§15.1

Exercises 15.1

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $\int_1^2\int_0^4 2xy\,dy\,dx$
2. $\int_0^2\int_{-1}^1 (x-y)\,dy\,dx$
3. $\int_{-1}^0\int_{-1}^1 (x+y+1)\,dx\,dy$
4. $\int_0^1\int_0^1 \left(1-\frac{x^2+y^2}{2}\right)dx\,dy$
5. $\int_0^3\int_0^2 (4-y^2)\,dy\,dx$
6. $\int_0^3\int_{-2}^0 (x^2y-2xy)\,dy\,dx$
7. $\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$
8. $\int_1^4\int_0^4 \left(\frac{x}{2}+\sqrt{y}\right)dx\,dy$
9. $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$
10. $\int_0^1\int_1^2 xye^x\,dy\,dx$
11. $\int_{-1}^2\int_0^{\pi/2} y\sin x\,dx\,dy$
12. $\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)\,dx\,dy$

Evaluate

#1 $\int_1^2\int_0^4 2xy\,dy\,dx$

$= \int_1^2 xy^2\Big|_0^4 dx = \int_1^2 16x\,dx = 8x^2\Big|_1^2$

$= 32-8 = 24$

#2 $\int_0^2\int_{-1}^1 x-y\,dy\,dx = \int_0^2 xy-\frac{1}{2}y^2\Big|_{-1}^1 dx = \int_0^2 x-(-x)-0\,dx$

$= \int_0^2 2x\,dx = x^2\Big|_0^2 = 4$

#4 $\int_0^1\int_0^1 1-\frac{x^2+y^2}{2}\,dx\,dy = \int_0^1 x-\frac{x^3}{6}+\frac{y^2}{2}x\Big|_0^1 dy = \int_0^1 \frac{5}{6}+\frac{y^2}{2}\,dy$

$= \frac{5}{6}y+\frac{y^3}{6}\Big|_0^1 = 1-0 = 1$

#6 $\int_0^3\int_{-2}^0 x^2y-2xy\,dy\,dx = \int_0^3 \frac{1}{2}x^2y^2-xy^2\Big|_{-2}^0 dx = \int_0^3\left(0-(2x^2-4x)\right)dx$

$= \int_0^3 -2x^2+4x\,dx = -\frac{2}{3}x^3+2x^2\Big|_0^3 = -18+18-0 = 0$

#7 $\int_0^1\int_0^1 \frac{y}{1+xy}\,\overbrace{dx}^{du}\,dy = \int_0^1 \ln(1+xy)\Big|_0^1 dy = \int_0^1 \ln(1+y)-\cancel{y\ln(1)}^{\,0}\,dy$

$\int \ln t\,dt$ FORMULA

u-sub
$u = 1+xy$
$du = y\,dx$

IBP $\int u\,dv = uv - \int v\,du$
$u = \ln(t)$ $dv = 1\,dt$
$du = \frac{1}{t}dt$ $v = t$

So $\int \ln t\,dt = t\ln t - \int dt$
$= t\ln t - t + C$

$= (1+y)\ln(1+y)-(1+y)\Big|_0^1$

$= [2\ln 2-2]-[1\cancel{\ln(1)}^{\,0}-1] = 2\ln 2-1$

#8 $\int_1^4\int_0^4 \frac{x}{2}+\sqrt{y}\,dx\,dy = \int_1^4 \frac{x^2}{4}+\sqrt{y}x\Big|_0^4 dy = \int_1^4 4+4\sqrt{y}\,dy = 4y+\frac{8}{3}y^{3/2}\Big|_1^4$

$= 16+\frac{8^2}{3}-4-\frac{8}{3} = 12+\frac{8(8-1)}{3} = \frac{36+56}{3} = 92/3 \approx 30.667$

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#9 $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx = \int_0^{\ln 2} e^{2x+y}\Big|_1^{\ln 5} dx$

$= \int_0^{\ln 2} e^{2x}(e^{\ln 5}-e^1)\,dx$

$= \frac{1}{2}e^{2x}(5-e)\Big|_0^{\ln 2} = (5-e)\left[\frac{1}{2}e^{2\ln 2}-\frac{1}{2}e^0\right]$

$= (5-e)\left[2-\frac{1}{2}\right] = \boxed{\frac{3}{2}(5-e)}$

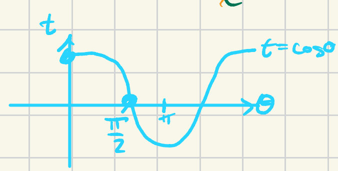
#10 $\int_0^1\int_1^2 xye^x\,dy\,dx = \int_0^1 \frac{1}{2}y^2xe^x\Big|_1^2 dx = \int_0^1 xe^x\left(2-\frac{1}{2}\right)dx$

$= \frac{3}{2}\int_0^1 xe^x\,dx = \frac{3}{2}xe^x\Big|_0^1 - \frac{3}{2}\int_0^1 e^x\,dx = \frac{3}{2}(1\cancel{e}-0) - \frac{3}{2}(\cancel{e^1}-e^0)$

IBP $\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^x\,dx$
$du = dx \quad v = e^x$

$= 3/2 * e^0 = \boxed{3/2}$



#11 $\int_{-1}^2\int_0^{\pi/2} y\sin x\,dx\,dy = \int_{-1}^2 -y\cos x\Big|_0^{\pi/2} dy = \int_{-1}^2 -y(0-1)\,dy = \int_{-1}^2 y\,dy$

$= \frac{1}{2}y^2\Big|_{-1}^2 = 2-\frac{1}{2} = \boxed{3/2}$

So like, are all the integrals 3/2 or something…?

#12 $\int_\pi^{2\pi}\int_0^\pi \sin x+\cos y\,dx\,dy = \int_\pi^{2\pi} -\cos x + x\cos y\Big|_0^\pi dy = \int_\pi^{2\pi}[\cancel{-1}+\pi\cos y]-(\cancel{-1}+0)\,dy$

$= \int_\pi^{2\pi} \pi\cos y\,dy = \pi\sin y\Big|_\pi^{2\pi} = \pi(0-0) = \boxed{0}$ welp

at least it wasn't 2/3 !!!

13. $\int_1^4\int_1^e \frac{\ln x}{xy}\,dx\,dy$
14. $\int_{-1}^2\int_1^2 x\ln y\,dy\,dx$

#13 $\int_1^4\int_1^e \frac{\ln x}{xy}\,dx\,dy = \int_1^4 \underbrace{\frac{1}{2}(\ln x)^2}_{*} * \frac{1}{y}\Big|_1^e dy$

IBP $\int u\,dv = uv - \int v\,du$

$u = \ln x \quad dv = \frac{1}{x}dx$
$du = \frac{1}{x}dx \quad v = \ln x$

$= \int_1^4\left[\frac{1}{2}(\cancel{\ln e})^2 - \frac{1}{2}(\cancel{\ln 1})^2\right]*\frac{1}{y}\,dy = \int_1^4 \frac{1}{2}*\frac{1}{y}\,dy = \frac{1}{2}\ln(y)\Big|_1^4$

$= \frac{1}{2}\ln 4 - \frac{1}{2}\cancel{\ln 1} = \boxed{\ln 2}$

$a\ln b = \ln(b^a)$ so $\frac{1}{2}\ln 4 = \ln(\sqrt{4}) = \ln 2$

$* = \int \frac{\ln x}{x}\,dx = (\ln x)^2 - \underbrace{\int \ln x\cdot\frac{1}{x}\,dx}_{*}$ So $2* = (\ln x)^2 + C$

$* = \frac{1}{2}(\ln x)^2 + C$ ✓

13. $\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy$

14. $\int_{-1}^{2} \int_{1}^{2} x \ln y\, dy\, dx$

see #7 FORMULA

#14 $\int_{-1}^{2}\int_{1}^{2} x\ln y\, dy\, dx = \int_{-1}^{2} x(y\ln y - y)\Big|_1^2\, dx$

$= \int_{-1}^{2} x[(2\ln 2 - 2) - (1\cancel{\ln 1}^{0} - 1)]\, dx = \int_{-1}^{2} (2\ln 2 - 1)\, x\, dx$

$= (2\ln 2 - 1)\frac{1}{2}x^2\Big|_{-1}^{2} = (2\ln 2 - 1)(2 - \frac{1}{2}) = (2\ln 2 - 1)(3/2) = 3\ln 2 - \frac{3}{2}$

Evaluating Double Integrals over Rectangles

In Exercises 15–22, evaluate the double integral over the given region R.

