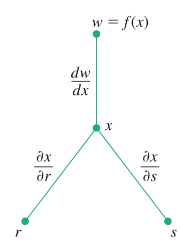
$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

#22 $\frac{\partial w}{\partial p}$ for $w=f(x,y,z,v)$, $x=g(p,q)$, $y=h(p,q)$, $z=j(p,q)$, $v=k(p,q)$




$$\frac{\partial w}{\partial p}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial p}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial p}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

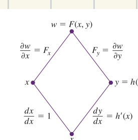
**Chain Rule**




$$\frac{\partial w}{\partial r}=\frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{dw}{dx}\frac{\partial x}{\partial s}$$

**FIGURE 14.24** Branch diagram for differentiating $f$ as a composite function of $r$ and $s$ with one intermediate variable.




$$\frac{dw}{dx}=F_x\cdot 1+F_y\cdot\frac{dy}{dx}$$

**FIGURE 14.25** Branch diagram for differentiating $w = F(x, y)$ with respect to $x$. Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

### Implicit Differentiation

Assuming that the equations in Exercises 25–28 define $y$ as a differentiable function of $x$, use Theorem 8 to find the value of $dy/dx$ at the given point.

**25.** $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

**26.** $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

**27.** $x^2 + xy + y^2 - 7 = 0, \quad (1, 2)$

**28.** $xe^y + \sin xy + y - \ln 2 = 0, \quad (0, \ln 2)$

**THEOREM 8—A Formula for Implicit Differentiation** Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

#25 $x^3 - 2y^2 + xy = 0$, @(1,1) Find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

$$\frac{dy}{dx} = -\frac{3x^2 + y}{-4y + x} \quad @ (1,1) \quad \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{4}{-3} = \frac{4}{3}$$

#26 $xy + y^2 - 3x - 3 = 0$ @ (-1,1)

$$\frac{dy}{dx} = -\frac{y - 3}{x + 2y} \quad @ (-1,1) \quad \left.\frac{dy}{dx}\right|_{(-1,1)} = -\frac{-4}{1} = 4$$

#28 $xe^y + \sin xy + y - \ln 2 = 0$ @ $(0, \ln 2)$

$$\frac{dy}{dx} = -\frac{e^y + y\cos xy}{xe^y + x\cos xy + 1} \quad @ (0, \ln 2) \quad \left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -\frac{2 + \ln 2 \cos(0)}{0 \cdot 2 + \ln 2 \cdot \cos(0) + 1} = -\frac{2 + \ln 2}{\ln 2 + 1} = -1 - \frac{1}{\ln 2 + 1}$$

Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.

**29.** $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

**30.** $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0, \quad (2, 3, 6)$

**31.** $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0, \quad (\pi, \pi, \pi)$

**32.** $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$

#29 $z^3 - xy + yz + y^3 - 2 = 0$ @ (1,1,1) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ @ pt.

(assume $z = z(x,y)$ & use formula $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$)

$$\frac{\partial z}{\partial x} = -\frac{-y}{3z^2 + y} \quad @ (1,1,1) \quad \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -\frac{-1}{3+1} = \frac{1}{4}$$

$$\frac{\partial z}{\partial y} = -\frac{-x + z + 3y^2}{3z^2 + y} \quad @ (1,1,1) \quad \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = -\frac{3}{4}$$

#30 $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ @ (2,3,6)

$$\frac{\partial z}{\partial x} = -\frac{-\frac{1}{x^2}}{-\frac{1}{z^2}} = -\frac{z^2}{x^2} \quad @ (2,3,6) \quad \left.\frac{\partial z}{\partial x}\right|_{(2,3,6)} = -\frac{36}{4} = -8$$

$$\frac{\partial z}{\partial y} = -\frac{-\frac{1}{y^2}}{-\frac{1}{z^2}} = -\frac{z^2}{y^2} \quad @ (2,3,6) \quad \left.\frac{\partial z}{\partial y}\right|_{(2,3,6)} = -\frac{36}{9} = -4$$

#32 $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ @ $(1, \ln 2, \ln 3)$

$$\frac{\partial z}{\partial x} = -\frac{e^y + \frac{2}{x}}{ye^z} \quad @ (1, \ln 2, \ln 3) \quad \left.\frac{\partial z}{\partial x}\right|_{(1,\ln 2,\ln 3)} = -\frac{2 + 2}{\ln 2 \cdot 3} = -\frac{4}{3\ln 2}$$

$$\frac{\partial z}{\partial y} = -\frac{xe^y + e^z}{ye^z} \quad @ (1, \ln 2, \ln 3) \quad \left.\frac{\partial z}{\partial y}\right|_{(1,\ln 2,\ln 3)} = -\frac{(1)2 + 3}{\ln 2 \cdot 3} = \frac{-5}{3\ln 2}$$

### Finding Partial Derivatives at Specified Points

**33.** Find $\partial w/\partial r$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$, $z = \sin(r + s)$.

**34.** Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u$, $y = u + v$, $z = \cos u$.

**35.** Find $\partial w/\partial v$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1$, $y = 2u + v - 2$.

**36.** Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y$, $x = u^2 + v^2$, $y = uv$.

**37.** Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = \ln 2, v = 1$ if $z = 5\tan^{-1} x$ and $x = e^u + \ln v$.

**38.** Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = 1, v = -2$ if $z = \ln q$ and $q = \sqrt{v + 3}\tan^{-1} u$.

#33 Find $\frac{\partial w}{\partial r}$ when $(r,s) = (1,-1)$ if $w = (x+y+z)^2$ and $x = r-s$, $y = \cos(r+s)$, $z = \sin(r+s)$

$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}}$$

$x(1,-1) = 2$
$y(1,-1) = \cos(0) = 1$
$z(1,-1) = \sin(0) = 0$

$$\frac{\partial w}{\partial r} = 2(x+y+z)*1 + 2(x+y+z)*(-\sin(r+s)) + 2(x+y+z)*\cos(r+s)$$

$$= 2(\underbrace{2+1+0}_{3})*1 + 2(3)(0) + 2(3)(1) = 6+6 = \boxed{12}$$

$(a+b+c)^2 = (a+b+c)(a+b+c)$
$= a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2$
$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
$= a^2 + b^2 + c^2 + 2(ab + ac + bc)$

Check. $w(r,s) = \left((r-s)^2 + \cos(r+s) + \sin(r+s)\right)^2$

$= (r-s)^4 + \cos^2(r+s) + \sin^2(r+s) + 2\left((r-s)^2\cos(r+s) + (r-s)^2\sin(r+s) + \cos(r+s)\sin(r+s)\right)$

$\frac{\partial w}{\partial r} = ?$ ABORT!! terrible plan probably 12.

#34 Find $\frac{\partial w}{\partial v}$ where $w = xy + \ln z$, $x = v^2/u$, $y = u+v$, $z = \cos u$ @ $(u,v) = (-1,2)$

$$\boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}}$$

$x(-1,2) = -4$
$y(-1,2) = 1$
$z(-1,2) = \cos(-1)$ (???)

$$\frac{\partial w}{\partial v} = y * \frac{2v}{u} + x*1 + \cancelto{0}{\frac{1}{z}*(0)}$$

ahh!

$$\left.\frac{\partial w}{\partial v}\right|_{(-1,2)} = 1*\frac{4}{-1} + (-4) = \boxed{-8}$$

#36 Find $\frac{\partial z}{\partial u}$ where $z = \sin xy + x\sin y$, $x = u^2 + v^2$, $y = uv$ @ $(u,v) = (0,1)$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

$x(0,1) = 1$
$y(0,1) = 0$

$$\frac{\partial z}{\partial u} = (y\cos(xy) + \sin y)*\cancelto{0}{2u} + (\cancelto{0}{x}\cos(xy) + \cancelto{0}{x}\cos y)*v$$

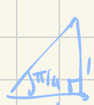
$$\left.\frac{\partial z}{\partial u}\right|_{(0,1)} = \boxed{0}$$

hate when that happens.

#38 Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ where $z = \ln q$, $q = \sqrt{v+3}\tan^{-1}(u)$ @ $(u,v) = (1,-2)$

$$\boxed{\frac{\partial z}{\partial u} = \frac{dz}{dq}*\frac{\partial q}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{dz}{dq}*\frac{\partial q}{\partial v}}$$

$q(1,-2) = \sqrt{1}\tan^{-1}(1) = \pi/4$



$\tan(\pi/4) = 1$
$\tan^{-1}(1) = \pi/4$

$$\frac{\partial z}{\partial u} = \frac{1}{q} * \frac{\sqrt{v+3}}{1+u^2} \quad @(1,-2) \quad \left.\frac{\partial z}{\partial u}\right|_{(1,-2)} = \frac{1}{\pi/4} * \frac{\sqrt{1}}{1+1} = \frac{4}{\pi}*\frac{1}{2} = \boxed{\frac{2}{\pi}}$$

$$\frac{\partial z}{\partial v} = \frac{1}{q}\tan^{-1}(u) * \frac{1}{2\sqrt{v+3}} \quad @(1,-2) \quad \left.\frac{\partial z}{\partial v}\right|_{(1,-2)} = \frac{1}{\pi/4} * \tan^{-1}(1) * \frac{1}{2\sqrt{1}} = \frac{\cancel{4}}{\cancel{\pi}} * \frac{\cancel{\pi}}{\cancel{4}} * \frac{1}{2} = \boxed{\frac{1}{2}}$$

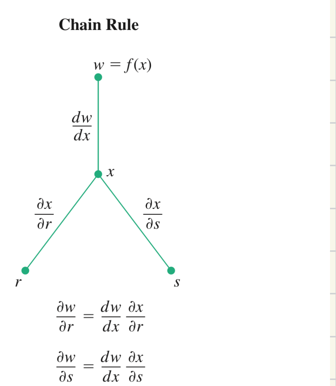



**FIGURE 14.24** Branch diagram for differentiating $f$ as a composite function of $r$ and $s$ with one intermediate variable.

Sal: you know what I really love?
Nobody: Is it calculus?
Sal: ... ...
Nobody: ...?
Sal: ... yes.