15. $\iint_R (6y^2 - 2x)\, dA$, R: $0 \le x \le 1$, $0 \le y \le 2$

16. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA$, R: $0 \le x \le 4$, $1 \le y \le 2$

17. $\iint_R xy\cos y\, dA$, R: $-1 \le x \le 1$, $0 \le y \le \pi$

18. $\iint_R y\sin(x + y)\, dA$, R: $-\pi \le x \le 0$, $0 \le y \le \pi$

#15 $\iint_R 6y^2 - 2x\, dA = \int_0^1\int_0^2 6y^2 - 2x\, dy\, dx$

$= \int_0^1 2y^3 - 2xy\Big|_0^2\, dx = \int_0^1 16 - 4x\, dx$

$= 16x - 2x^2\Big|_0^1 = 16 - 2 - 0 = 14$

#18 $\iint_R y\sin(x+y)\, dA = \int_{-\pi}^{0}\int_0^{\pi} y\sin(x+y)\, dy\, dx = \int_{-\pi}^{0}\left[-y\cos(x+y)\Big|_0^{\pi} - \int_0^{\pi} -\cos(x+y)\, dy\right] dx$

IBP $\int u\, dv = uv - \int v\, du$

$\boxed{\begin{array}{ll} u = y & dv = \sin(x+y)\,dy \\ du = dy & v = -\cos(x+y) \end{array}}$

$= \int_{-\pi}^{0} -\pi\cos(x+\pi) + \sin(x+y)\Big|_0^{\pi}\, dx$

$= \int_{-\pi}^{0} -\pi\cos(x+\pi) + \sin(x+\pi) - \sin(x)\, dx$

$= -\pi\sin(x+\pi) - \cos(x+\pi) + \cos x\Big|_{-\pi}^{0} = [-\pi\cancel{\sin(\pi)}^{0} - \cancel{\cos\pi}^{-1} + \cancel{\cos 0}^{1}] - [-\pi\cancel{\sin 0}^{0} - \cancel{\cos 0}^{1} + \cancel{\cos -\pi}^{-1}]$

$= [0 + 1 + 1] - [0 - 1 - 1] = 2 + 2 = 4$

worth.

#19. $\iint_R e^{x-y}\, dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\, dx\, dy$

$= \int_0^{\ln 2} e^{x-y}\Big|_0^{\ln 2}\, dy = \int_0^{\ln 2} e^{-y}(\cancel{e^{\ln 2}}^{2} - \cancel{e^0}^{1})\, dy$

$= \int_0^{\ln 2} e^{-y}\, dy = -e^{-y}\Big|_0^{\ln 2} = (-e^{\ln 2} - (-e^0))$

$= -2 + 1 = -1$

19. $\iint_R e^{x-y}\, dA$, R: $0 \le x \le \ln 2$, $0 \le y \le \ln 2$

20. $\iint_R xye^{xy^2}\, dA$, R: $0 \le x \le 2$, $0 \le y \le 1$

21. $\iint_R \frac{xy^3}{x^2 + 1}\, dA$, R: $0 \le x \le 1$, $0 \le y \le 2$

22. $\iint_R \frac{y}{x^2y^2 + 1}\, dA$, R: $0 \le x \le 1$, $0 \le y \le 1$

§15.2

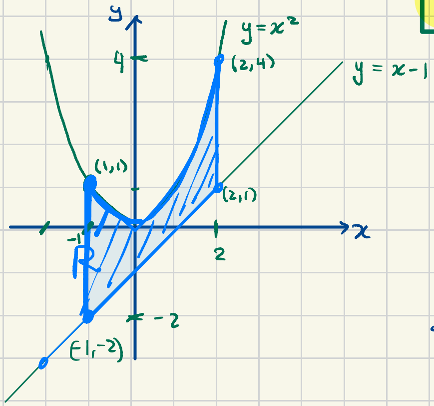
Exercises 15.2

Sketching Regions of Integration

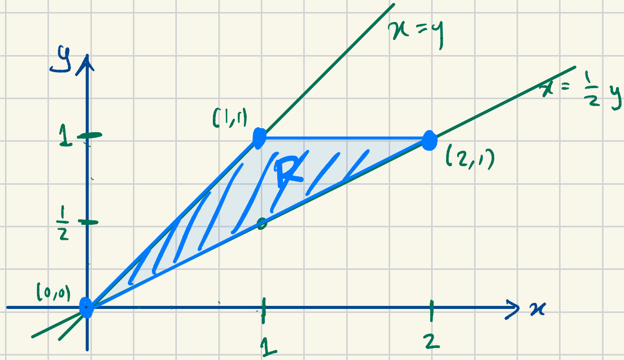
In Exercises 1–8, sketch the described regions of integration.

1. $0 \le x \le 3, \quad 0 \le y \le 2x$
2. $-1 \le x \le 2, \quad x - 1 \le y \le x^2$
3. $-2 \le y \le 2, \quad y^2 \le x \le 4$
4. $0 \le y \le 1, \quad y \le x \le 2y$
5. $0 \le x \le 1, \quad e^x \le y \le e$
6. $1 \le x \le e^2, \quad 0 \le y \le \ln x$
7. $0 \le y \le 1, \quad 0 \le x \le \sin^{-1} y$
8. $0 \le y \le 8, \quad \frac{1}{4}y \le x \le y^{1/3}$

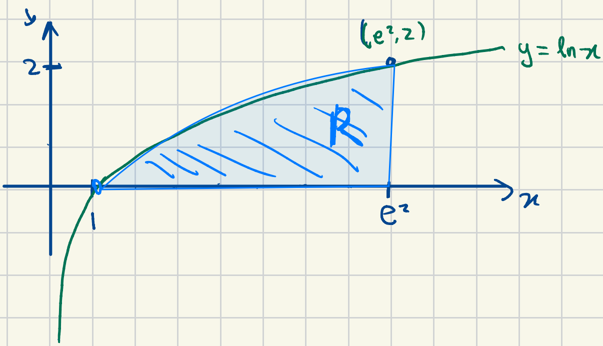
#2 $-1 \le x \le 2$, $x-1 \le y \le x^2$

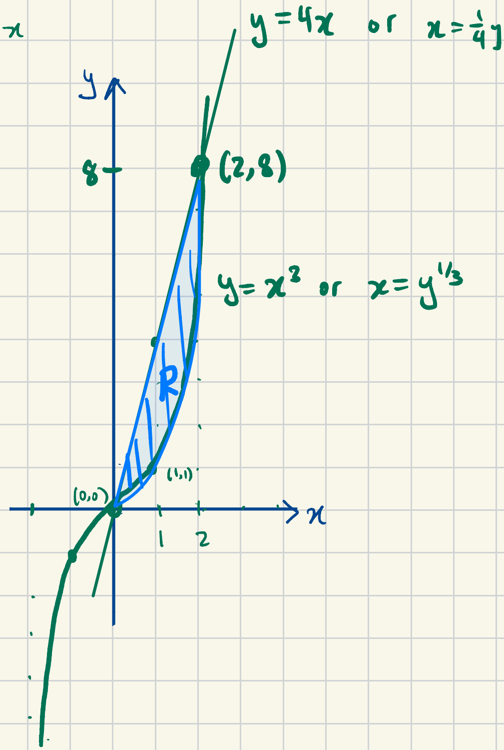


#4 $0 \le y \le 1$ $\quad y \le x \le 2y$



#6 $1 \le x \le e^2$, $0 \le y \le \ln x$



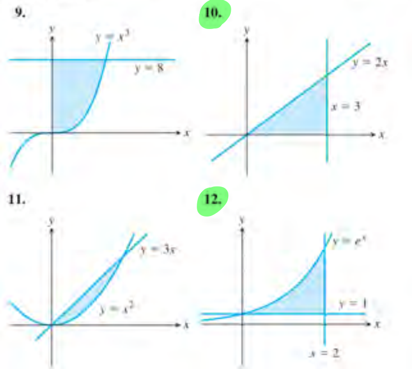


#8 $0 \le y \le 8$ $\quad \frac{1}{4}y \le x \le y^{1/3}$

Write an interated integral for Area $= \iint_R 1\, dA$ for (a) vertical and (b) horizontal cross sections

Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.



#10 (a) Vertical: For each $x \in [0,3]$, y is between 0 and $2x$. So

$$\text{Area} = \int_0^3 \int_0^{2x} 1\, dy\, dx$$

(b) horizontal: For each $y \in [0,6]$, x is between $\frac{1}{2}y$ and 3. So

$$\text{Area} = \int_0^6 \int_{1/2}^{3} 1\, dx\, dy$$

#12 (a) vert: $x \in [0,2]$ and y between 1 and e^x So Area $= \int_0^2 \int_1^{e^x} 1\, dy\, dx$

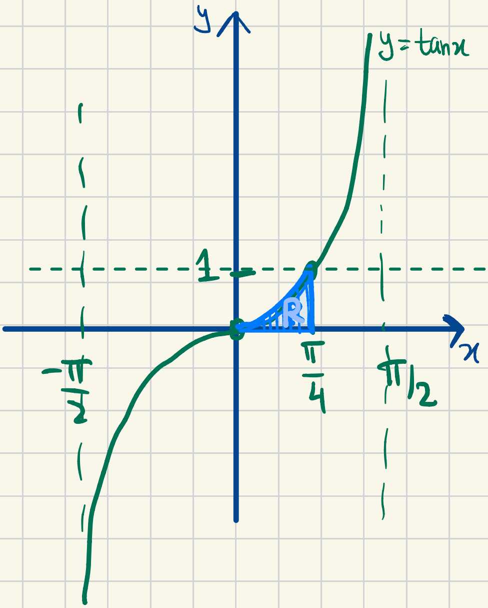
(b) horz: $y \in [1, e^2]$ and x between $\ln(y)$ and 2

So Area $= \int_1^{e^2} \int_{\ln y}^{2} 1\, dx\, dy$

Note
$y = e^x$
$\Leftrightarrow x = \ln y$

#14

13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$
14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$
15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$
16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$
17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$
18. Bounded by $y = x^2$ and $y = x + 2$



vert: $x \in [0, \pi/4]$ and y between 0 and $\tan x$

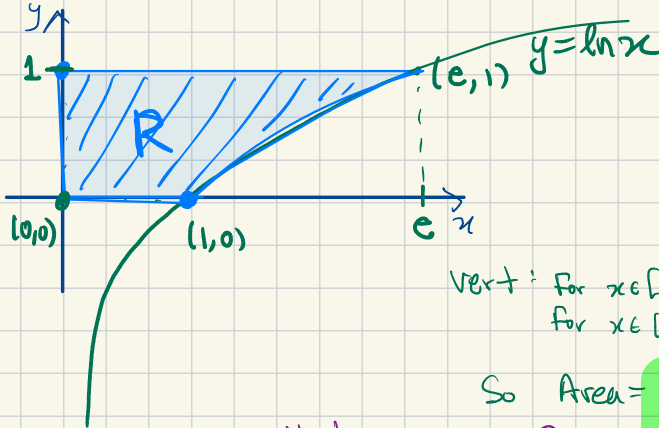
So Area $= \int_0^{\pi/4} \int_0^{\tan x} 1\, dy\, dx$

horz: $y \in [0,1]$ and x between $\tan^{-1} x$ and $\pi/4$ So

$$\text{Area} = \int_0^1 \int_{\tan^{-1} x}^{\pi/4} 1\, dx\, dy$$

13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$
14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$
15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$
16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$
17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$
18. Bounded by $y = x^2$ and $y = x + 2$

#16



vert: For $x \in [0,1]$ $y \in [0,1]$
for $x \in [1,e]$ y between $\ln x$ and 1.

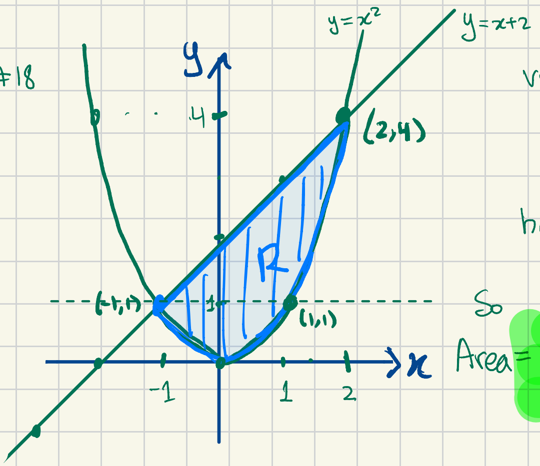
So Area $= \int_0^1 \int_0^1 1 \, dy \, dx + \int_0^1 \int_{\ln x}^1 1 \, dy \, dx$

yuck!! two integrals 

horz: For $y \in [0,1]$ x between 0 and e^y

So Area $= \int_0^1 \int_0^{e^y} 1 \, dx \, dy$

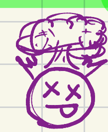
#18



vert: for $x \in [-1,2]$ y is between x^2 and $x+2$
So
Area $= \int_{-1}^2 \int_{x^2}^{x+2} 1 \, dy \, dx$

horz: For $y \in [0,1]$ x between $-\sqrt{y}$ and $+\sqrt{y}$
for $y \in [1,4]$ x between $y-2$ and $+\sqrt{y}$

So
Area $= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dy + \int_0^1 \int_{y-2}^{\sqrt{y}} 1 \, dx \, dy$

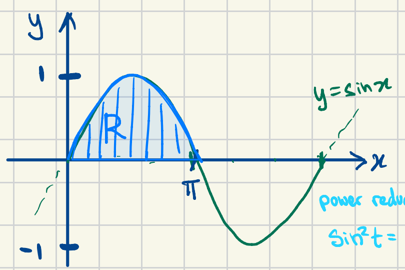


Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

19. $\int_0^{\pi} \int_0^{x} x \sin y \, dy \, dx$ 20. $\int_0^{\pi} \int_0^{\sin x} y \, dy \, dx$

#20 Sketch R & evaluate the integral



power reducing formula
$\sin^2 t = \frac{1-\cos 2t}{2}$

$$\text{Volume} = \int_0^{\pi} \int_0^{\sin x} y \, dy \, dx = \int_0^{\pi} \frac{1}{2} y^2 \Big|_0^{\sin x} dx$$

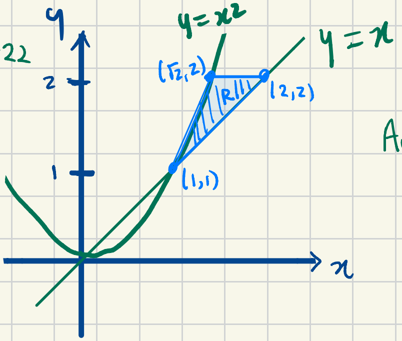
$$= \int_0^{\pi} \frac{1}{2} \sin^2 x \, dx = \int_0^{\pi} \frac{1}{2} \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{4} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi} = \frac{1}{4} \left[(\pi - 0) - (0 - 0) \right] = \pi/4$$

21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$ 22. $\int_1^2\int_y^{y^2} dx\,dy$

23. $\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$ 24. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx$