# §14.5

## Exercises 14.5

**Calculating Gradients**

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

**1.** $f(x, y) = y - x$, $(2, 1)$

**2.** $f(x, y) = \ln(x^2 + y^2)$, $(1, 1)$

**3.** $g(x, y) = xy^2$, $(2, -1)$

**4.** $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2}, 1)$

**5.** $f(x, y) = \sqrt{2x + 3y}$, $(-1, 2)$

**6.** $f(x, y) = \tan^{-1}\frac{\sqrt{x}}{y}$, $(4, -2)$

**DEFINITION** The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

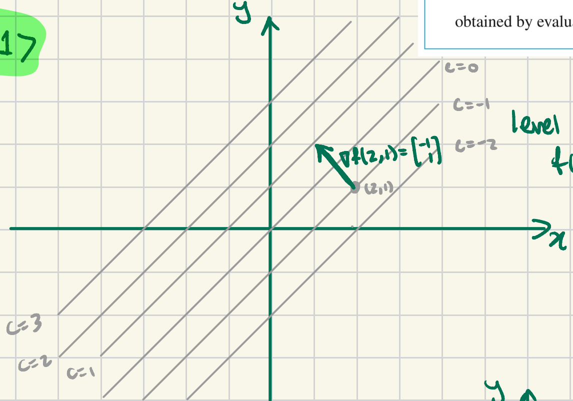
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.

#1 Find $\nabla f$ @ $(2,1)$ if $f(x,y) = y - x$

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}}$$

$\nabla f = \langle -1, 1\rangle$ for all $x,y$. So $\langle -1, 1\rangle$



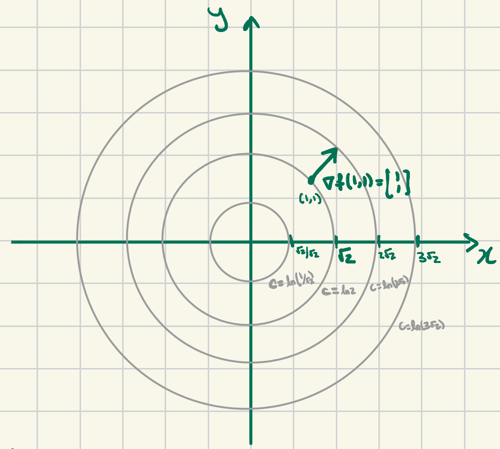

level curves of
$f(x,y) = c \iff y - x = c$
$\iff y = x + c$

#2 Find $\nabla f$ @ $(1,1)$ if $f(x,y) = \ln(x^2+y^2)$

$$\nabla f = \left(\frac{1}{x^2+y^2} * 2x\right)\hat{\imath} + \left(\frac{1}{x^2+y^2} * 2y\right)\hat{\jmath}$$

$$\nabla f\big|_{(1,1)} = \tfrac{1}{2} * 2\,\hat{\imath} + \tfrac{1}{2} * 2\,\hat{\jmath} = \langle 1, 1\rangle$$

level curve of
$f(x,y) = \ln(x^2+y^2)$ @ $(1,1)$
$c = \ln 2$
$\ln(x^2+y^2) = \ln 2$
$\boxed{x^2+y^2 = 2}$
circle of radius $\sqrt{2}$

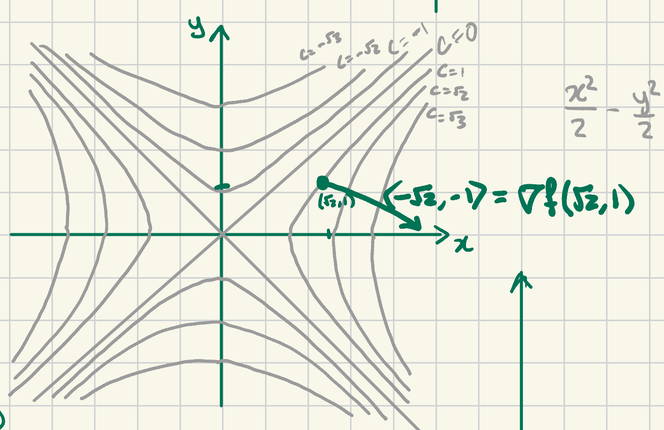



#4 Find $\nabla f$ @ $(\sqrt{2}, 1)$ if $f(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$

$$\nabla f = x\,\hat{\imath} + (-y)\,\hat{\jmath}$$

$$\nabla f\big|_{(\sqrt{2},1)} = \sqrt{2}\,\hat{\imath} - \hat{\jmath} = \langle \sqrt{2}, -1\rangle$$

$\frac{x^2}{2} - \frac{y^2}{2} = c$ hyperbola?

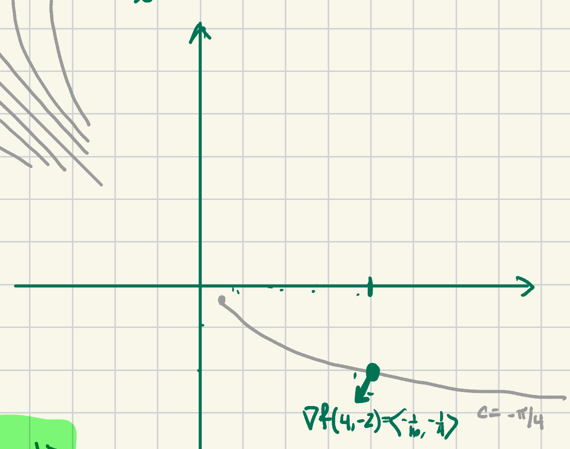



#6 Find $\nabla f$ at $(4,-2)$ if $f(x,y) = \tan^{-1}\left(\frac{\sqrt{x}}{y}\right)$

$$\nabla f = \left(\frac{1}{\left(\frac{\sqrt{x}}{y}\right)^2+1} * \frac{1}{2\sqrt{x}\,y}\right)\hat{\imath} + \left(\frac{1}{\left(\frac{\sqrt{x}}{y}\right)^2+1} * \frac{-\sqrt{x}}{y^2}\right)\hat{\jmath}$$

$$\nabla f\big|_{(4,-2)} = \frac{1}{\left(\frac{\sqrt{4}}{-2}\right)^2+1} * \frac{1}{2\sqrt{4}(-2)}\hat{\imath} + \left(\frac{1}{2} * \frac{-\sqrt{4}}{(-2)^2}\right)\hat{\jmath}$$

$$= \frac{1}{2} * \frac{-1}{8}\hat{\imath} + \frac{1}{2} * \frac{-1}{2}\hat{\jmath} = -\frac{1}{16}\hat{\imath} - \frac{1}{4}\hat{\jmath} = \left\langle -\frac{1}{16}, -\frac{1}{4}\right\rangle$$



In Exercises 7–10, find $\nabla f$ at the given point.

**7.** $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$

**8.** $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz, \quad (1, 1, 1)$

**9.** $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$

**10.** $f(x, y, z) = e^{x+y} \cos z + (y + 1) \sin^{-1} x, \quad (0, 0, \pi/6)$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

#7 Find $\nabla f$ @ $(1,1,1)$ where $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x$

$$\nabla f = \left(2x + \frac{z}{x}\right)\hat{i} + (2y)\hat{j} + (-4z + \ln x)\hat{k}$$

$$\nabla f\big|_{(1,1,1)} = 3\hat{i} + 2\hat{j} - 4\hat{k} = \langle 3, 2, -4 \rangle$$

#8 Find $\nabla f$ @ $(1,1,1)$ where $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1}(xz)$

$$\nabla f = \left(-6xz + \frac{1}{(xz)^2+1} * z\right)\hat{i} + (-6yz)\hat{j} + \left(6z^2 - 3(x^2+y^2) + \frac{1}{(xz)^2+1} * x\right)\hat{k}$$

$$\nabla f\big|_{(1,1,1)} = \left(-6 + \tfrac{1}{2}(1)\right)\hat{i} + (-6)\hat{j} + \left(6 - 3(2) + \tfrac{1}{2}(1)\right)\hat{k}$$

$$= -5.5\hat{i} - 6\hat{j} + \tfrac{1}{2}\hat{k} = \langle -5.5, -6, .5 \rangle$$

WANT $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$x = \sin u$
$dx = \cos u\, du$
$u = \sin^{-1} x$

Check: $\int \frac{d}{dx}\sin^{-1}(x)\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx$

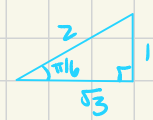
$= \int \frac{1}{\sqrt{1-\sin^2 u}} \cos u\, du$

$= \int 1\, du = u + C$

$= \sin^{-1}(x) + C$ ✓

#10 Find $\nabla f$ @ $(0,0,\pi/6)$ where $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1}(x)$

$$\nabla f = \left(e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}}\right)\hat{i} + \left(e^{x+y}\cos z + \sin^{-1}(x)\right)\hat{j} + \left(-e^{x+y}\sin z\right)\hat{k}$$




$\cos(\pi/6) = \frac{\sqrt{3}}{2}$

$\sin(\pi/6) = 1/2$

$$\nabla f\big|_{(0,0,\pi/6)} = \left(\frac{\sqrt{3}}{2} + 1\right)\hat{i} + \left(\frac{\sqrt{3}}{2} + 0\right)\hat{j} + \left(-1 * \tfrac{1}{2}\right)\hat{k}$$

$$= \left(\frac{\sqrt{3}}{2} + 1\right)\hat{i} + \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{k}$$

#11 Find $\left(\frac{df}{ds}\right)_{\vec{u}, P_0}$ where $f(x,y) = 2xy - 3y^2$, $P_0(5,5)$, $\vec{u} = 4\hat{i} + 3\hat{j}$

$$\boxed{\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \nabla f\big|_{P_0} \cdot \hat{u}}, \text{ ie, } \boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

### Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at $P_0$ in the direction of $\mathbf{u}$.