#22



$$\text{Area} = \int_1^2\int_y^{y^2} 1\,dx\,dy$$

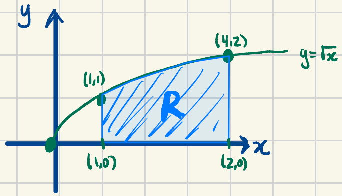
$$= \int_1^2 x\Big|_y^{y^2} dy = \int_1^2 y^2 - y\,dy = \tfrac{1}{3}y^3 - \tfrac{1}{2}y^2\Big|_1^2 = \left(\tfrac{8}{3}-2\right)-\left(\tfrac{1}{3}-\tfrac{1}{2}\right)$$

$$= \tfrac{7}{3} - \tfrac{3}{2} = \frac{14-9}{6} = 5/6$$

u-sub

$$\int e^{t/k}\,dt = ke^{t/k} + C$$

#24



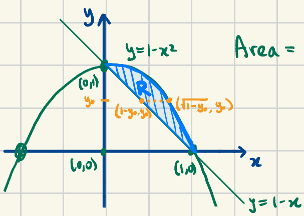
$$\text{Volume} = \int_1^4\int_0^{\sqrt{x}} \tfrac{3}{2}e^{y/\sqrt{x}}\,dy\,dx = \int_0^1 \sqrt{x}\,\tfrac{3}{2}e^{y/\sqrt{x}}\Big|_0^{\sqrt{x}} dx$$

$$= \int_0^1 \tfrac{3}{2}\sqrt{x}\left[e^{\sqrt{x}/\sqrt{x}} - e^0\right]dx = \int_0^1 \tfrac{3}{2}(e-1)\sqrt{x}\,dx$$

$$= \cancel{\tfrac{3}{2}}(e-1)x^{3/2}\cdot\cancel{\tfrac{2}{3}}\Big|_0^1 = (e-1)\left[1^{3/2}-0\right]$$

$$= e-1$$

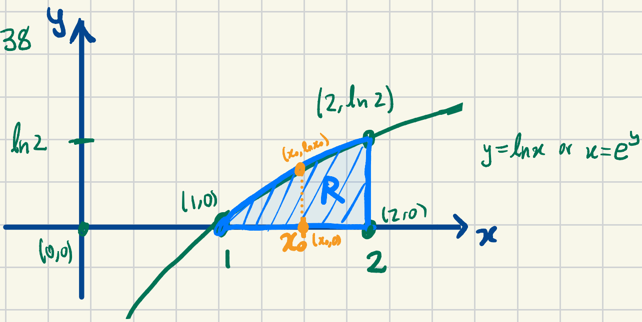
#36 Sketch the region and reverse the order of integration



$$\text{Area} = \int_0^1\int_{1-x}^{1-x^2} 1\,dy\,dx$$

$$= \int_0^1\int_{1-y}^{\sqrt{1-y}} 1\,dy\,dx$$

#38



$$\text{Area} = \int_0^{\ln 2}\int_{e^y}^2 1\,dx\,dy = \int_1^2\int_0^{\ln x} 1\,dx\,dy$$

Reversing the Order of Integration

In Exercises 33–46, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

33. $\int_0^1\int_2^{4-2x} dy\,dx$ 34. $\int_0^2\int_{y-2}^0 dx\,dy$

35. $\int_0^1\int_y^{\sqrt{y}} dx\,dy$ 36. $\int_0^1\int_{1-x}^{1-x^2} dy\,dx$

37. $\int_0^1\int_1^{e^x} dy\,dx$ 38. $\int_0^{\ln 2}\int_{e^y}^2 dx\,dy$

39. $\int_0^{3/2}\int_0^{9-4x^2} 16x\,dy\,dx$ 40. $\int_0^2\int_0^{4-y^2} y\,dx\,dy$

41. $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\,dx\,dy$ 42. $\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\,dy\,dx$

43. $\int_1^e\int_0^{\ln x} xy\,dy\,dx$ 44. $\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2\,dy\,dx$

45. $\int_0^3\int_1^{e^y} (x+y)\,dx\,dy$ 46. $\int_0^{\sqrt{3}}\int_0^{\tan^{-1} y} \sqrt{xy}\,dx\,dy$

§15.3

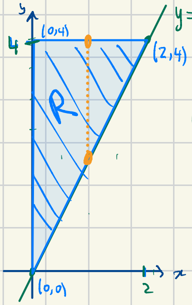
Exercises 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
9. The lines $y = x$, $y = x/3$, and $y = 2$
10. The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
11. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
12. The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

#2



(vertical)

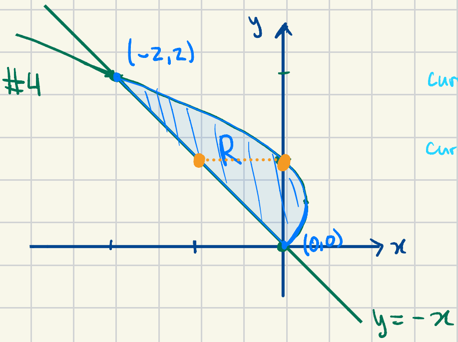
$$\text{Area} = \int_0^2 \int_{2x}^4 1\, dy\, dx$$

$$= \int_0^2 y \Big|_{2x}^4 dx = \int_0^2 4 - 2x\, dx$$

$$= 4x - x^2 \Big|_0^2 = (8-4) - (0-0)$$

$$= \boxed{4}$$

#4



Curve 1:
$x = y - y^2 = y(y-1)$

Curve 2:
$x = -y$

Intersection:
$y - y^2 = -y \Rightarrow 2y - y^2 = 0$
$\Rightarrow y(2-y) = 0$ @ $y = 0, 2$

R region (x,y) for $y \in [0,2]$,
x between $-y$ and $y - y^2$

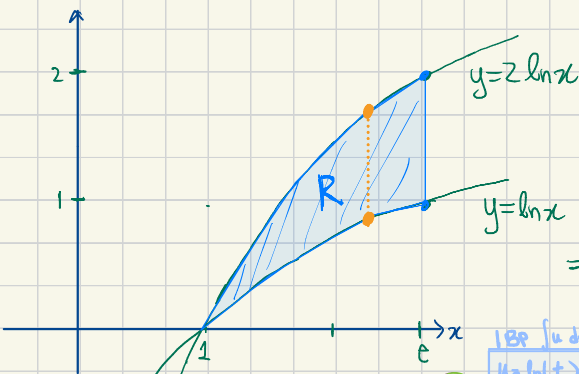
(horizontal)

$$\text{Area} = \int_0^2 \int_{-y}^{y-y^2} 1\, dx\, dy = \int_0^2 x \Big|_{-y}^{y-y^2} dy$$

$$= \int_0^2 (y - y^2) - (-y)\, dy = \int_0^2 2y - y^2\, dy = y^2 - \tfrac{1}{3}y^3 \Big|_0^2$$

$$= \left(4 - \tfrac{8}{3}\right) - (0-0) = \frac{12-8}{3} = \boxed{4/3}$$

#6



(vertical)

$$\text{Area} = \int_1^e \int_{\ln x}^{2\ln x} 1\, dy\, dx = \int_1^e y \Big|_{\ln x}^{2\ln x} dx$$

$$= \int_1^e 2\ln x - \ln x\, dx = \int_1^e \ln x\, dx$$

$$= x\ln x - x \Big|_1^e = (e\,\cancel{\ln e}^{\,1} - e) - (1\,\cancel{\ln(1)}^{\,0} - 1)$$

$$= e - e + 1 = \boxed{1}$$

IBP $\int u\, dv = uv - \int v\, du$

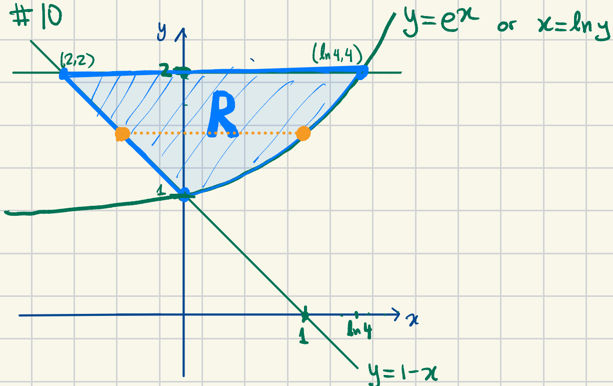
$$\boxed{\begin{array}{ll} u = \ln(t) & dv = 1\, dt \\ du = \frac{1}{t} dt & v = t \end{array}}$$

So $\int \ln t\, dt = t\ln t - \int dt$
$= t\ln t - t + C$

!?

don't know what I was expecting - but it wasn't that!

#10



(horizontal)

Exercises 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

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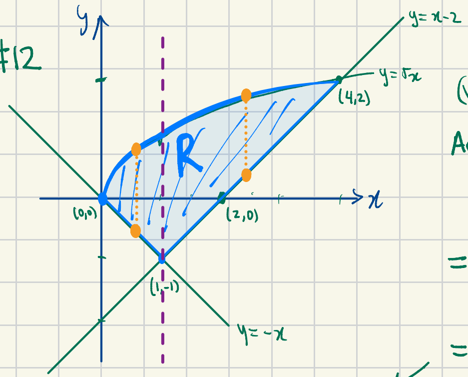
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
9. The lines $y = x$, $y = x/3$, and $y = 2$
10. The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
11. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
12. The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

$$\text{Area} = \int_1^2 \int_{1-y}^{\ln y} 1 \, dx \, dy = \int_1^2 \ln y - (1-y) \, dy = \int_1^2 \ln y - 1 + y \, dy$$

PREV PAGE

$$= y \ln y - y + \tfrac{1}{2} y^2 \Big|_1^2 = \left(2 \ln 2 - \cancel{2} + \cancel{2}\right) - \left(1 \cancel{\ln(1)}^{0} - 1 + \tfrac{1}{2}\right) = \boxed{2 \ln 2 + 1/2}$$

#12



(vertical - but not simple!)

$$\text{Area} = \int_0^1 \int_{-x}^{\sqrt{x}} 1 \, dy \, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 \sqrt{x} - (-x) \, dx + \int_1^4 \sqrt{x} - (x-2) \, dx$$

$$= \int_0^4 \sqrt{x} \, dx + \int_0^1 x \, dx + \int_1^4 -x + 2 \, dx$$

$$= \tfrac{2}{3} x^{3/2} \Big|_0^4 + \tfrac{1}{2} x^2 \Big|_0^1 + -\tfrac{1}{2} x^2 + 2x \Big|_1^4$$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 13–18 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

13. $\int_0^6 \int_{y^2/3}^{2y} dx\,dy$

14. $\int_0^3 \int_{-x}^{x(2-x)} dy\,dx$

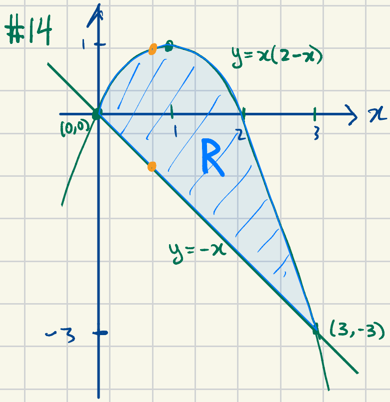
15. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx$

16. $\int_{-1}^2 \int_{y^2}^{y+2} dx\,dy$

17. $\int_{-1}^0 \int_{-2x}^{1-x} dy\,dx + \int_0^2 \int_{-x/2}^{1-x} dy\,dx$

18. $\int_0^2 \int_{x^2-4}^0 dy\,dx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx$

#14



curve 1: $y = -x$

curve 2: $y = x(2-x)$

intersection $-x = x(2-x)$

$\Rightarrow 0 = x + 2x - x^2$

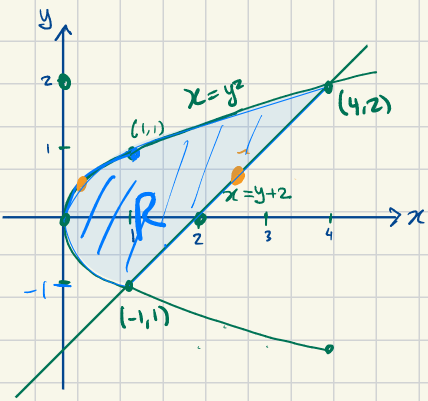
$\Rightarrow 0 = x(3-x)$ @ $x = 0, 3$

$$\text{Area} = \int_0^3 \int_{-x}^{x(2-x)} 1\,dy\,dx = \int_0^3 x(2-x) - (-x)\,dx$$

$$= \int_0^3 \cancel{x(2-x)} 2x + x\,dx = \int_0^3 3x - x^2\,dx = \frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_0^3$$

$$= \frac{27}{2} - 9 = \frac{27-18}{2} = \boxed{9/2}$$

#16



$$\text{Area} = \int_{-1}^2 \int_{y^2}^{y+2} 1\,dx\,dy = \int_{-1}^2 x \Big|_{y^2}^{y+2} dx$$

$$= \int_{-1}^2 (y+2) - y^2\,dy = \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^2$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - 3 - \frac{1}{2} = 4.5 = \boxed{9/2}$$

Finding Average Values

19. Find the average value of $f(x, y) = \sin(x + y)$ over

a. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$.

b. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$.

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\,dA.$$

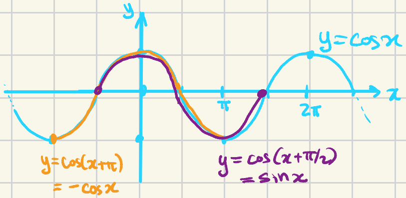
#19 (a) $R_1 = [0,\pi] \times [0,\pi]$, $\text{Area}(R_1) = \pi^2$

so $\text{Avg}_{R_1}(f) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \sin(x+y)\,dy\,dx$

$$= \frac{1}{\pi^2} \int_0^\pi -\cos(x+y) \Big|_0^\pi dx = \frac{1}{\pi^2} \int_0^\pi -\cos(x+\pi) + \cos(x)\,dx$$

$$= \frac{1}{\pi^2} \int_0^\pi 2\cos x\,dx = 2\sin x \Big|_0^\pi = 2(\overset{0}{\cancel{\sin\pi}} - \overset{0}{\cancel{\sin 0}})$$

$$= \boxed{0}$$



(b) $R_2 = [0,\pi] \times [0,\pi/2]$ $\text{Area}(R_2) = \pi^2/2$

Must've been neg. on other half of R_1!!

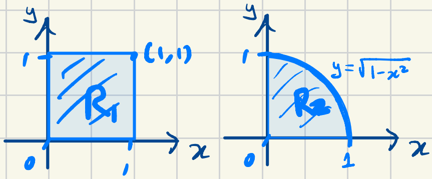
So $\text{Vol.} = \int_0^\pi \int_0^{\pi/2} \sin(x+y)\,dy\,dx = \int_0^\pi -\cos(x+y)\Big|_0^{\pi/2} dx = \int_0^\pi -\cos(x+\pi/2) + \cos(x)\,dx = \int_0^\pi \sin x + \cos x\,dx$

$$= -\cos x + \sin x \Big|_0^\pi = -\cos\pi + \overset{0}{\cancel{\sin\pi}} - (-\cos 0 + \overset{0}{\cancel{\sin 0}}) = -(-1) + 1 = 2$$

So $\text{Avg}_{R_2}(f) = \frac{1}{\pi^2/2} \cdot 2 = \boxed{4/\pi^2}$

20. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \le x \le 1, 0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.