**11.** $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$

**12.** $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$

**13.** $g(x, y) = \dfrac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

**14.** $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2), \quad P_0(1, 1), \quad \mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$

**15.** $f(x, y, z) = xy + yz + zx, \quad P_0(1, -1, 2), \quad \mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

**16.** $f(x, y, z) = x^2 + 2y^2 - 3z^2, \quad P_0(1, 1, 1), \quad \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

**17.** $g(x, y, z) = 3e^x \cos yz, \quad P_0(0, 0, 0), \quad \mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

**18.** $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2), \quad \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\nabla f = 2y\hat{i} + (2x - 6y)\hat{j}$$

$$\nabla f\big|_{(5,5)} = 10\hat{i} + (-20)\hat{j} = \langle 10, -20 \rangle \quad \text{\& } \vec{u} = \langle 4, 3 \rangle$$

$$\text{So } D_{\vec{u}} f\big|_{(5,5)} = \langle 10, -20 \rangle \cdot \langle 4, 3 \rangle = 40 - 60 = -20$$

**THEOREM 9—The Directional Derivative Is a Dot Product** If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \qquad (4)$$

the dot product of the gradient $\nabla f$ at $P_0$ and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

#12 Find $D_{\vec{u}} f\big|_{P_0}$ where $f(x,y) = 2x^2 + y^2$, $P_0(-1,1)$, $\vec{u} = 3\hat{i} - 4\hat{j}$

$$\nabla f = 4x\hat{i} + 2y\hat{j}$$

$$\nabla f\big|_{(-1,1)} = -4\hat{i} + 2\hat{j}, \text{ So } D_{\vec{u}} f\big|_{(-1,1)} = (-4\hat{i} + 2\hat{j}) \cdot (3\hat{i} - 4\hat{j}) = -12 - 8 = -20$$

$$\frac{\sqrt{3}}{\sqrt{3/4}} = \frac{\sqrt{3}}{\sqrt{3}/\sqrt{4}} = 2\frac{\sqrt{3}}{\sqrt{3}} = 2$$

#14 Find $D_{\vec{u}} h\big|_{P_0}$ where $h(x,y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$, $P_0(1,1)$, $u = 3\hat{i} - 2\hat{j}$

$$\nabla f = \left(\frac{1}{(y/x)^2+1} * \frac{-y}{x^2} + \frac{\sqrt{3}}{\sqrt{1-\left(\frac{xy}{2}\right)^2}} * \frac{y}{2}\right)\hat{i} + \left(\frac{1}{(y/x)^2+1} * \frac{1}{x} + \frac{\sqrt{3}}{\sqrt{1-\left(\frac{xy}{2}\right)^2}} * \frac{x}{2}\right)\hat{j}$$

$$\nabla f\big|_{(1,1)} = \left(\tfrac{1}{2} * (-1) * \cancel{\tfrac{\sqrt{3}}{\sqrt{3/4}}}^{\,2} * \tfrac{1}{2}\right)\hat{i} + \left(\tfrac{1}{2} * 1 * \cancel{\tfrac{\sqrt{3}}{\sqrt{3/4}}}^{\,2} * \tfrac{1}{2}\right)\hat{j} = -\tfrac{1}{2}\hat{i} + \tfrac{1}{2}\hat{j}$$

$$\text{So } D_{\vec{u}} f\big|_{(1,1)} = \langle -\tfrac{1}{2}, \tfrac{1}{2} \rangle \cdot \langle 3, -2 \rangle = -\tfrac{3}{2} - 1 = \frac{-5}{2}$$

#16 Find $D_{\vec{u}}f|_{P_0}$ where $f(x,y,z) = x^2+2y^2-3z^2$, $P_0(1,1,1)$, $\vec{u} = \langle 1,1,1\rangle$

$\nabla f = \langle 2x, 4y, -6z\rangle$

$\nabla f|_{(1,1,1)} = \langle 2,4,-6\rangle$ so $D_{\vec{u}}f|_{(1,1,1)} = \langle 2,4,-6\rangle \cdot \langle 1,1,1\rangle$
$= 2+4-6 = \boxed{0}$

#18 Find $D_{\vec{u}}f|_{P_0}$ where $f(x,y,z) = \cos xy + e^{yz} + \ln zx$, $P_0(1,0,\frac{1}{2})$ and $\vec{u} = \langle 1,2,2\rangle$

$\nabla f = \left(-\cancel{y}^{\,0}\sin xy + \frac{1}{x}\right)\hat{i} + \left(-x\cancel{\sin xy}^{\,\sin(0)=0} + ze^{yz}\right)\hat{j} + \left(\cancel{y}^{\,0}e^{yz} + \frac{1}{z}\right)\hat{k}$

$\nabla f|_{(1,0,1/2)} = 1\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}$ so $D_{\vec{u}}f|_{P_0} = \langle 1, \frac{1}{2}, 2\rangle \cdot \langle 1,2,2\rangle = 1+1+4 = \boxed{6}$

#19 Find the directions of maximum increase/decrease for $f(x,y) = x^2+xy+y^2$ @ $P_0(-1,1)$

$\nabla f = (2x+y)\hat{i} + (x+2y)\hat{j}$

$\nabla f|_{P_0} = \boxed{-\hat{i}+\hat{j}}$ max increase direction

$-\nabla f|_{P_0} = \boxed{\hat{i}-\hat{j}}$ max decrease direction

#20 "——" where $f(x,y) = x^2y + e^{xy}\sin y$, $P_0(1,0)$

$\nabla f = \left(2x\cancel{y}^{\,0} + \cancel{y}^{\,0}\sin y\, e^{xy}\right)\hat{i} + \left(x^2 + xe^{xy}\cancel{\sin y}^{\,\sin(0)=0} + e^{xy}\cos y\right)\hat{j}$

$\nabla f|_{P_0} = 0\hat{i} + \left(1^2 + 0 + \cancel{e^0}^{\,1}\cancel{\cos(0)}^{\,1}\right)\hat{j} = \boxed{0\hat{i}+2\hat{j}}$ max increase direction

and $-\nabla f|_{P_0} = \boxed{0\hat{i}-2\hat{j}}$ max decrease direction

#22 "——" where $g(x,y,z) = xe^y + z^2$, $P_0(1, \ln 2, \frac{1}{2})$

$\nabla g = e^y\hat{i} + xe^y\hat{j} + 2z\hat{k}$

$\nabla g|_{P_0} = e^{\ln 2}\hat{i} + (1)e^{\ln 2}\hat{j} + 2(\frac{1}{2})\hat{k}$

$= \boxed{2\hat{i}+2\hat{j}+\hat{k}}$ max increase direction

and $-\nabla g|_{P_0} = \boxed{-2\hat{i}-2\hat{j}-\hat{k}}$ max decrease direction

## Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at $P_0$ in the direction of **u**.

**11.** $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$

**12.** $f(x, y) = 2x^2 + y^2$, $P_0(-1, 1)$, $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$

**13.** $g(x, y) = \dfrac{x - y}{xy + 2}$, $P_0(1, -1)$, $\mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

**14.** $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$, $P_0(1, 1)$, $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$

**15.** $f(x, y, z) = xy + yz + zx$, $P_0(1, -1, 2)$, $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

**16.** $f(x, y, z) = x^2 + 2y^2 - 3z^2$, $P_0(1, 1, 1)$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

**17.** $g(x, y, z) = 3e^x\cos yz$, $P_0(0, 0, 0)$, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

**18.** $h(x, y, z) = \cos xy + e^{yz} + \ln zx$, $P_0(1, 0, 1/2)$, $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

**THEOREM 9—The Directional Derivative Is a Dot Product** If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \qquad (4)$$

the dot product of the gradient $\nabla f$ at $P_0$ and **u**. In brief, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

$\frac{d}{dx}\ln(cx) = \frac{1}{\cancel{c}x} \cdot \cancel{c}$
$= \frac{1}{x}$

also note
$\ln(cx) = \ln c + \ln x$
so $\frac{d}{dx}\ln(cx) = 0 + \frac{1}{x} = \frac{1}{x}$ ✓

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at $P_0$. Then find the derivatives of the functions in these directions.

**19.** $f(x, y) = x^2 + xy + y^2$, $P_0(-1, 1)$

**20.** $f(x, y) = x^2y + e^{xy}\sin y$, $P_0(1, 0)$

**21.** $f(x, y, z) = (x/y) - yz$, $P_0(4, 1, 1)$

**22.** $g(x, y, z) = xe^y + z^2$, $P_0(1, \ln 2, 1/2)$

**23.** $f(x, y, z) = \ln xy + \ln yz + \ln xz$, $P_0(1, 1, 1)$

**24.** $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z$, $P_0(1, 1, 0)$

Theorem: The function $f$ increases most rapidly in the direction of $\nabla f$, decreases most rapidly in the direction of $-\nabla f$, and does not change value in any direction $\vec{u}$ which is orthogonal to $\nabla f$.

#25 Sketch $f(x,y)=c$, $\nabla f|_{P_0}$ and the tangent line @ $P_0$ and give an equation for the tangent line @ $P_0$.

**Tangent Lines to Level Curves**

In Exercises 25–28, sketch the curve $f(x, y) = c$ together with $\nabla f$ and the tangent line at the given point. Then write an equation for the tangent line.

**25.** $x^2 + y^2 = 4, \quad (\sqrt{2}, \sqrt{2})$

**26.** $x^2 - y = 1, \quad (\sqrt{2}, 1)$

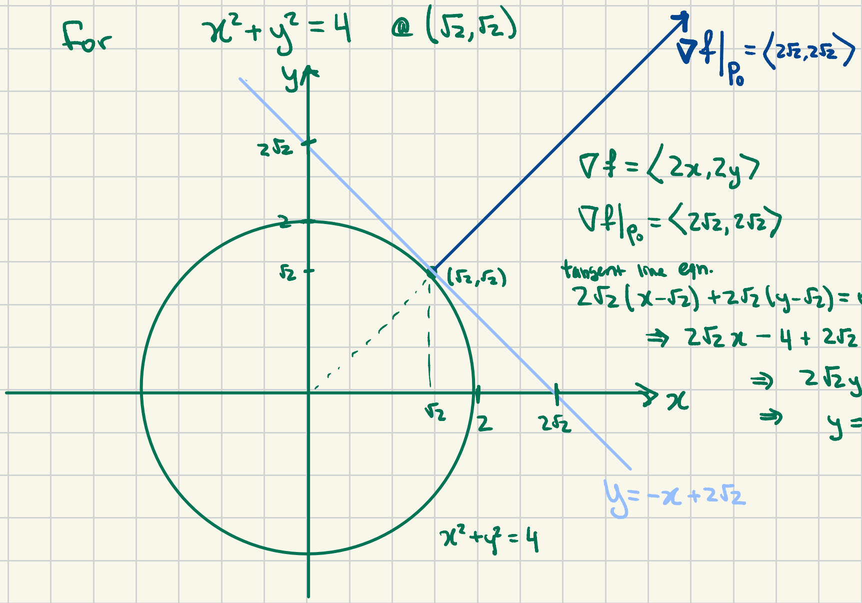
**27.** $xy = -4, \quad (2, -2)$

**28.** $x^2 - xy + y^2 = 7, \quad (-1, 2)$

**Tangent Line to a Level Curve**

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

for $x^2+y^2=4$ @ $(\sqrt{2},\sqrt{2})$




$\nabla f = \langle 2x, 2y\rangle$

$\nabla f|_{P_0} = \langle 2\sqrt{2}, 2\sqrt{2}\rangle$

tangent line eqn.