#20 $f(x,y) = xy$



Q: which is bigger? $Avg_{R_1}(f)$ or $\boxed{Avg_{R_2}(f)}$? Compute & verify.

guess b/c $\div$ smaller area, so larger total?

$$V_{R_1} = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^1 = \int_0^1 \tfrac{1}{2}x(1-0)\,dx = \int_0^1 \tfrac{1}{2}x \, dx$$

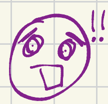
$$= \tfrac{1}{4}x^2\Big|_0^1 = 1/4 \qquad \text{So } Avg_{R_1}(f) = \frac{1}{\text{Area } R_1} V_{R_1} = \frac{1}{1} * \frac{1}{4} = \boxed{\frac{1}{4}}$$

$$V_{R_2} = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx = \int_0^1 \tfrac{1}{2}xy^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1-x^2)\,dx$$

$$= \int_0^1 \tfrac{1}{2}(x - x^3)\,dx = \tfrac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} * \tfrac{1}{4} = 1/8$$

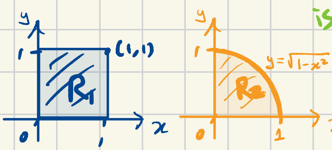
$$\text{So } Avg_{R_2}(f) = \frac{1}{\text{Area } R_2} V_{R_2} = \frac{1}{\pi/4} * \frac{1}{8} = \boxed{\frac{1}{2\pi} \approx \frac{1}{6}}$$

Well I'll be...



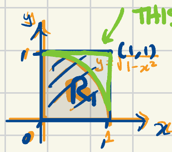
how's that work...

In retrospect, what seems to be happening

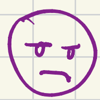


is that in the outer region of R_1, (The part not contained in R_2)

THIS BIT HERE



in the part of R_1 not contained in R_2 the value of $f(x,y) = xy$ is "particularly large" since



who woulda thunk it.

xy is small when x <u>or</u> y is small. And apparently this more than compensates for $\div$ by larger area.

#21 $h(x,y) = x^2+y^2$ over $R=[0,2]\times[0,2]$

21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, 0 \le y \le 2$.

22. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2\ln 2, \ln 2 \le y \le 2\ln 2$.

$$\text{Vol} = \int_0^2\int_0^2 x^2+y^2 \, dy\, dx = \int_0^2 x^2y + \tfrac{1}{3}y^3 \Big|_0^2$$

$$= \int_0^2 2x^2 + \tfrac{8}{3}\, dx = \tfrac{2}{3}x^3 + \tfrac{8}{3}x \Big|_0^2 = \tfrac{16}{3} + \tfrac{16}{3} = 32/3$$ and Area $R = 4$

So $\text{Avg}_R(h) = \frac{1}{\text{Area}R} * \text{Vol} = \frac{1}{4} * \frac{32}{3} = \boxed{8/3}$

#22 $f(x,y) = \frac{1}{xy}$ over $R = [\ln 2, 2\ln 2]\times[\ln 2, 2\ln 2]$

$\ln ab = \ln a + \ln b$

$\ln 2 + \cancel{\ln(\ln 2)} - \cancel{\ln(\ln 2)}$

$$\text{Vol} = \int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2} \frac{1}{xy}\, dy\, dx = \int_{\ln 2}^{2\ln 2} \frac{1}{x}\ln y \Big|_{\ln 2}^{2\ln 2} = \int_{\ln 2}^{2\ln 2} \frac{1}{x}\left[\ln(2\ln 2) - \ln(\ln 2)\right] dx$$

$\ln 2 + \cancel{\ln(\ln 2)} - \cancel{\ln(\ln 2)}$

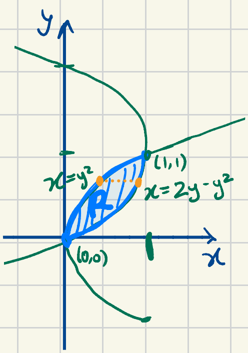
$$= \int_{\ln 2}^{2\ln 2} \frac{1}{x}\ln 2\, dx = \ln 2\left(\ln x \Big|_{\ln 2}^{2\ln 2}\right) = \ln 2\left[\ln(2\ln 2) - \ln 2\right] = (\ln 2)^2$$

and Area $R = (2\ln 2 - \ln 2)^2 = (\ln 2)^2$ wait... I know what's about to happen...

So $\text{Avg}_R(f) = \frac{1}{\text{Area}R} * \text{Vol} = \frac{1}{\cancel{(\ln 2)^2}} * \cancel{(\ln 2)^2} = \boxed{1}$ lol. knew it.

#26 $f(x,y) = 100(y+1)$

26. Regional population If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.



$$\text{Area} = \int_0^1\int_{y^2}^{2y-y^2} 1\, dx\, dy$$

$$= \int_0^1 (2y-y^2)-y^2\, dy = \int_0^1 2y - 2y^2\, dy = y^2 - \tfrac{2}{3}y^3\Big|_0^1 = 1 - 2/3 = 1/3$$

$$\text{Vol} = \int_0^1\int_{y^2}^{2y-y^2} 100(y+1)\, dx\, dy = \int_0^1\int_{y^2}^{2y-y^2} 100y\, dx dy + 100\,\text{Area}$$

$$= \int_0^1 100yx\Big|_{y^2}^{2y-y^2} dy + \underbrace{\frac{100}{3}}_{100*\text{Area}} = \int_0^1 100y(2y-y^2-y^2)\, dy + \frac{100}{3}$$

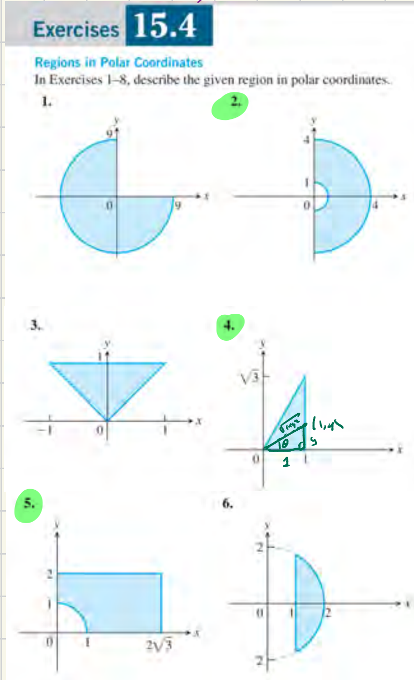
$$= \int_0^1 200y^2 - 200y^3\, dy + \frac{100}{3} = \frac{200}{3}y^3 - 50y^4\Big|_0^1 + \frac{100}{3} = \frac{200}{3} - 50 + \frac{100}{3}$$

$= \frac{300}{3} + 50 = 150$ So $\text{Avg}_R(f) = \frac{1}{\text{Area}}\text{Vol} = \frac{1}{1/3} * 150 = \boxed{450}$ people per sq. mile on AVERAGE.

Describe the region in polar coord.

$x = r\cos\theta$	$y = r\sin\theta$
$r^2 = x^2+y^2$	$\tan\theta = y/x$
$r \in [0,\infty)$	$\theta \in [0, 2\pi]$
	OR $\theta \in [-\pi, \pi]$

§15.4


Exercises 15.4

Regions in Polar Coordinates

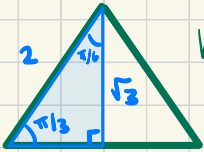
In Exercises 1–8, describe the given region in polar coordinates.

1. 2. 3. 4. 5. 6.

#2 $x^2+y^2 \in [1,4]$ and θ avoids quadrant II & III, so

$r \in [1,4]$ and $\theta \in [-\pi/2, \pi/2]$

#4 θ-value between x-axis ($0°$) and $60°$ ($\pi/3$) since



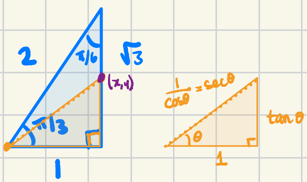
half an equilateral triangle

$x = r\cos\theta$

@ $x=1$

$\Rightarrow 1 = r\cos\theta$

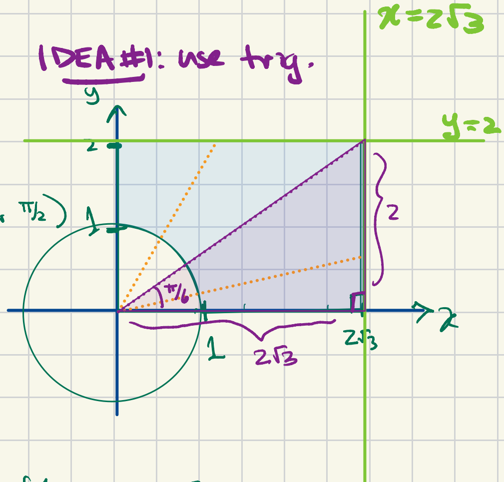
$\Rightarrow r = \sec\theta$.