$2\sqrt{2}(x-\sqrt{2}) + 2\sqrt{2}(y-\sqrt{2}) = 0$

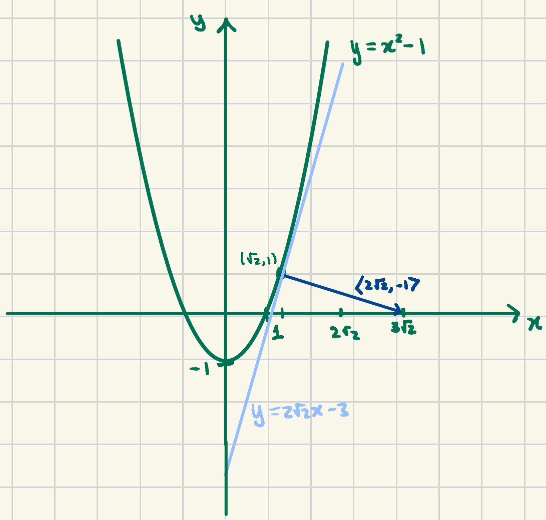
$\Rightarrow 2\sqrt{2}x - 4 + 2\sqrt{2}y - 4 = 0$

$\Rightarrow 2\sqrt{2}y = -2\sqrt{2}x + 8$

$\Rightarrow y = -x + \frac{8}{2\sqrt{2}} = -x + 2\sqrt{2}$

$4/\sqrt{2} = \frac{(\sqrt{2})^4}{\sqrt{2}} = \sqrt{2}^3$

#26 " — " for $x^2-y=1$ @ $(\sqrt{2},1)$




$\nabla f = \langle 2x, -1\rangle$

$\nabla f|_{P_0} = \langle 2\sqrt{2}, -1\rangle$

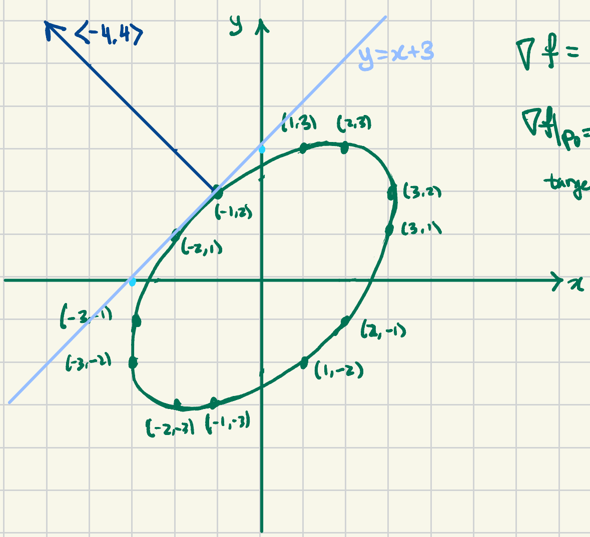
tangent line eqn.

$2\sqrt{2}(x-\sqrt{2}) + (-1)(y-1) = 0$

$\Rightarrow 2\sqrt{2}x - 4 - y + 1 = 0$

$\Rightarrow y = 2\sqrt{2}x - 3$

#28 " — " for $x^2 - xy + y^2 = 7$ @ $(-1,2)$




$\nabla f = \langle 2x-y, -x+2y\rangle$

$\nabla f|_{P_0} = \langle -4, 4\rangle$

tangent line eqn

$-4(x+1) + 4(y-2) = 0$

$\Rightarrow -4x - 4 + 4y - 8 = 0$

$\Rightarrow 4y = 4x + 12$

$\Rightarrow y = x + 3$

# §14.6

## Exercises 14.6

**Tangent Planes and Normal Lines to Surfaces**

In Exercises 1–8, find equations for the

**(a)** tangent plane and

**(b)** normal line at the point $P_0$ on the given surface.

1. $x^2 + y^2 + z^2 = 3$, $P_0(1, 1, 1)$
2. $x^2 + y^2 - z^2 = 18$, $P_0(3, 5, -4)$
3. $2z - x^2 = 0$, $P_0(2, 0, 2)$
4. $x^2 + 2xy - y^2 + z^2 = 7$, $P_0(1, -1, 3)$

**Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \qquad (1)$$

**Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \qquad (2)$$

#1 Find the equation of the tangent plane and normal line to

$x^2+y^2+z^2=3$ @ $P_0(1,1,1)$

(a) Eqn of tangent plane

$$f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$$

$\nabla f = \langle 2x, 2y, 2z\rangle$

$\nabla f(P_0) = \langle 2,2,2\rangle$

Eqn. of tangent plane is

$$2(x-1)+2(y-1)+2(z-1)=0$$

(b) Eqn. of normal line is

$$l(t) = \vec{P_0} + t\nabla f(P_0)$$

normal line is

$$l(t) = \langle 1,1,1\rangle + t\langle 2,2,2\rangle$$

#2 Find tangent plane of surface & normal line

$x^2+y^2-z^2=18$ @ $P_0(3,5,-4)$

(a) $\nabla f = \langle 2x, 2y, -2z\rangle$

$\nabla f(P_0) = \langle 6, 10, -8\rangle$ So eqn of tangent plane is

$$6(x-3)+10(y-5)-8(z+4)=0$$

(b) normal line is $l(t) = \langle 3,5,-4\rangle + t\langle 6,10,-8\rangle$

#4 Find tangent plane & normal line

$x^2+2xy-y^2+z^2=7$, $P_0(1,-1,3)$

(a) $\nabla f = \langle 2x-2y, 2x-2y, 2z\rangle$

$\nabla f(P_0) = \langle 4,4,6\rangle$ So eqn is $4(x-1)+4(y+1)+6(z-3)=0$

(b) and normal line is $l(t) = \langle 1,-1,3\rangle + t\langle 4,4,6\rangle$

In Exercises 9–12, find an equation for the plane that is tangent to the given surface at the given point.

9. $z = \ln(x^2 + y^2)$, $(1, 0, 0)$
10. $z = e^{-(x^2+y^2)}$, $(0, 0, 1)$
11. $z = \sqrt{y - x}$, $(1, 2, 1)$
12. $z = 4x^2 + y^2$, $(1, 1, 5)$

#10 Find eqn of tangent plane

$z = e^{-(x^2+y^2)}$ @ $P_0(0,0,1)$

Can use either

① $\nabla F = \vec{n}$ where $F(x,y,z)=k$ defines your surface

OR

② $z = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)$ where $z=f(x,y)$ defines surface

Option ②

Let $f(x,y) = e^{-x^2-y^2}$ so

$$Df = [-2xe^{-x^2-y^2} \quad -2ye^{-x^2-y^2}] \quad \text{and} \quad Df|_{P_0} = [0 \quad 0]$$

So tangent plane is $z = \frac{1}{e}$

#12 Find tangent plane of surface $z = 4x^2+y^2$ @ $P_0(1,1,5)$

option ② $Df = [8x \quad 2y]$ and $Df|_{P_0} = [8 \quad 2]$

and so tangent plane is $z = 5 + 8(x-1) + 2(y-1)$

### Tangent Lines to Intersecting Surfaces

In Exercises 13–18, find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

**13.** Surfaces: $x + y^2 + 2z = 4, \quad x = 1$
Point: $(1, 1, 1)$

**14.** Surfaces: $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$
Point: $(1, 1, 1)$

**15.** Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$
Point: $(1, 1, 1/2)$

**16.** Surfaces: $x + y^2 + z = 2, \quad y = 1$
Point: $(1/2, 1, 1/2)$

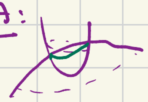
**17.** Surfaces: $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0,$
$x^2 + y^2 + z^2 = 11$
Point: $(1, 1, 3)$

**18.** Surfaces: $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$
Point: $\left(\sqrt{2}, \sqrt{2}, 4\right)$

# #14 Find the eqn. of the line tangent to the curve of intersection of the given surfaces at the specified point.

Surface 1: $xyz = 1$ & Surface 2: $x^2 + 2y^2 + 3z^2 = 6$, @ $P(1,1,1)$

IDEA:



tangent line is IN BOTH tangent planes, so it is orthogonal to BOTH gradients!
ie, use $\vec{v} = \vec{n}_1 \times \vec{n}_2$ and $\ell(t) = \vec{P} + t\vec{v}$

$F(x,y,z) = xyz - 1 = 0$ $\qquad$ $G(x,y,z) = x^2 + 2y^2 + 3z^2 - 6 = 0$

$\nabla F = \langle yz, xz, xy \rangle$ $\qquad$ $\nabla G = \langle 2x, 4y, 6z \rangle$

$\vec{n}_1 = \nabla F(P) = \langle 1,1,1 \rangle$ $\qquad$ $\vec{n}_2 = \nabla G(P) = \langle 2,4,6 \rangle$

$$\text{So} \quad \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = \hat{\imath}(6-4) - \hat{\jmath}(6-2) + \hat{k}(4-2)$$

$$= 2\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$$

Check: $\vec{v} \times n_1 = 2 - 4 + 2 = 0$ ✓
$\vec{v} \times n_2 = 4 - 16 + 12 = 0$ ✓

So

$$\ell(t) = \langle 1,1,1 \rangle + t\langle 2,-4,2 \rangle$$

# #16 tangent line of intersecting curve @ P

surface 1: $x + y^2 + z = 2$ $\qquad$ surface 2: $y = 1$ $\qquad$ & $P(1/2, 1, 1/2)$

$F(x,y,z) = x + y^2 + z - 2 = 0$ $\qquad$ $G(x,y,z) = y - 1 = 0$

$\nabla F = \langle 1, 2y, 1 \rangle$ $\qquad$ $\nabla G \equiv \langle 0,1,0 \rangle$

$\vec{n}_1 = \nabla F(P) = \langle 1,2,1 \rangle$ $\qquad$ $\vec{n}_2 = \langle 0,1,0 \rangle$

$$\text{\& so} \quad \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\imath}(-1) - \hat{\jmath}(0-0) + \hat{k}(1)$$

$$\vec{v} = \langle -1,0,1 \rangle$$

$$\text{So} \quad \ell(t) = \langle 1/2, 1, 1/2 \rangle + t\langle -1,0,1 \rangle$$

# #18 tangent line of intersecting curve @ P

Surface 1: $x^2 + y^2 = 4$ $\qquad$ surface 2: $x^2 + y^2 - z = 0$ $\qquad$ @ $P(\sqrt{2}, \sqrt{2}, 4)$

$F(x,y,z) = x^2 + y^2 - 4 = 0$ $\qquad$ $G(x,y,z) = x^2 + y^2 - z = 0$

$\nabla F = \langle 2x, 2y, 0 \rangle$ $\qquad$ $\nabla G = \langle 2x, 2y, -1 \rangle$

$\vec{n}_1 = \nabla F(P) = \langle 2\sqrt{2}, 2\sqrt{2}, 0 \rangle$ $\qquad$ $\vec{n}_2 = \nabla G(P) = \langle 2\sqrt{2}, 2\sqrt{2}, -1 \rangle$

$$\text{\& so} \quad \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix}$$

$$= \hat{\imath}(-2\sqrt{2}) - \hat{\jmath}(-2\sqrt{2}) + \hat{k}(8-8)$$

$$= \langle -2\sqrt{2}, 2\sqrt{2}, 0 \rangle$$

$$\text{So} \quad \ell(t) = \langle \sqrt{2}, \sqrt{2}, 4 \rangle + t\langle 2\sqrt{2}, -2\sqrt{2}, 0 \rangle$$

### Finding Linearizations

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

**25.** $f(x, y) = x^2 + y^2 + 1$ at **a.** $(0, 0)$, **b.** $(1, 1)$

**26.** $f(x, y) = (x + y + 2)^2$ at **a.** $(0, 0)$, **b.** $(1, 2)$

**27.** $f(x, y) = 3x - 4y + 5$ at **a.** $(0, 0)$, **b.** $(1, 1)$

**28.** $f(x, y) = x^3y^4$ at **a.** $(1, 1)$, **b.** $(0, 0)$

**29.** $f(x, y) = e^x \cos y$ at **a.** $(0, 0)$, **b.** $(0, \pi/2)$

**30.** $f(x, y) = e^{2y-x}$ at **a.** $(0, 0)$, **b.** $(1, 2)$

# #26 Find linearization of $f(x,y) = (x+y+2)^2$ @ (a) $P(0,0)$ and (b) $Q(1,2)$

(a) $Df = [f_x \ \ f_y] = [2x+2y+4 \quad 2x+2y+4]$

$Df|_P = [4 \ \ 4]$ & $Df|_Q = [10 \ \ 10]$, $f(P) = 4$, $f(Q) = 25$

So
(a) $L_1(x,y) = 4 + 4x + 4y$
(b) $L_2(x,y) = 25 + 10x + 10y$

tangent plane aka linearization.

$$L(x,y) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0)$$

where $z = f(x,y)$ defines surface

## Finding Linearizations

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

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**27.** $f(x, y) = 3x - 4y + 5$ at **a.** $(0, 0)$, **b.** $(1, 1)$

**28.** $f(x, y) = x^3y^4$ at **a.** $(1, 1)$, **b.** $(0, 0)$

**29.** $f(x, y) = e^x \cos y$ at **a.** $(0, 0)$, **b.** $(0, \pi/2)$

**30.** $f(x, y) = e^{2y-x}$ at **a.** $(0, 0)$, **b.** $(1, 2)$

tangent plane aka linearization.

$$\boxed{L(x,y) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)}$$

where $z = f(x,y)$ defines surface

#28 Find linearization @ (a) $P(1,1)$ (b) $Q(0,0)$

$f(x,y) = x^3y^4$

$Df = [3x^2y^4 \quad 4x^3y^3]$

$Df|_P = [3 \quad 4]$ & $Df|_Q = [0 \quad 0]$

and

$f(P) = 1$ & $f(Q) = 0$

So (a) $L_1(x,y) = 1 + 3(x-1) + 4(y-1)$

(b) $L_2(x,y) = 0$

#30 Find linearization @ (a) $P(0,0)$ (b) $Q(1,2)$

$f(x,y) = e^{2x-y}$

$Df = [2e^{2x-y} \quad -e^{2x-y}]$

$Df|_P = [2 \quad -1]$ & $Df|_Q = [2 \quad -1]$

and

$f(P) = 1$ & $f(Q) = 1$

So (a) $L_1(x,y) = 1 + 2x - y$

(b) $L_2(x,y) = 1 + 2(x-1) - (y-2)$

parallel planes

## Linearizations for Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 39–44 at the given points.

**39.** $f(x, y, z) = xy + yz + xz$ at **a.** $(1, 1, 1)$ **b.** $(1, 0, 0)$ **c.** $(0, 0, 0)$

**40.** $f(x, y, z) = x^2 + y^2 + z^2$ at **a.** $(1, 1, 1)$ **b.** $(0, 1, 0)$ **c.** $(1, 0, 0)$

**41.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at **a.** $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 2, 2)$

**42.** $f(x, y, z) = (\sin xy)/z$ at **a.** $(\pi/2, 1, 1)$ **b.** $(2, 0, 1)$

**43.** $f(x, y, z) = e^x + \cos(y + z)$ at **a.** $(0, 0, 0)$ **b.** $\left(0, \frac{\pi}{2}, 0\right)$ **c.** $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

**44.** $f(x, y, z) = \tan^{-1}(xyz)$ at **a.** $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 1, 1)$

#40 Find linearization @ (a) $P(1,1,1)$ (b) $Q(0,1,0)$ (c) $R(1,0,0)$

$f(x,y) = x^2+y^2+z^2$

$Df = [2x \quad 2y \quad 2z]$

$Df|_P = [2 \quad 2 \quad 2]$ & $Df|_Q = [0 \quad 2 \quad 0]$ & $Df|_R = [2 \quad 0 \quad 0]$

and

$f(P) = 3$ & $f(Q) = 1$ & $f(R) = 1$

So (a) $L_1(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$

(b) $L_2(x,y,z) = 1 + 2(y-1)$

(c) $L_3(x,y,z) = 1 + 2(x-1)$

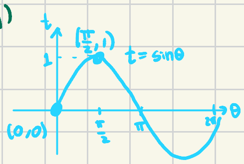
three variable linearization.

$$\boxed{L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)}$$

where $w = f(x,y,z)$ defines surface

#40 Find linearization @ (a) $P(\frac{\pi}{2},1,1)$ (b) $Q(2,0,1)$

$f(x,y) = \frac{\sin xy}{z}$




$Df = \left[\frac{y\sin xy}{z} \quad \frac{x\sin xy}{z} \quad \frac{-\sin xy}{z^2}\right]$

$Df|_P = [1 \quad \pi/2 \quad -1]$ & $Df|_Q = [0 \quad 0 \quad 0]$

and

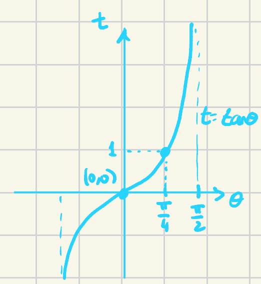
$f(P) = 1$ & $f(Q) = 0$

So (a) $L_1(x,y,z) = 1 + (x - \pi/2) + (y-1) + (z-1)$

(b) $L_2(x,y,z) = 0$

#44 Find linearization @ (a) $P(1,0,0)$ (b) $Q(1,1,0)$ (c) $R(1,1,1)$

$f(x,y) = \tan^{-1}(xyz)$




$Df = \left[\frac{yz}{1+x^2y^2z^2} \quad \frac{xz}{1+x^2y^2z^2} \quad \frac{xy}{1+x^2y^2z^2}\right]$

$Df|_P = [0 \quad 0 \quad 0]$ & $Df|_Q = [0 \quad 0 \quad 1]$

& $Df|_R = [1 \quad 1 \quad 1]$

$\tan(\frac{\pi}{4}) = 1$

$\Leftrightarrow \tan^{-1}(1) = \pi/4$

and

$f(P) = 0$ & $f(Q) = 0$ & $f(R) = \pi/4$

So (a) $L_1(x,y,z) = 0$

(b) $L_2(x,y,z) = z$

(c) $L_3(x,y,z) = (x-1) + (y-1) + (z-1)$

# §14.7

Find all crit points of $f(x,y)$

**Exercises 14.7**

**Finding Local Extrema**

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
3. $f(x, y) = x^2 + xy + 3x + 2y + 5$
4. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$
5. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
6. $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$
7. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$
8. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$
9. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$
10. $f(x, y) = x^2 + 2xy$
11. $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$
12. $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$

#2 $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

Step 1: Find $Df$ and set equal to zero row

$$Df = [f_x \ \ f_y] = [2y - 10x + 4 \quad 2x - 4y + 4] = [0 \quad 0]$$

Step 2: Solve resulting system of (possibly non-linear) eqns.

Solve.
$$\begin{cases} ① \ 2y - 10x + 4 = 0 \\ ② \ 2x - 4y + 4 = 0 \end{cases}$$

① says $y = 5x - 2$

sub into ② $2x - 4(5x-2) + 4 = 0$

$\Rightarrow 2x - 20x + 8 + 4 = 0$

$\Rightarrow -18x = -12$, $x = 12/18 = 2/3$

& sub back into ① to get $y = 5(2/3) - 2$
$= 10/3 - 2$
$= 4/3$

So crit point is $(x,y) = (2/3, 4/3)$

#4 $f(x,y) = 5xy - 7x^2 + 3x - 6y + 2$

Step 1:
$$Df = [5y - 14x + 3 \quad 5x - 6] = [0 \quad 0]$$

Step 2:
$$\begin{cases} ① \ 5y - 14x + 3 = 0 \\ ② \ 5x - 6 = 0 \end{cases}$$

② already says $x = 6/5$

So ① becomes

$5y - 14\left(\frac{6}{5}\right) + 3 = 0$

$\Rightarrow 5y - \frac{14 \cdot 6 - 3 \cdot 5}{5} = 0$

So $y = \frac{84 - 15}{25} = \frac{69}{25}$

and crit point is $\left(6/5, 69/25\right)$

#12 $f(x,y) = 1 - \sqrt[3]{x^2 + y^2}$

$$\left(\sqrt[3]{t}\right)' = \left(t^{1/3}\right)' = \frac{1}{3} t^{-2/3} = \frac{1}{3t^{2/3}}$$

Step 1: $Df = \left[\frac{2x}{3(x^2+y^2)^{2/3}} \quad \frac{2y}{3(x^2+y^2)^{2/3}}\right] = [0 \quad 0]$

Step 2: Solve
$$\begin{cases} ① \ \frac{2x}{3(x^2+y^2)^{2/3}} = 0 \\ ② \ \frac{2y}{3(x^2+y^2)^{2/3}} = 0 \end{cases}$$

But this system has NO SOLUTIONS.

however, $Df$ is DNE at $(0,0)$. So.