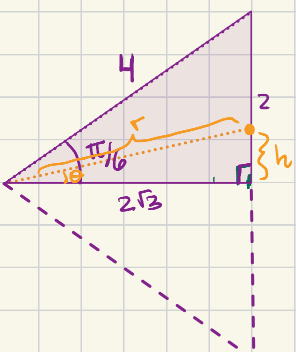
So $\theta \in [0, \pi/3]$

and $r \in [0, \sec\theta]$

IDEA #1: use trig.



#5 θ-value between x-axis ($0°$) and y-axis ($90°$ or $\pi/2$)



$\tan\theta = \frac{h}{2\sqrt{3}}$

$\Rightarrow h = 2\sqrt{3}\tan\theta$

$\cos\theta = \frac{2\sqrt{3}}{r}$

$\Rightarrow r = \frac{2\sqrt{3}}{\cos\theta} = 2\sqrt{3}\sec\theta$

So For $\theta \in [0, \pi/6]$, $r \in [1, 2\sqrt{3}\sec\theta]$

ALT IDEA: (use algebra) $x = r\cos\theta$, so $r = \frac{x}{\cos\theta} = x\sec\theta$

@ $x = 2\sqrt{3}$ then $r = 2\sqrt{3}\sec\theta$

So for $\theta \in [\pi/6, \pi/2]$, now use $y = r\sin\theta \Rightarrow r = \frac{y}{\sin\theta} = y\csc\theta$

@ $y=2$ get $r = 2\csc\theta$.

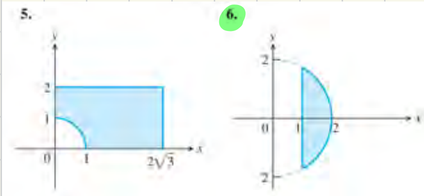
Punchline: $\theta \in [0, \pi/2]$

and if $0 \le \theta \le \pi/6$ then $r \in [1, 2\sqrt{3}\sec\theta]$

and if $\pi/6 \le \theta \le \pi/2$ then $r \in [1, 2\csc\theta]$

Ok That's simpler.

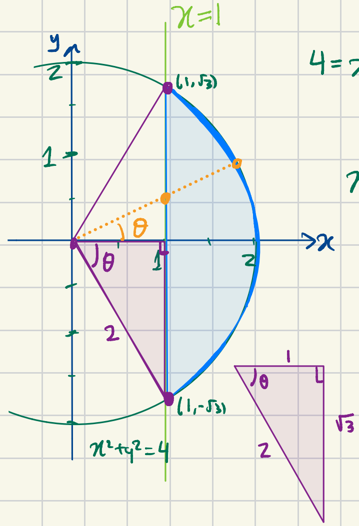
5. 6.



7. The region enclosed by the circle $x^2 + y^2 = 2x$

8. The region enclosed by the semicircle $x^2 + y^2 = 2y, y \geq 0$

#6 The θ-value is between $(1,-\sqrt{3})$ & $(1,\sqrt{3})$ so $\theta \in [-\pi/3, \pi/3]$.



$4 = x^2+y^2 = r^2 \Rightarrow r = \pm 2, r \geq 0$

so $r=2$ on circle.

$x = r\cos\theta \Rightarrow r = \frac{x}{\cos\theta} = x\sec\theta$

@ $x=1$ $r = \sec\theta$ on line $x=2$.

So $\boxed{\theta \in [-\pi/3, \pi/3] \text{ and } r \in [\sec\theta, 2]}$

So $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

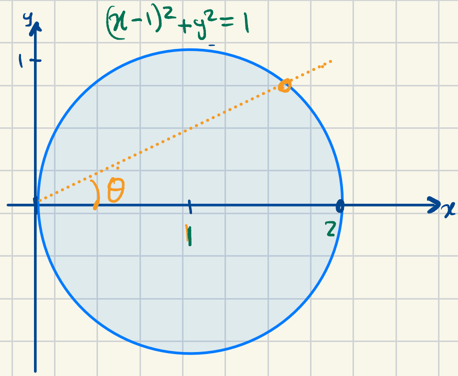
#7 $x^2+y^2=2x \Rightarrow x^2-2x+y^2=0$

(complete the square) $\Rightarrow x^2-2x+1-1+y^2=0$

$\Rightarrow (x-1)^2+y^2=1$ $(*)$

Circle of radius 1 w/ center $(+1,0)$.

$(x-1)^2+y^2=1$



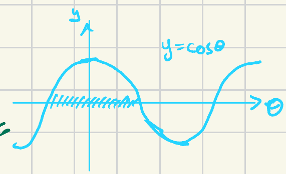
try @ $x = r\cos\theta$, $y = r\sin\theta$ into $(*)$

$(r\cos\theta-1)^2 + r^2\sin^2\theta = 1$

$\Rightarrow r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$

$r^2 \times 1$

$\Rightarrow r^2 - 2r\cos\theta = 0 \Rightarrow r^2 = 2r\cos\theta \overset{(r\neq 0)}{\Rightarrow} \frac{r^2}{r} = 2\cos\theta \Rightarrow r = 2\cos\theta$ //



So $\boxed{\theta \in [-\pi/2, \pi/2] \text{ and } r \in [0, 2\cos\theta]}$

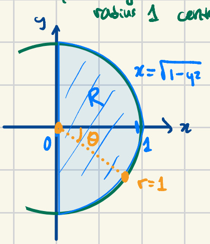
Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$ 10. $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

Evaluate the integral w/ $f(x,y) = x^2+y^2 = r^2$

#10 & R right half of circle of radius 1 centered at $(0,0)$



don't forget $dA = r\,dr\,d\theta$

$\text{Volume} = \iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, r\,dr\,d\theta$

$\text{Volume} = \int_{-\pi/2}^{\pi/2}\int_0^1 r^2 \times r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^3\,dr\,d\theta$

$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}r^4\Big|_0^1 d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} 1\,d\theta = \frac{1}{4}\theta\Big|_{-\pi/2}^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \pi/4$

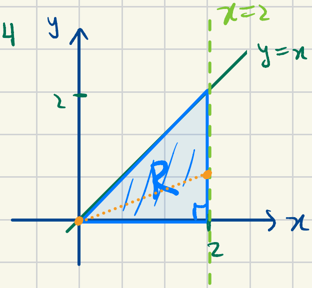
#12 R the circle of radius $r=a$ w/ center $(0,0)$, so $\theta\in[0,2\pi]$, $r\in[0,a]$

and $\text{Vol} = \int_0^a \int_0^{2\pi} 1\, r \, d\theta\, dr = \int_0^a r\,\theta \Big|_0^{2\pi} dr$

$= \int_0^a 2\pi r\, dr = \pi r^2 \Big|_0^a = \boxed{\pi a^2}$

lol. good to know (xx)

#14



Change $x=2$ to polar.

$x = r\cos\theta$

$\Rightarrow r = \frac{x}{\cos\theta} = x\sec\theta$

$\overset{@x=2}{\Rightarrow} r = 2\sec\theta$,

And $f(x,y) = y = r\sin\theta$.