Crit point is $(0,0)$

13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$
15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
17. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$
18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
19. $f(x, y) = 4xy - x^4 - y^4$
20. $f(x, y) = x^4 + y^4 + 4xy$
21. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$
22. $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
23. $f(x, y) = y \sin x$
24. $f(x, y) = e^{2x} \cos y$
25. $f(x, y) = e^{x^2+y^2-4x}$
26. $f(x, y) = e^y - ye^x$
27. $f(x, y) = e^{-y}(x^2 + y^2)$
28. $f(x, y) = e^x(x^2 - y^2)$
29. $f(x, y) = 2 \ln x + \ln y - 4x - y$
30. $f(x, y) = \ln (x + y) + x^2 - y$

#14 $f(x,y) = x^3 + 3xy + y^3$

Step 1: $Df = [3x^2 + 3y \quad 3y^2 + 3x] = [0 \quad 0]$

Step 2. Solve
$$\begin{cases} ① \ 3x^2 + 3y = 0 \\ ② \ 3y^2 + 3x = 0 \end{cases} \Rightarrow \begin{cases} ① \ x^2 + y = 0 \\ ② \ y^2 + x = 0 \end{cases}$$

① becomes
$-x^2 = y$
sub into ② get
$(-x^2)^2 + x = 0$

$\Rightarrow x^4 + x = x(x^3 + 1) = 0$

So $x = 0$ or $x^3 = -1$

Hence $\underline{x = 0, -1}$

If $x = 0$, Then from ① $y = 0$
else $x = -1$, Then from ① $y = -1$

So crit points are $(0,0), (-1,-1)$

Find all crit points of $f(x,y)$

13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$
15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
17. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$
18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
19. $f(x, y) = 4xy - x^4 - y^4$
20. $f(x, y) = x^4 + y^4 + 4xy$
21. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$
22. $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
23. $f(x, y) = y \sin x$
24. $f(x, y) = e^{2x} \cos y$
25. $f(x, y) = e^{x^2+y^2-4x}$
26. $f(x, y) = e^y - ye^x$
27. $f(x, y) = e^{-y}(x^2 + y^2)$
28. $f(x, y) = e^x(x^2 - y^2)$
29. $f(x, y) = 2 \ln x + \ln y - 4x - y$
30. $f(x, y) = \ln (x + y) + x^2 - y$

#16 $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

Step 1: Find Df & set to the zero row.

$$Df = [3x^2+6x \quad 3y^2-6y] = [0 \quad 0]$$

Step 2: Solve the resulting system.

$$\begin{cases} ① \ 3x^2+6x=0 \\ ② \ 3y^2-6y=0 \end{cases}$$

From ① $x^2+x = x(x+1)=0$ $\quad x=0,-1$

From ② $y^2-y = y(y-1)=0$ $\quad y=0,1$.

NOTE: ALL FOUR PAIRS are valid

$x$ either 0 or -1

$y$ either 0 or 1

(*) idea: ① & ② say something about $x$ OR $y$, not any relationship between $x$ & $y$ !!!

(can say $x$ & $y$ are INDEPENDENT vars)

So crit points are

$$\boxed{(0,0),\ (0,1),\ (-1,0),\ (-1,1)}$$

#18 $f(x,y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

Step 1: $Df = [6x^2 - 18x \quad 6y^2+6y-12] = [0 \quad 0]$

Step 2: Solve

$$\begin{cases} ① \ 6x^2-18x=0 \\ ② \ 6y^2+6y-12=0 \end{cases}$$

① says $x^2-3x=0 \Rightarrow x(x-3)=0 \Rightarrow x=0$ or $x=3$.

② says $y^2+y-2 = (y+2)(y-1)=0 \Rightarrow y=-2$ or $y=1$.

ALL four pairs are possible see (*) above So crit points are $\boxed{(0,-2),\ (0,1),\ (3,-2),\ (3,1)}$

#20 $f(x,y) = x^4 + y^4 + 4xy$

Step 1: $Df = [4x^3+4y \quad 4y^3+4x] = [0 \quad 0]$

Step 2: Solve $\begin{cases} ① \ 4x^3+4y=0 \\ ② \ 4y^3+4x=0 \end{cases} \Rightarrow \begin{cases} ① \ -x^3=y \\ ② \ -y^3=x \end{cases}$

Sub ① into ② to get

$-(-x^3)^3 = x \Rightarrow -x^6 = x \Rightarrow x^6+x=0 \Rightarrow x(x^5+1)=0$

$\Rightarrow x=0$ or $x^5=-1$

$\Rightarrow x=0$ or $x=-1$

Now sub $x=0$ or $x=-1$ back into ① to get

when $x=0$, $y=0$

when $x=-1$, $y=1$

So TWO solutions are $\boxed{(0,0)\ \&\ (-1,1)}$

(**) Idea: In this case, ① & ② are providing a relationship between $x$ & $y$, so the variables $x$ & $y$ are in some dependence relation

#22 $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

Step 1: $Df = \left[\frac{-1}{x^2}+y \quad \frac{-1}{y^2}+x\right] = [0 \quad 0]$

Step 2: Solve

$$\begin{cases} ① \ -\frac{1}{x^2}+y=0 \\ ② \ -\frac{1}{y^2}+x=0 \end{cases} \Rightarrow \begin{cases} ① \ y=\frac{1}{x^2} \\ ② \ x=\frac{1}{y^2} \end{cases}$$

Sub ① into ② to get

$x = \frac{1}{(1/x^2)} = x^2 \Rightarrow x^2-x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0$ or $x=1$

Sim to above, (**) sub into ① to get TWO Solutions

$$\boxed{(0,0),\ (1,1)}$$

Find all crit points of $f(x,y)$

#24 $f(x,y) = e^{2x}\cos y$ Step 1: $Df = [2e^{2x}\cos y \quad -e^{2x}\sin y] = [0 \ 0]$

Step 2: Solve $\begin{cases} ① \ 2e^{2x}\cos y = 0 \\ ② \ -e^{2x}\sin y = 0 \end{cases}$

Since $e^{2x} \neq 0$ for any $x \in \mathbb{R}$, we need ① $\cos y = 0$ and ② $\sin y = 0$.

BUT ① & ② are never both true since $\sin^2 y + \cos^2 y = 1$ (pythagorus).

So $f$ has NO CRIT PTS

#26 $f(x,y) = e^y - ye^x$ Step 1: $Df = [-ye^x \quad e^y - e^x] = [0 \ 0]$

Step 2: Solve $\begin{cases} ① \ -ye^x = 0 \\ ② \ e^y - e^x = 0 \end{cases}$

Since $e^x \neq 0$ for any $x \in \mathbb{R}$, ① says $\underline{y=0}$.

So ② says $e^0 = e^x \Rightarrow e^x = 1$.

So $\ln(e^x) = \ln(1) \Rightarrow \underline{x=0}$.

So only crit point is at $(0,0)$

13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$
15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
17. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$
18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
19. $f(x, y) = 4xy - x^4 - y^4$
20. $f(x, y) = x^4 + y^4 + 4xy$
21. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$
22. $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
23. $f(x, y) = y \sin x$
24. $f(x, y) = e^{2x} \cos y$
25. $f(x, y) = e^{x^2+y^2-4x}$
26. $f(x, y) = e^y - ye^x$
27. $f(x, y) = e^{-y}(x^2 + y^2)$
28. $f(x, y) = e^x(x^2 - y^2)$
29. $f(x, y) = 2 \ln x + \ln y - 4x - y$
30. $f(x, y) = \ln (x + y) + x^2 - y$

#28 $f(x,y) = e^x(x^2 - y^2)$ Step 1: $Df = [e^x(2x - 0) + e^x(x^2 - y^2) \quad e^x(-2y)]$

$= [e^x(x^2 - 2x - y^2) \quad -2ye^x] = [0 \ 0]$

Step 2: Solve $\begin{cases} ① \ e^x(x^2 - 2x - y^2) = 0 \\ ② \ -2ye^x = 0 \end{cases}$

NOTE: Since $e^x \neq 0$ for all $x \in \mathbb{R}$, $\begin{cases} ① \\ ② \end{cases}$ becomes $\begin{cases} ① \ x^2 - 2x - y^2 = 0 \\ ② \ -2y = 0 \end{cases}$

So by ②, $\underline{y=0}$.

Sub into ①, $x^2 - 2x = 0$

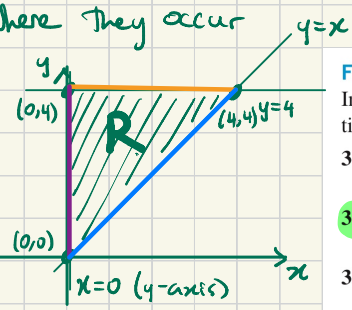
$\Rightarrow x(x-2) = 0$

So $\underline{x=0 \text{ or } x=2}$.

So TWO crit points at $(0,0)$ & $(2,0)$

Find absolute MAX/MIN on R & where they occur

#32 $D(x,y) = x^2 - xy + y^2 + 1$ on R




IDEA: ① Find 2-D crit points in R using $Df = [0 \ 0]$

② Find 1-D crit points on boundary segments by specializing away one of the vars

③ Evaluate at the 0-D vertices of R & any pts from ① or ②

### Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant
32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$
33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant
34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 3$
35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$
36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1$, $0 \le y \le 1$
37. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (see accompanying figure)

① $DD = [2x - y \quad 2y - x] = [0 \ 0]$ so solve $\begin{cases} 2x - y = 0 \\ 2y - x = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ x = 2y \end{cases} \Rightarrow x = 4x \Rightarrow x = 0$, $\Rightarrow y = 0$

Get $(0,0)$.

② ① $y=x$ $f_1(x) = D(x,x) = x^2 - x^2 + x^2 + 1 = x^2 + 1$

$f_1'(x) = 2x = 0 \Rightarrow x=0, y=0$ get $(0,0)$

@ $x=0$ $f_2(y) = D(0,y) = y^2 + 1$, $f_2'(y) = 2y = 0 \Rightarrow y=0, x=0$ get $(0,0)$

@ $y=4$ $f_3(x) = D(x,4) = x^2 - 4x + 16 + 1 = x^2 - 4x + 17$

$f_3'(x) = 2x - 4 = 0 \Rightarrow x=2, y=4$ get $(0,4)$.

③

| $(x,y)$ | $D(x,y)$ |
|---|---|
| $(0,0)$ | 1 |
| $(0,4)$ | 17 |
| $(4,4)$ | 17 |

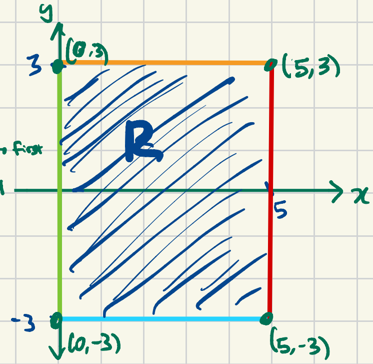
MIN value of 1 @ $(0,0)$

MAX value of 17 @ $(0,4)$ & $(4,4)$

Find absolute MAX/MIN & where they occur.

#34 $f(x,y) = x^2 + xy + y^2 - 6x$ on R

See previous page for IDEA.




**Finding Absolute Extrema**

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant

32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$

33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant

34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 3$

35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$

36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1$, $0 \le y \le 1$

37. $f(x, y) = (4x - x^2)\cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (see accompanying figure)

① $Df = [2x+y-6 \quad 2y+x] = [0 \quad 0]$

Solve $\begin{cases} 2x+y-6=0 \\ 2y+x=0 \end{cases} \Rightarrow$ $x=-2y$ from 2nd eqn, sub into first $-4y+y-6=0 \Rightarrow y=-2, x=4$

get $(4,-2)$

② @ $x=0$ $f_1(y) = f(0,y) = y^2 \rightarrow f_1'(y) = 2y = 0$

so $y=0$, get $(0,0)$

@ $x=5$ $f_2(y) = f(5,y) = 25+5y+y^2-30 = y^2+5y-5$ so $f_2'(y) = 2y+5=0$

$\Rightarrow y = -5/2$ and $x=5$

get $(5, -5/2)$

@ $y=-3$ $f_3(x,-3) = x^2-3x+9-6x = x^2-9x+9$ so $f_3'(x) = 2x-9=0 \Rightarrow x = 9/2$ and $y=-3$.

get $(9/2, -3)$

@ $y=3$ $f_4(x,3) = x^2+3x+9-6x = x^2-3x+9$ So, $f_4'(x) = 2x-3=0 \Rightarrow x=3/2$, $y=3$

get $(3/2, 3)$

③ Evaluate

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(4,-2)$ | $\boxed{-12}$ MIN |
| $(0,0)$ | $0$ |
| $(5,-5/2)$ | $-11.25$ |
| $(9/2,-3)$ | $-11.25$ |
| $(3/2,3)$ | $6.75$ |
| $(0,3)$ | $9$ |
| $(5,3)$ | $\boxed{19}$ MAX |
| $(0,-3)$ | $9$ |
| $(5,-3)$ | $-11$ |

So MAX value of 19 at (5,3) and MIN value of -12 at (4,-2)

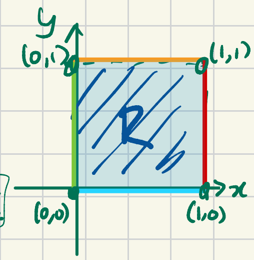
#36 $f(x,y) = 48xy - 32x^3 - 24y^2$ on R

① $Df = [48y - 96x^2 \quad 48x - 48y] = [0 \quad 0]$ Solve $\begin{cases} 48y - 96x^2 = 0 \\ 48x - 48y = 0 \end{cases} \Rightarrow \begin{cases} y = 2x^2 \\ y = x \end{cases}$

So $x = 2x^2 \Rightarrow x(2x-1)=0$

$\Rightarrow \boxed{x=0}$ or $\boxed{x=1/2}$ and $\boxed{y=0}$ and $\boxed{y=1/2}$

get $(0,0)$ & $(1/2, 1/2)$




② @ $x=0$ $f_1(y) = f(0,y) = -24y^2$, $f_1'(y) = -48y = 0 \Rightarrow y=0, x=0$

get $(0,0)$

@ $x=1$ $f_2(y) = f(1,y) = 48y - 32 - 24y^2$, so $f_2'(y) = 48 - 48y = 0 \Rightarrow y=1, x=1$

get $(1,1)$

@ $y=0$ $f_3(x) = f(x,0) = -32x^3$, So $f_3'(x) = -96x^2 = 0 \Rightarrow x=0, y=0$

get $(0,0)$

@ $y=1$ $f_4(x) = f(x,1) = 48x - 32x^3 - 24$, so $f_4'(x) = 48 - 96x^2 = 0 \Rightarrow x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$, $y=1$

get $(1/\sqrt{2}, 1)$, $\cancel{(-1/\sqrt{2}, 1)}$ not in R

③

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(0,0)$ | $0$ |
| $(1/2,1/2)$ | $\boxed{2}$ MAX |
| $(1,1)$ | $-8$ |
| $(1/\sqrt{2},1)$ | $-1.37$ |
| $\cancel{(-1/\sqrt{2},1)}$ | not in R |
| $(1,0)$ | $\boxed{-32}$ MIN |
| $(0,1)$ | $-24$ |

So MAX value of $f$ is 2 @ $(1/2, 1/2)$ and MIN value of $f$ is $-32$ @ $(1,0)$

# §14.8

Find the extreme values of $f(x,y)$ subject to $g(x,y)=k$.

#2 $f(x,y)=xy$, subject to $g(x,y)=x^2+y^2-10=0$.

IDEA: Solve the Legrange Equations $\begin{cases}\nabla f=\lambda\nabla g\\ g(x,y)=k\end{cases}$

Set up LG-eqns.

$\nabla f=\langle y,x\rangle$, $\nabla g=\langle 2x,2y\rangle$

Solve $\begin{cases}① \; y=\lambda 2x\\ ② \; x=\lambda 2y\\ ③ \; x^2+y^2-10=0\end{cases}$ Sub ① into ② $x=\lambda 2(\lambda 2x)$ $\Rightarrow 4\lambda^2x-x=0 \Rightarrow x(4\lambda^2-1)=0$

So $\boxed{\text{Case I}}$ $x=0$ or $\boxed{\text{Case II}}$ $\lambda=1/2$ or $\boxed{\text{Case III}}$ $\lambda=-1/2$

Case I. If $x=0$, then ③ says $y^2=10$ or $y=\pm\sqrt{10}$ so get $(0,\pm\sqrt{10})$

Case II. If $\lambda=\frac{1}{2}$ then $y=x$ so $x^2=5$ and $x=\pm\sqrt{5}$ so get $(\sqrt{5},\sqrt{5}),(-\sqrt{5},-\sqrt{5})$

Case III. If $\lambda=-\frac{1}{2}$ then $y=-x$ so $x^2=5$ and $x=\pm\sqrt{5}$ so get $(\sqrt{5},-\sqrt{5}),(-\sqrt{5},\sqrt{5})$

Evaluate

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(0,\pm\sqrt{10})$ | 0 |
| $(\pm\sqrt{5},\pm\sqrt{5})$ | 5 |
| $(\pm\sqrt{5},\mp\sqrt{5})$ | -5 |

MAX value is 5 at $(\sqrt{5},\sqrt{5})$ and $(-\sqrt{5},-\sqrt{5})$
MIN value is -5 at $(\sqrt{5},-\sqrt{5})$ and $(-\sqrt{5},\sqrt{5})$

## Exercises 14.8

**Two Independent Variables with One Constraint**

1. **Extrema on an ellipse** Find the points on the ellipse $x^2+2y^2=1$ where $f(x,y)=xy$ has its extreme values.
2. **Extrema on a circle** Find the extreme values of $f(x,y)=xy$ subject to the constraint $g(x,y)=x^2+y^2-10=0$.
3. **Maximum on a line** Find the maximum value of $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$.
4. **Extrema on a line** Find the local extreme values of $f(x,y)=x^2y$ on the line $x+y=3$.
5. **Constrained minimum** Find the points on the curve $xy^2=54$ nearest the origin.
6. **Constrained minimum** Find the points on the curve $x^2y=2$ nearest the origin.
7. Use the method of Lagrange multipliers to find
   a. **Minimum on a hyperbola** The minimum value of $x+y$, subject to the constraints $xy=16$, $x>0$, $y>0$
   b. **Maximum on a line** The maximum value of $xy$, subject to the constraint $x+y=16$.

   Comment on the geometry of each solution.
8. **Extrema on a curve** Find the points on the curve $x^2+xy+y^2=1$ in the $xy$-plane that are nearest to and farthest from the origin.
9. **Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi$ cm$^3$.

#4 $f(x,y)=x^2y$, subject to $g(x,y)=x+y=3$. Set up $\nabla f=\langle 2xy, x^2\rangle$ $\nabla g=\langle 1,1\rangle$

Solve $\begin{cases}① \; 2xy=\lambda\\ ② \; x^2=\lambda\\ ③ \; x+y=3\end{cases}$ Use ① & ② w/ $\lambda=\lambda$ to get $2xy=x^2 \Rightarrow x^2-2xy=0 \Rightarrow x(x-2y)=0$

So $x=0$ (CASE I.) or $x=2y$ (CASE II.)

In Case I: $x=0, y=3, \lambda=0$ get $(0,3)$

In Case II: $\begin{cases}① \; 4y^2=\lambda\\ ② \; 3y=3\end{cases} \Rightarrow y=1$ & $\lambda=4$. get $(2,1)$.

Evaluate

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(0,3)$ | 0 ← MIN |
| $(2,1)$ | 4 ← MAX |

So MAX of $f$ is 4 at $(2,1)$
MIN of $f$ is 0 at $(0,3)$

#6. $d(x,y)=(x^2+y^2)^{1/2}$ but use $f(x,y)=x^2+y^2$, subject to $g(x,y)=x^2y=2$.

Set up $\nabla f=\langle 2x,2y\rangle$ & $\nabla g=\langle 2xy, x^2\rangle$

Solve $\begin{cases}① \; 2x=\lambda 2xy\\ ② \; 2y=\lambda x^2\\ ③ \; x^2y=2\end{cases}$ Try ① first $2x-\lambda 2xy=0 \Rightarrow 2x(1-\lambda y)=0$

So Case I $x=0$ or Case II $\lambda y=1$

In Case I. $x=0$ so ② $y=0$, but ③ fails ✗

In Case II. $\begin{cases}① \; 2x=2x\\ ② \; 2(1/\lambda)=\lambda x^2\\ ③ \; x^2(1/\lambda)=2\end{cases}$ ② says $2=\lambda^2x^2$ ③ says $2=x^2/\lambda$ so $\lambda^2x^2=x^2/\lambda$ $\Rightarrow \lambda^3=1 \Rightarrow \lambda=1$ and $x=\pm\sqrt{2}$. and original ② then $2y=(\pm\sqrt{2})^2 \Rightarrow y=1$.

So get $(\sqrt{2},1)$ & $(-\sqrt{2},1)$

Evaluate

| $(x,y)$ | $f(x,y)$ | $d(x,y)$ |
|---|---|---|
| $(\sqrt{2},1)$ | $2+1=3$ | $\sqrt{3}$ |
| $(-\sqrt{2},1)$ | 3 | $\sqrt{3}$ |

So MIN distance of $\sqrt{3}$ at $(\sqrt{2},1)$ & $(-\sqrt{2},1)$

(no MAX distance since $x^2y=2$ is UNBOUNDED)

Find MAX/MIN of $f(x,y)$ subject to $g(x,y)=k$, and where they occur.