So $\theta \in [0, \pi/4]$ and $r\in[0, 2\sec\theta]$,

$\text{Vol} = \iint_R f(x,y)\, dA = \int_0^{\pi/4}\int_0^{2\sec\theta} r\sin\theta * r\, dr\, d\theta = \int_0^{\pi/4} \sin\theta * \frac{1}{3} r^3 \Big|_0^{2\sec\theta} d\theta$

$= \int_0^{\pi/4} \frac{1}{3}\sin\theta * 8\sec^3\theta\, d\theta = \int_0^{\pi/4} \frac{8}{3} * \frac{1}{\cos^3\theta}\sin\theta\, d\theta = \int_1^{1/\sqrt{2}} -\frac{8}{3}u^{-3}\, du = \frac{-8}{3} * \frac{-1}{2} u^{-2}\Big|_1^{1/\sqrt{2}}$

u-sub box: $u = \cos\theta$, $du = -\sin\theta\, d\theta$

If $\theta = \pi/4$, $u = 1/\sqrt{2}$

$= \frac{4}{3}\left(\left(\frac{1}{\sqrt{2}}\right)^{-2} - 0\right) = \frac{4}{3} * 2$

$= \boxed{8/3}$

#18 R is the circle of radius $r=1$ centered at $(0,0)$. So $\theta\in[0,2\pi]$ and $r[-1,1]$. and $f(x,y) = \frac{2}{(1+x^2+y^2)^2} = \frac{2}{(1+r^2)^2}$.

So $\text{Vol} = \int_0^{2\pi}\int_0^1 \frac{1}{(1+r^2)^2} * r\, dr\, d\theta = \int_0^{2\pi}\int_1^2 \frac{1}{2u^2}\, du\, d\theta = \int_0^{2\pi} \frac{-1}{2}u^{-1}\Big|_1^2 d\theta$

u-sub box: $u = 1+r^2$, $du = 2r\, dr$; $\frac{1}{2}du = r\,dr$

If $r=1$ then $u=2$; $r=0$ then $u=1$

$= \int_0^{2\pi} \frac{-1}{2}\left(\frac{1}{2} - 1\right) d\theta = \frac{1}{4}\int_0^{2\pi} 1\, d\theta$

$= \frac{1}{4} * (2\pi - 0) = \frac{2\pi}{4} = \boxed{\pi/2}$

#20 R circle $r=1$ w/ center $(0,0)$.

So $\theta\in[0,2\pi]$ and $r\in[0,1]$.

and $f(x,y) = \ln(x^2+y^2+1) = \ln(r^2+1)$

$\int \ln t\, dt = t\ln t - t + C$

So $\text{Vol} = \iint_R f(x,y)\, dA = \int_0^{2\pi}\int_0^1 \ln(r^2+1) * r\, dr\, d\theta = \int_0^{2\pi}\int_1^2 \frac{1}{2}\ln(u)\, du\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(u\ln u - u\Big|_1^2\right) du$

$= \frac{1}{2}\int_0^{2\pi} (2\ln 2 - 2) - (1\cancel{\ln 1}^{0} - 1)\, d\theta = \left(\ln 2 - \frac{1}{2}\right)\int_0^{2\pi} 1\, d\theta = \boxed{2\pi\left(\ln 2 - \frac{1}{2}\right)}$

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\, dx$

10. $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$

11. $\int_0^2\int_0^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

13. $\int_0^6\int_0^y x\, dx\, dy$

14. $\int_0^2\int_0^x y\, dy\, dx$

15. $\int_1^{\sqrt{3}}\int_1^x dy\, dx$

16. $\int_{\sqrt{2}}^2\int_{\sqrt{4-y^2}}^y dx\, dy$

17. $\int_{-1}^0\int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx$

18. $\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\, dy\, dx$

19. $\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$

20. $\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\, dx\, dy$

21. $\int_0^1\int_x^{\sqrt{2-x^2}} (x+2y)\, dy\, dx$

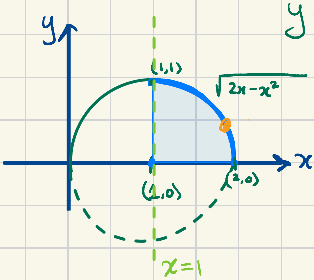
22. $\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$

Evaluate the integral

top-right quarter of circle w/ r=1 center @(1,0).

#22

22. $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} \, dy \, dx$



$y = \sqrt{2x - x^2} = \sqrt{1-(1-x)^2}$

$x=1$ line in polar is $x = r\cos\theta$

$\Rightarrow r = x\sec\theta \Rightarrow r = \sec\theta$

Whole Circle is $y^2+(x-1)^2=1$.

$\Rightarrow r^2\sin^2\theta + (r\cos\theta - 1)^2 = 1$

$\Rightarrow r^2\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + \cancel{1} = \cancel{1}$

$\underbrace{\quad}_{r^2}$

$\Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

So $\theta \in [0, \pi/4]$ and $r \in [\sec\theta, 2\cos\theta]$

$$\text{So Vol} = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4} * r\,dr\,d\theta = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-3}\,dr\,d\theta = \int_0^{\pi/4} -\frac{1}{2}r^{-2}\Big|_{\sec\theta}^{2\cos\theta} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} \frac{1}{4\cos^2\theta} - \frac{1}{\sec^2\theta}\,d\theta = \int_0^{\pi/4} \frac{-1}{8}\sec^2\theta + \frac{1}{2}\cos^2\theta\,d\theta = -\frac{1}{8}\tan\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\Big|_0^{\pi/4}$$

$\int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt$

$= \frac{1}{2}t + \frac{1}{4}\sin 2t + C$

$$= \left(-\cancel{\tfrac{1}{8}}(1) + \frac{1}{4}*\frac{\pi}{4} + \cancel{\tfrac{1}{8}}\right) - (-0+0+0) = \boxed{\pi/16}$$

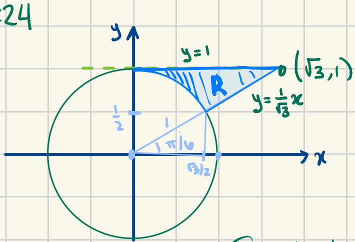
Convert to Cartesian coordinates

#24

In Exercises 23–26, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

23. $\int_0^{\pi/2}\int_0^1 r^3 \sin\theta\cos\theta\,dr\,d\theta$

24. $\int_{\pi/6}^{\pi/2}\int_1^{\csc\theta} r^2\cos\theta\,dr\,d\theta$



$y = r\sin\theta \Leftrightarrow r = y\csc\theta$

So $r = \csc\theta \Leftrightarrow y = 1$.

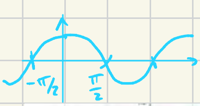
$f(r,\theta) = r\cos\theta$ Since $dA = r * dr\,d\theta$

$= \sqrt{x^2+y^2}$

$$\text{So Vol} = \boxed{\int_0^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^1 \sqrt{x^2+y^2}\,dy\,dx + \int_{\sqrt{3}/2}^{\sqrt{3}}\int_{1/\sqrt{3}\,x}^1 \sqrt{x^2+y^2}\,dy\,dx}$$

#28 To find θ intersection set $r=r$ so

$1+\cos\theta = 1 \Rightarrow \cos\theta = 0 \Rightarrow \theta = -\pi/2, \pi/2$



Then For $\theta \in [-\pi/2, \pi/2]$ notice $\cos\theta \geq 0$

So $1 \leq 1+\cos\theta$ So bounds are

$r \in [1, 1+\cos\theta]$.

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

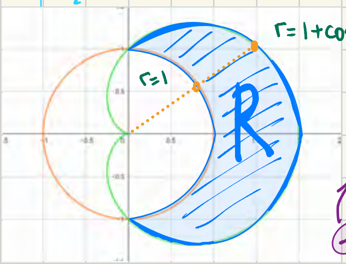
28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

29. One leaf of a rose Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3$, $0 \leq \theta \leq 2\pi$. The region looks like a snail shell.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.



And

$$\text{Vol} = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} 1 * r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}r^2\Big|_1^{1+\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - 1\right] d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{2}\cos^2\theta\,d\theta$$

$$= \sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\Big|_{-\pi/2}^{\pi/2} = 2\left(1 + \frac{1}{4}\frac{\pi}{2} + \frac{1}{8}*0\right) = \boxed{2 + \pi/4}$$

function is odd