**8. Extrema on a curve** Find the points on the curve $x^2 + xy + y^2 = 1$ in the $xy$-plane that are nearest to and farthest from the origin.

**9. Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi$ cm$^3$.

#8. $d(x,y)=\sqrt{x^2+y^2}$ but use $f(x,y)=x^2+y^2$ subj. to $g(x,y)=x^2+xy+y^2=1$

Set up: $\nabla f = \langle 2x, 2y\rangle$ $\nabla g = \langle 2x+y, 2y+x\rangle$

Solve $\begin{cases} ① \ 2x=\lambda(2x+y) \\ ② \ 2y=\lambda(2y+x) \\ ③ \ x^2+xy+y^2=1 \end{cases}$ Case 1: $x=0$. Then ③ becomes $y^2=1$ so $y=\pm1$ and get two points $(0,1)$ & $(0,-1)$.

notice that if $(a,b)$ solution to ①,②,③ then so is $(b,a)$. and visa versa.

Case 2: $y=0$. By symmetry of the problem, get two points w/ $x$ & $y$ swapped $(1,0),(-1,0)$

Case 3: $x\neq0$ and $y\neq0$. Then ① & ② become $\lambda=\frac{2x}{2x+y}=\frac{2y}{2y+x}$ $\Rightarrow 2x(2y+x)=2y(2x+y)$

$\Rightarrow \cancel{4xy}+2x^2=\cancel{4xy}+2y^2$

$\Rightarrow x^2=y^2$

$\Rightarrow x=\pm y$.

So ③ becomes (a) @ $x=-y$ $x^2-x^2+x^2=1$ So $x^2=1 \Rightarrow x=\pm1$

(b) @ $x=y$ $x^2+x^2+x^2=1$ or $3x^2=1 \Rightarrow x=\pm1/\sqrt{3}$

So get four more points $(\pm1/\sqrt{3}, \pm1/\sqrt{3})$ and $(\pm1, \mp1)$.

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(0,1)$ | 1 |
| $(0,-1)$ | 1 |
| $(1,0)$ | 1 |
| $(-1,0)$ | 1 |
| $(\pm1,\mp1)$ | 2 |
| $(\pm1/\sqrt{3},\pm1/\sqrt{3})$ | 2/3 |

So $d(x,y)=\sqrt{f(x,y)}$ has $\boxed{\text{MAX value } \sqrt{2} \text{ @ } (1,-1) \text{ and } (-1,1)}$

and $\boxed{\text{MIN value } \sqrt{2}/\sqrt{3} \text{ @ } (1/\sqrt{3},1/\sqrt{3}) \text{ and } (-1/\sqrt{3},-1/\sqrt{3})}$

**Three Independent Variables with One Constraint**

**17. Minimum distance to a point** Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

**18. Maximum distance to a point** Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

**19. Minimum distance to the origin** Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

**20. Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.

**21. Minimum distance to the origin** Find the points on the surface $z^2 = xy + 4$ closest to the origin.

**22. Minimum distance to the origin** Find the point(s) on the surface $xyz = 1$ closest to the origin.

#18 Find point on sphere $x^2+y^2+z^2=4$ farthest from $P(1,-1,1)$.

Soln: constraint is $g(x,y,z)=x^2+y^2+z^2=4$ and $d(x,y,z)=\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}$

but use $d^2(x,y)=f(x,y)$ for L-G method eqns.

Set up: $\nabla f=\langle 2x-2, 2y+2, 2z-2\rangle$

$\nabla g=\langle 2x, 2y, 2z\rangle$

L-G EQNS. $\begin{cases} 2x-2=\lambda 2x \\ 2y+2=\lambda 2y \\ 2z+2=\lambda 2z \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{cases} x-\lambda x=1 \\ y-\lambda y=-1 \\ z-\lambda z=1 \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{1-\lambda} \\ y=\frac{-1}{1-\lambda} \\ z=\frac{1}{1-\lambda} \\ ④ \ x^2+y^2+z^2=4 \end{cases}$

Sub $x,y,z$ into constraint ④ to get

$3\left(\frac{1}{1-\lambda}\right)^2=4 \Rightarrow \frac{1}{1-\lambda}=\pm\frac{2}{\sqrt{3}} \Rightarrow \lambda=1\pm\frac{\sqrt{3}}{2}$

So $1-\lambda=\pm\sqrt{3}/2$ and $(x,y,z)=(\pm2/\sqrt{3}, \mp2/\sqrt{3}, \pm2/\sqrt{3})$

By observation, $(1,-1,1)$ is in the same direction as $(2/\sqrt{3},-2/\sqrt{3},2/\sqrt{3})$, so the farthur of the two is $\boxed{(-2/\sqrt{3}, 2/\sqrt{3}, -2/\sqrt{3})}$

Find MAX/MIN of Function w/ given constraint.

**Three Independent Variables with One Constraint**

**17. Minimum distance to a point** Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

**18. Maximum distance to a point** Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

**19. Minimum distance to the origin** Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

**20. Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.

**21. Minimum distance to the origin** Find the points on the surface $z^2 = xy + 4$ closest to the origin.

**22. Minimum distance to the origin** Find the point(s) on the surface $xyz = 1$ closest to the origin.

#20 $d(x,y,z) = f^2(x,y,z)$, $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = xy - z + 1 = 0$

Set up $\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle y, x, -1 \rangle$

Solve $\begin{cases} ① \; 2x = \lambda y \\ ② \; 2y = \lambda x \\ ③ \; 2z = -\lambda \\ ④ \; xy - z + 1 = 0 \end{cases}$

Solve ① for $x = \frac{\lambda}{2} y$ & sub into ②

$2y = \lambda(\frac{\lambda}{2} y) \Rightarrow y(4 - \lambda^2) = 0 \Rightarrow y = 0$ or $\lambda = \pm 2$.

If $y = 0$ Then ① $x = 0$ and ④ $z = 1$. Get $(0,0,1)$

If $\lambda = 2$ Then ①&② $x = y$ and ③ $z = -1$ and ④ $xy = -2$ is impossible $\cancel{\infty}$

If $\lambda = -2$ then ①&② $x = -y$ and ③ $z = 1$ and ④ $xy = 0$ So get $(0,0,1)$.

So only one solution @ $(0,0,1)$. So $\boxed{\text{MIN of } d(x,y) \text{ is } 1 \text{ at } (0,0,1)}$

#22 $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ but use $f(x,y,z) = x^2+y^2+z^2$ w/ constraint $g(x,y,z) = xyz = 1$.

Set up. $\nabla f = \langle 2x, 2y, 2z \rangle$ and $\nabla g = \langle yz, xz, xy \rangle$

Solve $\begin{cases} ① \; 2x = \lambda yz \\ ② \; 2y = \lambda xz \\ ③ \; 2z = \lambda xy \\ ④ \; xyz = 1 \end{cases}$

IDEA: Try to show that any solution to LG-EQNS must be of the form $(x,y,z) = (x, \pm x, \pm x)$

Note ④ only holds if $x \neq 0$ and $y \neq 0$ and $z \neq 0$. So by ①, $\lambda \neq 0$ as well.

Use ①&② to set $\frac{2x}{2y} = \frac{\lambda yz}{\lambda xz} \Rightarrow \frac{x}{y} = \frac{y}{x} \Rightarrow x^2 = y^2$. Similarly, ①&③ obtain $x^2 = z^2$.

So $|x| = |y| = |z|$ and So Candidates are $(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$ or else ④ fails.

All of these have $\boxed{\text{MAX } d(x,y,z) \text{ of } \sqrt{3} \text{ at the Four points } (1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)}$

**25. Minimizing a sum of squares** Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

**26. Maximizing a product** Find the largest product the positive numbers $x$, $y$, and $z$ can have if $x + y + z^2 = 16$.

**27. Rectangular box of largest volume in a sphere** Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

#25. Find three real numbers with smallest sum of squares whose sum is 9.

$f(x,y,z) = x^2+y^2+z^2$ and $g(x) = x+y+z = 9$.

Set up: $\nabla f = \langle 2x, 2y, 2z \rangle$ and $\nabla g = \langle 1, 1, 1 \rangle$

Solve $\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x+y+z = 9 \end{cases}$ So $3(\frac{\lambda}{2}) = 9 \Rightarrow \lambda = 6$, $\boxed{x = y = z = 3.}$

#26 $f(x,y,z) = xyz$ and $g(x,y,z) = x + y + z^2 = 16$

Set up $\nabla f = \langle yz, xz, xy \rangle$ and $\nabla g = \langle 1, 1, 2z \rangle$

Solve $\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = 2z\lambda \\ x+y+z^2 = 16 \end{cases}$

Case I. If $z = 0$, then $\lambda = 0$ and $xy = 0$ so $x = 0$ or $y = 0$

get $(16,0,0)$ and $(0,16,0)$.

Case II. If $z \neq 0$ then ① $y = \frac{\lambda}{z}$ and ② $x = \frac{\lambda}{z}$ so ③ becomes $\frac{\lambda^2}{z^2} = 2z\lambda \Rightarrow \lambda = 2z^3$ and ④ becomes

$2(\frac{\lambda}{z}) + z^2 = 16 \Rightarrow z^3 - 16z + 2\lambda = 0. \Rightarrow z^3 - 16z + 4z^3 = 0 \Rightarrow 5z^3 - 16z = 0 \Rightarrow z(5z^2 - 16) = 0$

$\xrightarrow{(z \neq 0)} 5z^2 - 16 = 0 \Rightarrow z = \pm 4/\sqrt{5}$

get $(32/5, 32/5, \pm 4/\sqrt{5})$

$x = y = \frac{2(4/\sqrt{5})^{\cancel{3}\,2}}{\cancel{4/\sqrt{5}}} = \frac{32}{5}$

Evaluate

| $(x,y,z)$ | $f(x,y,z)$ |
| --- | --- |
| $(16,0,0)$ | $0$ |
| $(0,16,0)$ | $0$ |
| $(32/5, 32/5, \pm 4/\sqrt{5})$ | $\pm 2^{12}/5^{5/2}$ $\approx \pm 73.25$ |

$\boxed{\text{So MAX value of } f(x,y,z) \text{ is } \frac{2^{12}}{5^{5/2}} = \frac{4096}{25\sqrt{5}} \approx 73.25}$